Fourier Transform

Biomedical Signals and Systems

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Outline

Periodic Signal

- A signal is periodic, if for some positive value of T

$$x(t+T) = x(t).$$

- The fundamental period of $x(t)$ is the minimum positive non-zero value of T such that the above equation is satisfied.
- The value $f_0 = 1//T$ (or $\omega_0 = 2\pi/T$) is called as the fundamental frequency.
- Complex periodic exponential:

$$e^{j2\pi t/T} = e^{j2\pi f_0 t} = e^{j\omega_0 t}$$

- The set of harmonically related complex eponentials:

$$\phi_k(t) = e^{j2\pi k f_0 t},\ k = 0, \pm 1, \pm 2, \ldots$$

Fourier Series

- The Fourier series of a periodic continuous time signal is

$$x(t) = \sum_{k=-\infty}^{\infty} a_k \phi_k(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

where

$$a_k = \frac{1}{T}\int_T x(t) e^{-j2\pi k f_0 t} dt = \frac{1}{T}\int_T x(t) e^{-j2\pi k t/T} dt$$

Aperiodic Signal

- Consider the following periodic rectangular pulse function: (over a single period)

$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

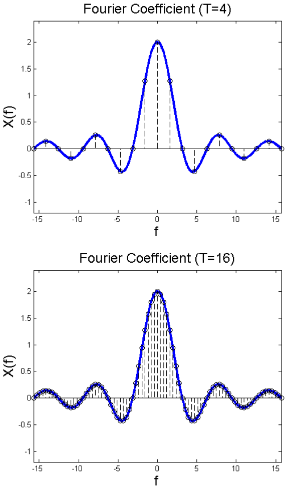
$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-j2\pi k f_0 t} dt = \frac{2\sin(2\pi k f_0 T_1)}{2\pi k f_0 T}$$

where $f_0 = 1/T$.

- Look at the samples of an envelop function

$$Ta_k = \frac{2\sin(2\pi f T_1)}{2\pi f}|_{f=kf_0}$$

Aperiodic Signal



Aperiodic Signal I

- How to represent an aperiodic signal?
- The idea is to think an aperiodic signal as the limit of a periodic signal as the period becomes arbitrary large.
- Assume that $x(t) = 0$, if $|t| > T_1$. We can construct a periodic signal $\tilde{x}(t)$ for which $x(t)$ is one period.
- Consider the Fourier series representation of $\tilde{x}(t)$:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t}$$

Aperiodic Signal II

where

$$
\begin{aligned}
a_k &= \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)e^{-j2\pi k f_0 t}dt \\
&= \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-j2\pi k f_0 t}dt,\ \left(\because \tilde{x}(t) = x(t), |t| < \frac{T}{2}\right) \\
&= \frac{1}{T}\int_{-\infty}^{\infty} x(t)e^{-j2\pi k f_0 t}dt,\ \left(\because x(t) = 0, |t| > \frac{T}{2}\right)
\end{aligned}
$$

Aperiodic Signal III

- Therefore, defining the the envelope $X(f)$ of a_k as

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

we have, for the coefficients Ta_k ,

$$a_k = \frac{1}{T} X(f)|_{f=kf_0} = \frac{1}{T} X(kf_0)$$

- We can express $\tilde{x}(t)$ in terms of $X(f)$ as

$$\begin{aligned} \tilde{x}(t) &= \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k f_0 t} = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(kf_0) e^{j2\pi k f_0 t} \\ &= \sum_{k=-\infty}^{\infty} X(kf_0) e^{j2\pi k f_0 t} f_0, \quad \because f_0 = \frac{1}{T} \end{aligned}$$

Aperiodic Signal IV

- Consider the limiting case $T \to \infty$:

$$T \to \infty \Rightarrow f_0 = \frac{1}{T} \to df,\ kf_0 \approx f$$

$$\begin{aligned} x(t) &= \lim_{T\to\infty} \tilde{x}(t) \\ &= \lim_{f_0 \to df} \sum_{k=-\infty}^{\infty} X(kf_0) e^{j2\pi k f_0 t} f_0 \\ &\approx \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \end{aligned}$$

Fourier Transform

- For a given signal x(t), its Fourier transform and inverse Fourier transform are

$$
\begin{aligned}
X(f) &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \\
x(t) &= \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df
\end{aligned}
$$

Joseph Fourier

https://en.wikipedia.org/wiki/Joseph_Fourier

Outline

Fourier Transform

- The Fourier transform of a function, $x(t)$, is defined as

$$\mathcal{F}\{x(t)\} \equiv X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \tag{1}$$

- The inverse Fourier transform is defined as

$$x(t) = \mathcal{F}^{-1}\{X(f)\} = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \tag{2}$$

- The Fourier transform pairs are often represented as

$$x(t) \quad \Longleftrightarrow \quad X(f)$$
$$x(t) \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad X(f)$$

Fourier Transform

- $X(f)$ is a complex function of f.
- $X(f)$ can be represented by its real part, $\mathrm{Re}\{X(f)\}$, and imaginary part, $\mathrm{Im}\{X(f)\}$.

$$X(f) = \mathrm{Re}\{X(f)\} + j\,\mathrm{Im}\{X(f)\} \tag{3}$$

- $X(f)$ can be represented in polar form by its magnitude and phase

$$X(f) = |X(f)|e^{j\angle X(f)} \tag{4}$$

where

$$|X(f)| = \sqrt{\mathrm{Re}\{X(f)\}^2 + \mathrm{Im}\{X(f)\}^2} \tag{5}$$

$$\angle X(f) = \tan^{-1}\frac{\mathrm{Im}\{X(f)\}}{\mathrm{Re}\{X(f)\}} \tag{6}$$

Fourier Transform

- Another commonly used definitions of the Fourier transform pairs are

$$\mathcal{F}\{x(t)\} \equiv X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt \tag{7}$$

$$x(t) = \mathcal{F}^{-1}\{X(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \tag{8}$$

where $\omega = 2\pi f$.

