Lecture #2
Probability

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Ching-Han Hsu, Ph.D.
Department of Biomedical Engineering
and Environmental Sciences
National Tsing Hua University

Random Experiment and Sample Space

Definition (Random Experiment)

An experiment that can result in different outcomes, even though it is repeated in the same manner every time, is called a random experiment.

Definition (Sample Space)

The set of all possible outcomes of a random experiment is called the **sample space** of the experiment. The sample space is denoted as S (or Ω).

Sample Spaces: Tossing Coin

Example (Coin Tossing Once)

A coin is tossed once. Then $S = \{H, T\}$, where H stands for head and T for tail.

Example (Coin Tossing Twice)

If the same coin is tossed twice, then $S = \{HH, HT, TH, TT\}$.

Example (Coin Tossing Twice)

If we are interested in the number of heads occurring, then $S = \{0, 1, 2\}$.

Sample Space: Drawing Cards

Example (Drawing cards from a Bridge Deck)

A card is drawn from a well shuffled bridge deck. The possible sample spaces are as follows:

- the set of suit $S_1 = \{C, D, H, S\}$
- the set of denomination $S_2 = \{A, 2, 3, \ldots, J, Q, K\}$
- the set of 52 points

$$S_3 = \{AC, 2C, \ldots, KC, \ldots, AS, \ldots, KS\}$$

Note: Club (C), Diamond (D), Heart (H) and Spade (S).

Sample Spaces: Discrete and Continuous

Definition (Discrete Sample Space)

A sample space S is said to be discrete if it contains at most countable number of elements.

Definition (Continuous Sample Space)

A sample space S is said to be continuous if its elements constitute a continuum: all points in an interval, or all points in the plane, or all points in the k-dimensional space, and so on.

Continuous Sample Spaces: Choosing a Point

Example (From an Interval)

A point is chosen form the interval $[0, 1]$. The sample space is

$$S = \{x : 0 \leq x \leq 1\}$$

and contains an uncountable number of points.

Example (From a Region)

If a point is chosen form the square bounded by points $(0, 0), (1, 0), (0, 1)$ and $(1, 1)$, then the sample space

$$S = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

Note that this space is also uncountable.

Sample Spaces: Life Length of a Light Bulb

Example (Continuous Sample Space)

The time until failure of a light bulb manufactured at certain plant is recorded. One can simply takes

$$S = \{t : t \geq 0\},$$

which is a continuous sample space.

Example (Discrete Sample Space)

If the time is recorded to the nearest hour, then

$$S = \{0, 1, 2, \ldots\},$$

which is a discrete sample space.

Note that the type of a sample space is subject to how the measurement is interpreted.

Events

Definition (Event)

An **event** is a set of outcomes and hence a subset of the sample space of a random experiment.

- The **empty set**, denoted by $\emptyset$, and S are also subsets (i.e. events).
- Events are denoted by the capital letters, like $A, B, C, \ldots$.

Definition (Mutually Exclusion)

Two events, denoted as A and B, such that

$$A \cap B = \emptyset$$

are said to be **mutually exclusive** (or **disjoint**).

Events: Birth Month

Example

Which month does a person's birthday fall? An obvious choice for the sample space is

$$S = \{Jan, Feb, Mar, Apr, May, Jun, Jul, Aug, Sep, Oct, Nov, Dec\}$$

Example

The outcomes correspond to a long month, i.e., a month with 31 days. This is the event

$$L = \{Jan, Mar, May, Jul, Aug, Oct, Dec\}.$$

Events: Birth Month (Cont'd)

Example

For example, R is the event that corresponds to the months that have the letter r in their (full) name

$$R = \{Jan, Feb, Mar, Apr, Sep, Oct, Nov, Dec\}.$$

Example

The long months that contain the letter r are

$$L \cap R = \{Jan, Mar, Oct, Dec\}.$$

The set $L \cap R$ is called the intersection of L and R and occurs if both L and R occur. This is also an event!!