Fourier Transform: Dirichlet conditions

- For most good functions or well-behaved functions, the integral converges and the Fourier transform exist.
- One set of conditions that guarantees the convergence of the transform is called the Dirichlet conditions:
  1. $x(t)$ is absolutely integrable
  $$\int_{-\infty}^{\infty} |x(t)| dt < \infty$$
  2. $x(t)$ has a finite number of discontinuities and a finite number of maxima and minima during every finite interval.
- If $x(t)$ has finite energy (square integrable)
$$\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty$$
then we are guaranteed that $X(f)$ is finite and exists.

Fourier Transform: rectangular function

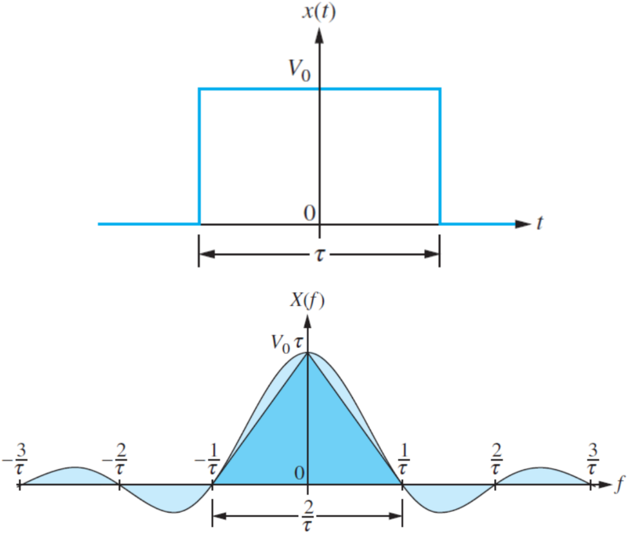
Example

Find the Fourier transform of

$$x(t) = \begin{cases} V_0, & -\frac{\tau}{2} < x < \frac{\tau}{2} \\ 0, & \text{elsewhere} \end{cases} \tag{9}$$

$$\begin{aligned} X(f) &= \int_{-\frac{\tau}{2}}^{\frac{\tau}{2}} V_0 e^{-j2\pi f t} dt \\ &= \frac{V_0}{j2\pi f}\left(e^{j\pi f\tau} - e^{-j\pi f\tau}\right) \\ &= V_0 \frac{\sin(\pi f \tau)}{\pi f} \end{aligned}$$

Fourier Transform: rectangular function



Fourier Transform: triangular function

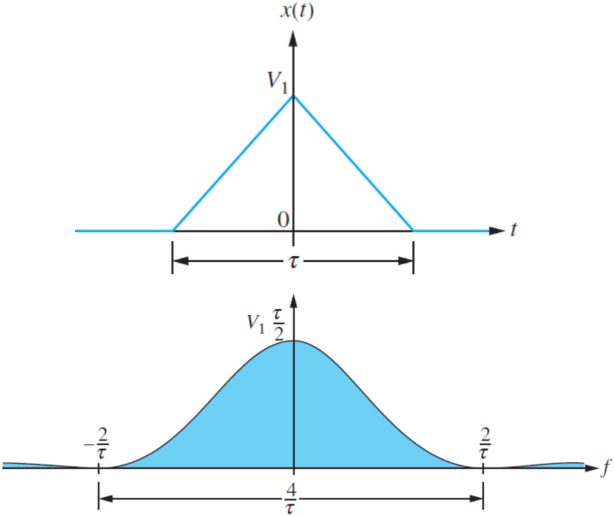
Example

Find the Fourier transform of

$$x(t) = \begin{cases} V_1 \left(1 - 2\frac{|t|}{\tau}\right), & -\frac{\tau}{2} < x < \frac{\tau}{2} \\ 0, & \text{elsewhere} \end{cases} \tag{10}$$

$$\begin{aligned} X(f) &= V_1 \int_{-\frac{\tau}{2}}^{0} \left(1 + 2\frac{t}{\tau}\right) e^{-j2\pi f t} dt + V_1 \int_{0}^{\frac{\tau}{2}} \left(1 - 2\frac{t}{\tau}\right) e^{-j2\pi f t} dt \\ &= \frac{2V_1}{\tau} \left[\frac{\sin(\pi f \tau/2)}{\pi f}\right]^2 \end{aligned}$$

Fourier Transform: triangular function



Fourier Transform: Causal Exponential Function

Example

Find the Fourier transform of

$$x(t) = e^{-\alpha t}u(t), \alpha > 0 \tag{11}$$

$$\begin{aligned} X(f) &= \int_0^\infty e^{-\alpha t} e^{-j2\pi f t} dt = \int_0^\infty e^{-(\alpha + j2\pi f)t} dt \\ &= \frac{1}{\alpha + j2\pi f}, \ -\infty < f < \infty \\ |X(f)| &= \frac{1}{\sqrt{\alpha^2 + 4\pi^2 f^2}} \\ \angle X(f) &= -\tan^{-1}\left\{\frac{2\pi f}{\alpha}\right\} \end{aligned}$$

Fourier Transform: Causal Exponential Function

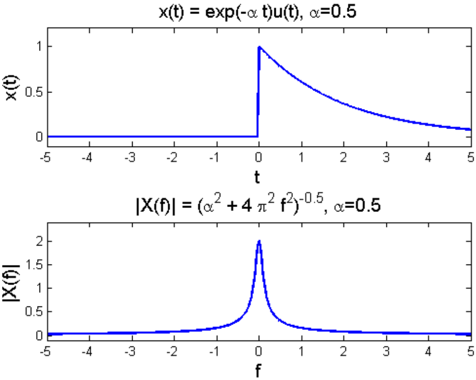


Figure 1: Plot of the causal exponential function, (top) $x(t) = e^{-\alpha t}u(t)$ and its magnitude response, (b) $|X(f)| = \frac{1}{\sqrt{\alpha^2+4\pi^2 f^2}}$, given that $\alpha = 0.5$

Delta Function

Example

Consider the delta function $x(t) = \delta(t)$.

- The Fourier transform is

$$\begin{aligned} X(f) &= \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi f t} dt \\ &= e^{-j2\pi f 0} \int_{-\infty}^{\infty} \delta(t) dt \\ &= 1 \end{aligned}$$

Delta Function

Example

Find the inverse Fourier transform of

$$X(f) = \delta(f - f_0)$$

- The inverse Fourier transform is

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df = \int_{-\infty}^{\infty} \delta(f - f_0) e^{j2\pi f t} df \\ &= e^{j2\pi f_0 t} \int_{-\infty}^{\infty} \delta(f - f_0) df \\ &= e^{j2\pi f_0 t} \end{aligned}$$

$$e^{\pm j2\pi f_0 t} \iff \delta(f \mp f_0)$$

Ideal Low-Pass Filter I

- Consider the signal $x(t)$ whose Fourier transform is

$$X(f) = \begin{cases} 1, & |f| < f_c \\ 0, & |f| > f_c \end{cases}$$

where f_c denotes the cut-off frequency.

- The inverse Fourier transform is

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df = \int_{-f_c}^{f_c} 1 \cdot e^{j2\pi f t} df \\ &= \frac{1}{j2\pi t} \left[e^{j2\pi f_c t} - e^{-j2\pi f_c t} \right] = \frac{j2\sin(2\pi f_c t)}{j2\pi t} \\ &= \frac{\sin(2\pi f_c t)}{\pi t} = 2f_c \cdot \frac{\sin(2\pi f_c t)}{2\pi f_c t} \end{aligned}$$

Ideal Low-Pass Filter II

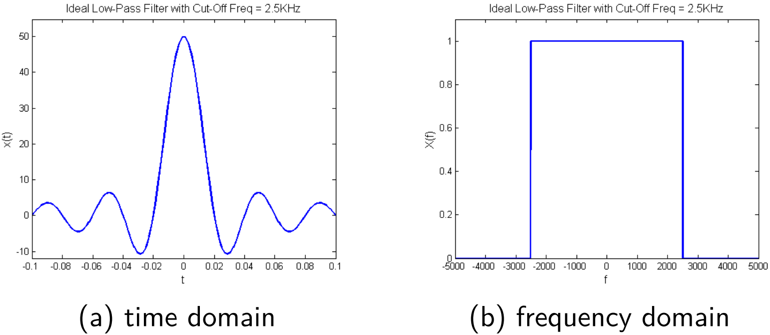


Figure 2: The ideal low-pass filter with $f_c = 2500$Hz.

Outline

Linearity I

- Given that

$$x(t) \iff X(f) \quad y(t) \iff Y(f),$$

then

$$ax(t) + by(t) \iff aX(f) + bY(f) \tag{12}$$

for any functions x, y and constants a, b.

- Fourier transform is a linear operator.
- Find the Fourier transform of

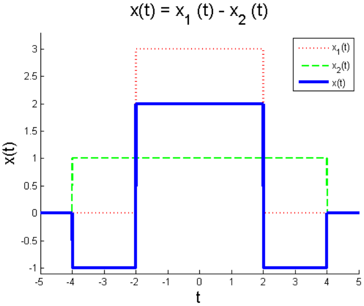
$$x(t) = \begin{cases} 2, & -2 < t < 2 \\ -1, & 2 < |t| < 4 \\ 0, & \text{elsewhere} \end{cases} \tag{13}$$

Linearity II

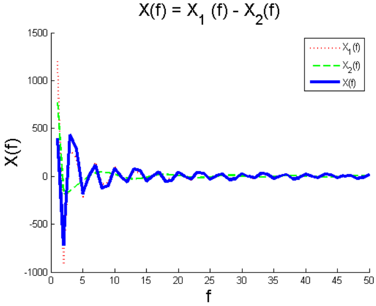
- Let $x(t) = x_1(t) + x_2(t)$, where $x_1(t) = \begin{cases} 3, & -2 < t < 2 \\ 0, & \text{elsewhere} \end{cases}$ and $x_2(t) = \begin{cases} 1, & -4 < t < 4 \\ 0, & \text{elsewhere} \end{cases}$
- Using the linearity property, we find

$$\begin{aligned} X(f) &= X_1(f) - X_2(f) \\ &= 3\frac{\sin(4\pi f)}{\pi f} - \frac{\sin(8\pi f)}{\pi f} \\ &= \frac{\sin(4\pi f)}{\pi f}\left[3 - 2\cos(4\pi f)\right] \end{aligned}$$

Linearity III



(a) time domain



(b) frequency domain

Fourier Transforms of sine and cosine

Example

Given the fact that

$$e^{\pm j2\pi f_0 t} \iff \delta(f \mp f_0)$$

what are the Fourier transforms of sine and cosine?