Intersection, Union, Complement

Some of the basic set operations are summarized below in terms of events:

- The union of two events is the event that consists of all outcomes that are contained in either of the two events, denoted by $A \cup B$.
- The intersection of two events is the event that consists of all outcomes that are contained in both of the two events, denoted by $A \cap B$.
- The complement of an event E in a sample space is the set of outcomes in the sample space that are not in the event, denoted by E' or E^c.

Theorem (DeMorgan's laws)

For any two events A and B we have

$$
\begin{aligned}
(A \cup B)' &= A' \cap B' \\
(A \cap B)' &= A' \cup B'
\end{aligned}
$$

Axioms of Probability

Definition (Axioms of Probability)

Probability is a number that is assigned to each member of a collection of events from a random experiment that satisfies the following properties: If S is the sample space and E is any event in a random experiment,

1. $P(S) = 1$
2. $0 \leq P(E) \leq 1$
3. For two events E_1 and E_2 with $E_1 \cap E_2 = \emptyset$

$$P(E_1 \cup E_2) = P(E_1) + P(E_2)$$

Probability and Set Operations

$$
\begin{aligned}
P(A \cup B) &= P(A) + P(B) - P(A \cap B) \\
P(A \cup B) &\leq P(A) + P(B) \\
P(\cup_{i=1}^{n} A_i) &= \sum_{i=1}^{n} P(A_i) - \sum_{i<j-2}^{n} P(A_i \cap A_j) \\
&\quad + \sum_{i<j<k-3}^{n} P(A_i \cap A_j \cap A_k) \\
&\quad + \cdots \\
&\quad + (-1)^{n-1} P(A_1 \cap A_2 \cap \cdots \cap A_n) \\
P(\cup_{i=1}^{n} A_i) &\leq \sum_{i=1}^{n} P(A_i)
\end{aligned}
$$

Review the textbook for more details about set operations.

Advanced and Optional Topic

Definition (Sigma Field)

A class (or collection) ξ of subsets of S is said to be a **sigma field** (**σ-field**), if it satisfies the following properties:

1. $S \in \xi$,
2. if $E \in \xi$, then $E^c \in \xi$ (or $E' \in \xi$),
3. (closure under countable union) if $E_1, E_2, \ldots \in \xi$, then $\cup_{i=1}^{\infty} E_i \in \xi$,
4. (closure under countable intersection) if $E_1, E_2, \ldots \in \xi$, then $\cap_{i=1}^{\infty} E_i \in \xi$,

Ex. (Coin Tossing Once) A coin is tossed once with sample space $S = \{H, T\}$. A σ-field associated with S can be defined as follows:

$$\xi = \{S, \emptyset, \{H\}, \{T\}\}$$

Advanced and Optional Topic

Definition (Axioms of Probability)

Let S be a set of outcomes and ξ be the associated collection of events. A set function P defined on ξ is called a probability if the following axioms are satisfied:

Axiom I. $0 \leq P(E) \leq 1, \ \forall E \in \xi$,

Axiom II. $P(S) = 1$,

Axiom III. if $\{E_i\}$ is a sequence of mutually exclusive events ($E_i \cap E_j = \emptyset$ for $i \neq j$), then

$$P(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i).$$

Note that this is a formal definition of probability.

Example: Coin Tossing

Example

A coin is tossed and the up face is record.

Sol.

- The sample sample $S = \{H, T\}$,
- The σ-field $\xi = \{S, \emptyset, \{H\}, \{T\}\}$.
- The probability is defined as $P(H) = p = 1 - P(T), 0 \leq p \leq 1$.
- If the coin is fair, $p = \frac{1}{2}$.

Example: Rolling a Pair of Dice

Example

Suppose that a pair of fair dice is rolled. The random experiment has 36 outcomes. The sample space is

$$S = \{(i,j),\ i = 1,\ldots,6,\ j = 1,\ldots,6\}$$

Since the dice are fair, all pairs are equally likely and

$$P(i,j) = \frac{1}{36},\ \forall i,j.$$

What is the probability that the sum of face values is at least 5?