$$\cos(2\pi f_0 t) = \frac{1}{2}(e^{j2\pi f_0 t} + e^{-j2\pi f_0 t}) \iff \frac{1}{2}[\delta(f - f_0) + \delta(f + f_0)]$$

$$\sin(2\pi f_0 t) = \frac{1}{2j}(e^{j2\pi f_0 t} - e^{-j2\pi f_0 t}) \iff \frac{1}{2j}[\delta(f - f_0) - \delta(f + f_0)]$$

Conjugate Symmetry I

- If $x(t)$ is real, then

$$x(t) = x^*(f) \iff X(f) = X^*(-f) \tag{14}$$

- Proof:

$$\begin{aligned} X^*(-f) &= \left[\int_{-\infty}^{\infty} x(t)e^{-j2\pi(-f)t}dt\right]^* \\ &= \int_{-\infty}^{\infty} \left[x(t)e^{j2\pi ft}\right]^* dt = \int_{-\infty}^{\infty} x^*(t)e^{-j2\pi ft}dt \\ &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt \\ &= X(f) \end{aligned}$$

Conjugate Symmetry II

- If $x(t)$ is real, then $\mathrm{Re}\{X(f)\}$ and $|X(f)|$ are even functions of f, while $\mathrm{Im}\{X(f)\}$ and $\angle X(f)$ are odd functions of f
- A real function $x(t)$ is completely specified from knowing $X(f)$ for $f \geq 0$.

Time Reversal I

- Let the Fourier transform of $x(t)$ be $X(f)$. Then, the Fourier transform of $x(-t)$ is $X(-f)$.

$$\begin{aligned} x(t) &\iff X(f) \\ x(-t) &\iff X(-f) \end{aligned}$$

- Proof:

$$\begin{aligned} \mathcal{F}\{x(-t)\} &= \int_{-\infty}^{\infty} x(-t)e^{-j2\pi ft}dt \\ &= \int_{-\infty}^{\infty} x(t)e^{j2\pi ft}dt \\ &= \int_{-\infty}^{\infty} x(t)e^{-j2\pi(-f)t}dt \\ &= X(-f) \end{aligned}$$

Time Reversal: exponential function I

- Consider the exponential functions:

$$
\begin{aligned}
x_1(t) &= e^{-at}u(t),\ a > 0 \\
X_1(f) &= \int_0^{\infty} e^{-(a+j2\pi f)t} dt = \frac{1}{a + j2\pi f} \\
&= \frac{a}{a^2 + 4\pi^2 f^2} - j\frac{2\pi f}{a^2 + 4\pi^2 f^2} \\
&= \frac{1}{\sqrt{a^2 + 4\pi^2 f^2}} \angle \tan^{-1}\left(-\frac{2\pi f}{a}\right)
\end{aligned}
$$

Time Reversal: exponential function II

$$
\begin{aligned}
x_2(t) &= e^{at}u(-t),\ a > 0 \\
X_2(f) &= \int_{-\infty}^{0} e^{(a-j2\pi f)t} dt = \frac{1}{a - j2\pi f} \\
&= \frac{a}{a^2 + 4\pi^2 f^2} + j\frac{2\pi f}{a^2 + 4\pi^2 f^2} \\
&= \frac{1}{\sqrt{a^2 + 4\pi^2 f^2}} \angle \tan^{-1}\left(\frac{2\pi f}{a}\right)
\end{aligned}
$$

Time Shift

- Let the Fourier transform of $x(t)$ be $X(f)$. Then, a time shift to the right by t_0 adds a negative value $-j2\pi f t_0$ to the Fourier transform of $x(t)$

$$\begin{aligned} x(t) &\iff X(f) \\ x(t-t_0) &\iff X(f)e^{-j2\pi f t_0} \end{aligned}$$

- Proof:

$$\begin{aligned} \mathcal{F}\{x(t-t_0)\} &= \int_{-\infty}^{\infty} x(t-t_0)e^{-j2\pi f t}dt \\ &= e^{-j2\pi f t_0}\int_{-\infty}^{\infty} x(t-t_0)e^{-j2\pi f(t-t_0)}d(t-t_0) \\ &= e^{-j2\pi f t_0}\int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f \tau}d\tau \\ &= X(f)e^{-j2\pi f t_0} \end{aligned}$$

Frequency Shift

- Let the Fourier transform of $x(t)$ be $X(f)$. Then, multiplication of a time signal $x(t)$ by $e^{j2\pi f_0}$ shifts the Fourier transform by f_0 to the right

$$\begin{aligned} x(t) &\Longleftrightarrow X(f) \\ e^{j2\pi f_0} x(t) &\Longleftrightarrow X(f - f_0) \end{aligned}$$

- Proof:

$$\begin{aligned} \mathcal{F}\{e^{j2\pi f_0} x(t)\} &= \int_{-\infty}^{\infty} e^{j2\pi f_0} x(t) e^{-j2\pi f t} dt \\ &= e^{-j2\pi f t_0} \int_{-\infty}^{\infty} x(t) e^{-j2\pi (f - f_0) t} dt \\ &= X(f - f_0) \end{aligned}$$

- The function $e^{j2\pi f_0} x(t)$ is not a real signal.

Modulation Property

- Consider the following manipulation:

$$\begin{aligned}
x(t) &\iff X(f) \\
x(t)e^{j2\pi f_0} &\iff X(f - f_0) \\
x(t)e^{-j2\pi f_0} &\iff X(f + f_0) \\
x(t)\left[\frac{e^{j2\pi f_0} + e^{-j2\pi f_0}}{2}\right] &\iff \frac{X(f - f_0) + X(f + f_0)}{2} \\
= x(t)\cos(2\pi f_0) &
\end{aligned}$$

- The signal $x(t)\cos(2\pi f_0)$ is an amplitude modulated (AM) signal. The amplitude modulation by $\cos(2\pi f_0)$ shifts $X(f)$ to the left and right , centering at $\pm f_0$ where f_0 is the frequency of the sinusoidal carrier.

Time and Frequency Scaling

$$
\begin{aligned}
x(t) &\iff X(f) \\
x(\alpha t) &\iff \frac{1}{|\alpha|} X\left(\frac{f}{\alpha}\right)
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{F}\{x(\alpha t)\} &= \int_{-\infty}^{\infty} x(\alpha t) e^{-j2\pi f t} dt \\
&= \begin{cases} \frac{1}{\alpha} \int_{-\infty}^{\infty} x(\alpha t) e^{-j2\pi \frac{f}{\alpha}(\alpha t)} d(\alpha t), & \alpha > 0 \\ \frac{-1}{\alpha} \int_{-\infty}^{\infty} x(\alpha t) e^{-j2\pi \frac{f}{\alpha}(\alpha t)} d(\alpha t), & \alpha < 0 \end{cases} \\
&= \frac{1}{|\alpha|} X\left(\frac{f}{\alpha}\right)
\end{aligned}
$$

Differentiation

- Let $x(t)$ be a signal with Fourier transform $X(f)$
- By differentiating both sides of the inverse Fourier transform, we obtain

$$
\begin{aligned}
\frac{dx(t)}{dt} &= \frac{d}{dt}\int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \\
&= j2\pi f \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df \\
\mathcal{F}\left\{\frac{d}{dt}x(t)\right\} &= (j2\pi f)X(f) \\
\mathcal{F}\left\{\frac{d^n}{dt^n}x(t)\right\} &= (j2\pi f)^n X(f)
\end{aligned}
$$

- The operation of differentiation in the time domain is replaced by multiplication with $j2\pi$ in the frequency domain.

Integration

- Let $x(t)$ be a signal with Fourier transform $X(f)$
- Then the integration of $x(t)$, its corresponding Fourier transform become

$$\int_{\infty}^{t} x(\tau)d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j2\pi f}X(f) + \frac{X(0)\delta(f)}{2} \tag{15}$$

- The impulse term reflects the dc or average value that can result form integration.

Fourier transform of unit-step function

Let us determine the Fourier transform, $X(f)$, of the unit-step function, $x(t) = u(t)$.

- Recall that

$$g(t) = \delta(t) \overset{\mathcal{F}}{\longleftrightarrow} G(f) = 1$$

- Then the integration of $\delta(t)$, its corresponding Fourier transform becomes

$$x(t) = \int_{\infty}^{t} \delta(\tau) d\tau \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j2\pi f} + \frac{\delta(f)}{2} \tag{16}$$

- The impulse term reflects the dc or average value that can result form integration.

Convolution I

- Convolution in time domain is equivalent to multiplication in the frequency domain

$$x(t) * h(t) \iff X(f) \cdot H(f) \tag{17}$$

- Convolution of two functions $x(t)$ and $h(t)$ is defined by

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau \tag{18}$$

Convolution II

- The Fourier transform of $y(t)$ is

$$\begin{aligned} Y(f) &= \int_{-\infty}^{\infty} y(t)e^{-j2\pi ft}dt \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(t-\tau)h(\tau)e^{-j2\pi ft}d\tau dt \\ &= \int_{-\infty}^{\infty} h(\tau)\left[\int_{-\infty}^{\infty} x(t-\tau)e^{-j2\pi f(t-\tau}d(t-\tau)\right]e^{-j2\pi f\tau}d\tau \\ &= X(f)\int_{-\infty}^{\infty} h(\tau)e^{-j2\pi f\tau}d\tau \\ &= X(f)\cdot H(f) \end{aligned}$$

Product of two time signals

- Multiplication in time domain is equivalent to convolution in the frequency domain

$$x(t) \cdot h(t) \iff \int_{-\infty}^{\infty} X(f-\phi)H(\phi)d\phi \tag{19}$$

- The Fourier transform of the product of two function is given by

$$\begin{aligned}
&\int_{-\infty}^{\infty} x(t)h(t)e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty} x(t)\left[\int_{-\infty}^{\infty} H(\phi)e^{j2\pi\phi t}d\phi\right]e^{-j2\pi ft}dt \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} x(t)e^{-j2\pi(f-\phi)t}dt\right]H(\phi)d\phi \\
&= \int_{-\infty}^{\infty} X(f-\phi)H(\phi)d\phi
\end{aligned}$$

Parseval's Theorem I

- The energy of a square-integrable signal is

$$\int_{-\infty}^{\infty} |x(t)|^2 dt \tag{20}$$

- Parseval's theorem states that the expression for energy in the time domain and the frequency domain are numerical identical:

$$\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df \tag{21}$$

Parseval's Theorem II

$$
\begin{aligned}
\int_{-\infty}^{\infty} |x(t)|^2 dt &= \int_{-\infty}^{\infty} x(t)x^*(t)dt \\
&= \int_{-\infty}^{\infty} x(t) \left[\int_{-\infty}^{\infty} X^*(f)e^{-j2\pi ft} df \right] dt \\
&= \int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty} x(t)e^{-j2\pi ft} dt \right] X^*(f)df \\
&= \int_{-\infty}^{\infty} X(f)X^*(f)df \\
&= \int_{-\infty}^{\infty} |X(f)|^2 df
\end{aligned}
$$