Example: Rolling a Pair of Dice (Cont'd)

Solution

$$
\begin{aligned}
& P\{(i,j) : i + j \geq 5,\ i,j \in [1,2,3,4,5,6]\} \\
&= 1 - P\{(i,j) : i + j \leq 4,\ i,j \in [1,2,3,4,5,6]\} \\
&= 1 - P\{(1,1),(1,2),(1,3),(2,1),(2,2),(3,1)\} \\
&= 1 - \frac{6}{36} = \frac{5}{6}
\end{aligned}
$$

Axioms of Probability

Proposition

For the empty set $\emptyset$, $P(\emptyset) = 0$.

Proof.

If S is the sample space and E is any event in a random experiment, since $\emptyset \cap \emptyset = \emptyset$ and $\emptyset \cup \emptyset = \emptyset$, it follows from the Axiom III that

$$P(\emptyset) = P(\emptyset \cup \emptyset) = P(\emptyset) + P(\emptyset) = 2P(\emptyset).$$

Hence, $P(\emptyset) = 0$. □

Axioms of Probability

Proposition

For any event, $P(E') = 1 - P(E)$.

Proof.

Since $E \cap E' = \emptyset$ and $E \cup E' = S$, it follows from the Axiom II and III that

$$P(S) = 1 = P(E \cup E' = S) = P(E) + P(E').$$

Hence, $P(E') = 1 - P(E)$. $\square$

Conditional Probability

Definition

The **conditional probability** of an event B given an event A, denoted as $P(B|A)$, is

$$P(B|A) = P(A \cap B)/P(A), \quad A > 0 \tag{1}$$

Theorem (Multiplication Rule)

$$P(A \cap B) = P(A|B)P(B) = P(B|A)P(A) \tag{2}$$

Example: Conditional Probability

Example (Coin Tossing)

Consider three independent tossings of a fair coin. Let X be the number of heads and Y be the number of tails before the first head.

$Y \backslash X$	0	1	2	3
0	0	htt	hth, hht	hhh
1	0	tht	thh	0
2	0	tth	0	0
3	ttt	0	0	0

$Y \backslash X$	0	1	2	3	$P(Y = y)$
0	0	$1/8$	$2/8$	$1/8$	$4/8$
1	0	$1/8$	$1/8$	0	$2/8$
2	0	$1/8$	0	0	$1/8$
3	$1/8$	0	0	0	$1/8$
$P(X = x)$	$1/8$	$3/8$	$3/8$	$1/8$	1

Example: Conditional Probability

Let us compute the conditional distribution of X given $Y = y,\ y = 0, 1, 2, 3$.

$$P(X = x|Y = 0) = \begin{cases} 1/4, & x = 1, 3 \\ 1/2, & x = 2 \\ 0, & \text{otherwise} \end{cases}$$

$$P(X = x|Y = 1) = \begin{cases} 1/2, & x = 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

$$P(X = x|Y = 2) = \begin{cases} 1, & x = 1 \\ 0, & \text{otherwise} \end{cases}$$

$$P(X = x|Y = 3) = \begin{cases} 1, & x = 0 \\ 0, & \text{otherwise} \end{cases}$$

Example: Conditional Probability

Example

A day's production of 850 parts contains 50 parts that do not meet customer requirements. Two parts are selected randomly without replacement from the batch. What is the probability that the second part is defective given that the first part is defective?

- Let A denote the event that the first part selected is defective, $P(A) = \frac{50}{850}$.
- Let B denote the event that the second part selected is defective, $P(B|A) = \frac{49}{849}$.
- What is the probability that first and second selected parts are both defective,

$$P(A \cap B) = P(B|A)P(A) = \frac{50}{850}\frac{49}{849}$$

Example: Multiplication Rule

Example

The probability that the first stage of machining operation meets specification is 0.90. Failures are due to metal vibrations, fixture alignment, blade condition, etc. Given that the first stage meets specifications, the second stage of machining meets specification is 0.95. What is the probability that both stages meet specifications?