Outline

Gaussian Function

Example

Find the Fourier transform of the Gaussian function

$$g(t) = e^{-\pi t^2} = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} \Big|_{\mu=0,\sigma^2=1/(2\pi)}$$

- The Fourier transform is

$$\begin{aligned} G(f) &= \mathcal{F}\{g(t)\} \\ &= \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-j2\pi f t} dt \\ &= e^{\pi(jf)^2} \cdot \underbrace{\int_{-\infty}^{\infty} e^{-\pi(t^2+2jft+(jf)^2)} dt}_{=1} \\ &= e^{-\pi f^2} \end{aligned}$$

Gaussian Function I

Example

Find the Fourier transform of the Gaussian function

$$x(t) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}}$$

- Known that

$$g(t) = e^{-\pi t^2} \iff G(f) = e^{-\pi f^2}$$

- Observe that:

$$x(t) = \frac{1}{\sqrt{2\pi\sigma^2}} g\left[\frac{(t-\mu)}{\sqrt{2\pi\sigma^2}}\right]$$

Gaussian Function II

- By scaling property

$$h(\alpha t) \iff \frac{1}{|\alpha|} H\left(\frac{f}{\alpha}\right)$$

$$g\left[\frac{1}{\sqrt{2\pi\sigma^2}} t\right] \iff \sqrt{2\pi\sigma^2} e^{-\pi(\sqrt{2\pi\sigma^2} f)^2} = \sqrt{2\pi\sigma^2} e^{-2\pi^2\sigma^2 f^2}$$

- By time-shifting property

$$h(t - t_0) \iff H(f) e^{-j2\pi f t_0}$$

$$g\left[\frac{(t-\mu)}{\sqrt{2\pi\sigma^2}}\right] \iff \sqrt{2\pi\sigma^2} e^{-2\pi^2\sigma^2 f^2} e^{-j2\pi f \mu}$$

Gaussian Function III

- Since Fourier transform is linear, we have

$$\begin{aligned} ah(t) &\iff aH(f) \\ \frac{1}{\sqrt{2\pi\sigma^2}} g\left[\frac{(t-\mu)}{\sqrt{2\pi\sigma^2}}\right] &\iff \frac{1}{\sqrt{2\pi\sigma^2}}\sqrt{2\pi\sigma^2} e^{-2\pi^2\sigma^2 f^2} e^{-j2\pi f\mu} \\ &= e^{-2\pi^2\sigma^2 f^2} e^{-j2\pi f\mu} \end{aligned}$$

- The Fourier transform of $x(t)$ is

$$\begin{aligned} X(f) &= \mathcal{F}\{x(t)\} \\ &= \mathcal{F}\left\{\frac{1}{\sqrt{2\pi\sigma^2}} g\left[\frac{(t-\mu)}{\sqrt{2\pi\sigma^2}}\right]\right\} \\ &= e^{-2\pi^2\sigma^2 f^2} e^{-j2\pi f\mu} \end{aligned}$$

Outline

Autocorrelation

- The autocorrelation function, $r_{xx}(\tau)$, of an energy signal $x(t)$ is defined by

$$r_{xx}(\tau) = \int_{-\infty}^{\infty} x(t)x(t-\tau)dt \tag{22}$$

- For a power signal, the autocorrelation is given by

$$r_{xx}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t-\tau)dt, \; -\infty < \tau < \infty \tag{23}$$

- Autocorrelation is an even function

$$r_{xx}(-\tau) = r_{xx}(\tau) \tag{24}$$

Example: Autocorrelation

Find the autocorrelation of the energy signal $x(t) = e^{-t}u(t)$.

$$
\begin{aligned}
r_{xx}(\tau) &= \int_{-\infty}^{\infty} x(t)x(t-\tau)dt \\
&= \int_{-\infty}^{\infty} \left[e^{-t}u(t)\right]\left[e^{-(t-\tau)}u(t-\tau)\right]dt \\
&= e^{\tau}\int_{0}^{\infty} e^{-2t}u(t-\tau)dt \\
\tau < 0 \qquad & r_{xx}(\tau) = e^{\tau}\int_{0}^{\infty} e^{-2t}u(t-\tau)dt = \frac{1}{2}e^{\tau} \\
\tau > 0 \qquad & r_{xx}(\tau) = e^{\tau}\int_{t}^{\infty} e^{-2t}u(t-\tau)dt = \frac{1}{2}e^{-\tau} \\
r_{xx}(\tau) &= \frac{1}{2}e^{-|\tau|}, \ -\infty < \tau < \infty
\end{aligned}
$$

Example: Autocorrelation

Find the autocorrelation of the power signal $x(t) = \cos 2\pi t$.

$$
\begin{aligned}
r_{xx}(\tau) &= \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \cos 2\pi t x \cos 2\pi(t-\tau) dt \\
&= \lim_{T\to\infty} \frac{1}{4T} \int_{-T}^{T} \left[\cos 2\pi(2t-\tau) + \cos 2\pi\tau\right] dt \\
&= \lim_{T\to\infty} \frac{1}{4T} \int_{-T}^{T} \cos 2\pi(2t-\tau) dt \\
&+ \cos 2\pi\tau \cdot \left[\lim_{T\to\infty} \frac{1}{4T} \int_{-T}^{T} dt\right] \\
&= \frac{1}{2} \cos 2\pi\tau
\end{aligned}
$$

Cross-Correlation

- The cross-correlation function, $r_{xy}(\tau)$, between two energy signals $x(t)$ and $y(t)$ is defined by

$$r_{xy}(\tau) = \int_{-\infty}^{\infty} x(t)y(t-\tau)dt \tag{25}$$

- For a power signal, the cross-correlation is given by

$$r_{xy}(\tau) = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)y(t-\tau)dt, \ -\infty < \tau < \infty \tag{26}$$

- Cross-correlation has the following property

$$r_{xy}(\tau) = r_{yx}(-\tau) \tag{27}$$

Example: Cross-Correlation

Example

Find the cross-correlation $r_{xy}(\tau)$ and $r_{yx}(\tau)$, where $x(t) = \cos t$ and $y(t) = e^{-t}u(t)$. Also show that $r_{xy}(\tau) = r_{yx}(-\tau)$.

$$
\begin{aligned}
r_{xy}(\tau) &= \int_{-\infty}^{\infty} x(t)y(t-\tau)dt = \int_{-\infty}^{\infty} \cos t \left[e^{-(t-\tau)}u(t-\tau)\right] dt \\
&= e^{\tau} \int_{\tau}^{\infty} e^{-t} \cos t dt = \frac{1}{2}(\cos\tau - \sin\tau) \\
r_{yx}(\tau) &= \int_{-\infty}^{\infty} y(t)x(t-\tau)dt = \int_{-\infty}^{\infty} e^{-t}u(t)\cos(t-\tau)dt \\
&= \int_{0}^{\infty} e^{-t}\cos(t-\tau)dt = \frac{1}{2}(\cos\tau + \sin\tau) \\
r_{xy}(\tau) &= r_{yx}(-\tau)
\end{aligned}
$$

Outline

Fourier Transform Properties

Property	Time Domain	$\Leftrightarrow$	Frequency Domain
Definition	$x(t) = \int_{-\infty}^{\infty} X(f)e^{j2\pi ft}df$	$\Leftrightarrow$	$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$
Linearity	$ax(t) + bg(t)$	$\Leftrightarrow$	$aX(f) + bG(f)$
Zero Time	$x(0)$	$\Leftrightarrow$	$\int_{-\infty}^{\infty} X(f)df$
Zero Freq	$\int_{-\infty}^{\infty} x(t)dt$	$\Leftrightarrow$	$X(0)$
Real	$x(t) = x^*(t)$	$\Leftrightarrow$	$X(f) = X^*(-f)$
Even	$x(t) = x(-t)$	$\Leftrightarrow$	$X(f) = X(-f)$
Odd	$x(t) = -x(-t)$	$\Leftrightarrow$	$X(f) = -X(-f)$
Duality	$X(t)$	$\Leftrightarrow$	$x(-f)$
Time Shift	$x(t - t_0)$	$\Leftrightarrow$	$e^{-2\pi f_0 t}X(f)$
Freq Shift	$x(t)e^{2\pi f_0 t}$	$\Leftrightarrow$	$X(f - f_0)$
Modulation	$x(t)\cos(2\pi f_0 t)$	$\Leftrightarrow$	$\frac{X(f-f_0)+X(f+f_0)}{2}$

Fourier Transform Properties

Property	Time Domain	$\Leftrightarrow$	Frequency Domain
Scale	$x(\alpha t)$	$\Leftrightarrow$	$\frac{1}{\|\alpha\|} X\left(\frac{f}{\alpha}\right)$
Reversal	$x(-t)$	$\Leftrightarrow$	$X(-f)$
Times t	$tx(t)$	$\Leftrightarrow$	$-\frac{1}{j2\pi}\frac{d}{df}X(f)$
Integration	$\int_{-\infty}^{t} x(\tau)d\tau$	$\Leftrightarrow$	$\frac{X(f)}{j2\pi f} + \frac{X(0)\delta(f)}{2}$
Differentiation	$\frac{d}{dt}x(t)$	$\Leftrightarrow$	$j2\pi f X(f)$
Convolution	$\int_{-\infty}^{\infty} x(t-\tau)h(\tau)d\tau$	$\Leftrightarrow$	$X(f)H(f)$
Multiplication	$x(t)h(t)$	$\Leftrightarrow$	$\int_{-\infty}^{\infty} X(f-\phi)h(\phi)d\phi$

Outline

Why Uncertainty Principle?

- What is the uncertainty principle about?
- It says $x(t)$ and $X(f)$ cannot be both concentrated.
- Let us recall the scaling property of Fourier transform:

$$\begin{aligned} x(t) &\iff X(f) \\ x(\alpha t) &\iff \frac{1}{\alpha} X\left(\frac{f}{\alpha}\right),\ \alpha > 0 \end{aligned}$$

- That is, "reducing the support of a signal by scaling by a factor a increases the support of its Fourier transform by a factor a^{-1} ", and vice versa.

Normalized Energy Function

- Consider a time function $g(t0$ with total energy

$$||x(t)||^2 = \int |x(t)|^2 dt$$

- Define

$$\sigma^2(t) = \frac{1}{||x(t)||^2} \int (t - t_0)^2 |x(t)|^2 dt$$

as a spread measure of (random) variable t.

- Note that $p(t) = |x(t)|^2 / ||x(t)||^2$ is a probability density function as
  1. $p(t) \geq 0$
  2. $\int p(t) dt = 1$
- If we choose $t_0 = \frac{1}{||x||^2} \int t |x(t)|^2 dt$, i.e., the mean value m_t of t, $\sigma^2(t)$ can be interpreted as the variance of t.

Heisenberg's Uncertainty Principle

Suppose that $x(t)$ is a function in L^2 of the line with all of the following properties: $||x(t)||^2 = 1$, $tx(t) \in L^2$, and $fX(f) \in L^2$. Define the following numbers:

- The average value of t : $m_t = \int_{-\infty}^{\infty} t|x(t)|^2 dt$
- The uncertainty of t: $\sigma_t = \left[\int_{-\infty}^{\infty} (t - m_t)^2 |x(t)|^2 dt\right]^{\frac{1}{2}}$
- The average value of f : $m_f = \int_{-\infty}^{\infty} f|X(f)|^2 df$
- The uncertainty of f: $\sigma_f = \left[\int_{-\infty}^{\infty} (f - m_f)^2 |X(f)|^2 df\right]^{\frac{1}{2}}$

Then

$$\sigma_t \sigma_f \geq \frac{1}{4\pi}$$

Heisenberg's Uncertainty Principle

- The fact that $|x(t)|^2 = 1$, that is, $\int_{-\infty}^{\infty} |x(t)|^2 = 1$ makes the function $|x(t)|^2$ a probability density function for t considered as a random variable.
- The definition of the average value of t and the uncertainty of t are the standard probabilistic definitions of mean m_t and standard deviation.
- By Parseval's theorem $\int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} |X(f)|^2 df = 1$, this indicates that $|X(f)|^2$ is also a probability density function for f as a random variable, and we can compute its mean m_f and standard deviation σ_f.

Proof of Uncertainty Principle I

- Assume that both the average time and the average frequency are zero, i.e., $m_t = m_f = 0$.
- Given that $\int_{-\infty}^{\infty} |x(t)|^2 dt = 1$, consider the following integration by parts

$$\begin{aligned}
\int u dv &= uv - \int v du \\
u &= |x(t)|^2 = x(t)\overline{x(t)} \\
du &= \left[x'(t)\overline{x(t)} + x(t)\overline{x'(t)}\right] dt = \left[2\text{Re}(x(t)\overline{x'(t)})\right] dt \\
v &= t \\
dv &= dt
\end{aligned}$$

Proof of Uncertainty Principle II

- Now consider the total energy of $x(t)$. We have

$$\begin{aligned}
1 &= \int_{-\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} u dv \\
&= t|x(t)|^2|_{-\infty}^{\infty} - 2\text{Re} \int_{-\infty}^{\infty} t(x(t)\overline{x'(t)})dt \\
&= -2\text{Re} \int_{-\infty}^{\infty} t(x(t)\overline{x'(t)})dt \\
&\leq 2 \left| \int_{-\infty}^{\infty} t(x(t)\overline{x'(t)})dt \right| \\
&\leq 2 \left(\int_{-\infty}^{\infty} |tx(t)|^2 dt \right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |x'(t)|^2 dt \right)^{\frac{1}{2}}
\end{aligned}$$

The last line above is justified by the Cauchy-Schwarz inequality.

Proof of Uncertainty Principle III

- We can obverse that

$$\begin{aligned}
\int_{-\infty}^{\infty} |tx(t)|^2 dt &= \int_{-\infty}^{\infty} (t-0)^2 |x(t)|^2 dt = \sigma_t^2 \\
\mathcal{F}\left\{x'(t)\right\} &= (j2\pi f) X(f) \\
\int_{-\infty}^{\infty} |x'(t)|^2 dt &= \int_{-\infty}^{\infty} |j2\pi f X(f)|^2 df \\
&= 4\pi^2 \int_{-\infty}^{\infty} (f-0)^2 |X(f)|^2 df = 4\pi^2 \sigma_f^2
\end{aligned}$$

- Finally,

$$\begin{aligned}
1 &\leq 2\left(\int_{-\infty}^{\infty} |tx(t)|^2 dt\right)^{\frac{1}{2}} \left(\int_{-\infty}^{\infty} |x'(t)|^2 dt\right)^{\frac{1}{2}} \\
&= 2\left(\sigma_t^2\right)^{\frac{1}{2}} \left(4\pi^2 \sigma_f^2\right)^{\frac{1}{2}} = 4\pi \sigma_t \sigma_f
\end{aligned}$$

Proof of Uncertainty Principle IV

- This proves the Heisenberg's uncertainty principle

$$\sigma_t \sigma_f \geq \frac{1}{4\pi}$$

- Under what condition will the equality hold??!!
- Some examples:

Table 1: Examples of Uncertainty Principle

$x(t)$	$X(f)$	σ_t	σ_f	$\sigma_t\sigma_f$
$\begin{cases}\sqrt{\frac{3}{2}}(1-\lvert t\rvert), & -1 \leq t \leq 1 \\ 0, & \text{elsewhere}\end{cases}$	$\sqrt{\frac{3}{2}}\frac{1-\cos 2\pi f}{2\pi^2 f^2}$	$\frac{1}{\sqrt{10}}$	$\frac{\sqrt{3}}{2\pi}$	$\frac{1}{4\pi}\sqrt{\frac{6}{5}}$
$\frac{\sqrt{2\pi}}{\pi(1+t^2)}$	$\sqrt{2\pi}e^{-2\pi\lvert f\rvert}$	1	$\frac{1}{2\sqrt{2\pi}}$	$\frac{\sqrt{2}}{4\pi}$
$\sqrt[4]{2}e^{-\pi t^2}$	$\sqrt[4]{2}e^{-\pi f^2}$	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{2\sqrt{\pi}}$	$\frac{1}{4\pi}$