- Let A and B denote the events that the first and second stages meet specifications, respectively.
- $P(A) = 0.90$ and $P(B|A) = 0.95$.
- The probability that both stages meet specifications:

$$P(A \cap B) = P(B|A)P(A) = 0.95 \times 0.9 = 0.855.$$

Total Probability

Theorem (Total Probability)

For any events A and B,

$$\begin{aligned} P(B) &= P(B \cap A) + P(B \cap A') \\ &= P(B|A)P(A) + P(B|A')P(A') \end{aligned} \tag{3}$$

Theorem (Total Probability)

Assume $A_1, A_2, \ldots, A_k$ are k mutually exclusive and exhaustive sets (i.e., $\cup_{i=1}^{k} A_i = S$). Then

$$\begin{aligned} P(B) &= P(B \cap A_1) + P(B \cap A_2) + \cdots + P(B \cap A_k) \\ &= P(B|A_1)P(A_1) + \cdots + P(B|A_k)P(A_k) \end{aligned} \tag{4}$$

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Example: Total Probability Rule

Example

In semiconductor manufacturing, the probability is 0.1 that a chip subject to high levels of contamination causes a product failure. The probability is 0.005 that a chip subject to low (not high) levels of contamination causes a product failure. In a particular production run, 20% of the chips are subject to high levels of contamination. What is the probability that a product using one of these chips fails?

Prob. of Failure	Level of Contam.	Prob. of Level
0.1	High	0.2
0.005	Low (not High)	0.8

Example: Total Probability Rule

- Let F denote the event that the product fails, and H denote the event that the chip is exposed to high level of contamination. Then

Prob. of Failure	Prob. of Level
$P(F\|H) = 0.1$	$P(H) = 0.2$
$P(F\|H') = 0.005$	$P(H') = 0.8$

- The probability that a product using one of these chips fails is

$$
\begin{aligned}
P(F) &= P(F|H)P(H) + P(F|H')P(H') \\
&= 0.1 \times 0.2 + 0.005 \times 0.8 = 0.024
\end{aligned}
$$

Independence

Definition (Independence)

Two events are **independent** if any one of the following equivalent statements is true:

$$\begin{aligned} &(1) \qquad P(A|B) = P(A) \\ &(2) \qquad P(B|A) = P(B) \\ &(3) \qquad P(A \cap B) = P(A)P(B) \end{aligned} \tag{5}$$

Definition (Independence of Multiple Events)

The events $E_1, E_2, \ldots, E_n$ are independent if and only if for any subset of these events $E_{i_1}, E_{i_2}, \ldots, E_{i_k}$,

$$P(E_{i_1} \cap E_{i_2} \cap \ldots E_{i_k}) = P(E_{i_1}) \times P(E_{i_2}) \times \cdots P(E_{i_k}) \tag{6}$$

Example: Independence

Example

A day's production of 850 parts contains 50 parts that do not meet customer requirements. Two parts are selected randomly **with replacement** from the batch. What is the probability that the second part is defective (event B) given that the first part is defective (event A)?

- The probability that the first part is defective, is $P(A) = \frac{50}{850}$.
- The probability of B does not depend on whether or not the part was defective, $P(B|A) = P(B) = \frac{50}{850}$.
- What is the probability that first and second selected parts are both defective,

$$P(A \cap B) = P(B|A)P(A) = \frac{50}{850}\frac{50}{850} = 0.035$$

Bayes' Theorem

Theorem (Bayes' Theorem)

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}, \quad P(B) > 0 \tag{7}$$

Theorem (Bayes' Theorem)

If $E_1, E_2, \ldots, E_k$ are k mutually exclusive and exhaustive events and B is any event,

$$P(E_1|B) = \frac{P(B|E_1)P(E_1)}{\sum_{i=1}^{k} P(B|E_i)P(E_i)}, \quad P(B) > 0 \tag{8}$$

Medical Diagnostic

Example

The probability that the test correctly identifies someone with the illness as positive is 0.99. The probability that the test correctly identifies someone without the illness as negative is 0.95. The incidence of the illness in the general population is 0.0001. You take the test, and the result is positive. What is the probability that you have the illness?

- Let D denote the event that you have illness, and $P(D) = 0.0001$. The probability that you are healthy (no illness) is $P(D') = 1 - P(D)$.
- Let S denote the event that the test signal positive, $P(S|D) = 0.99$.
- The probability that the test correctly identifies someone without the illness as negative is $P(S'|D') = 0.95$.

Medical Diagnostic

- The probability that the test **falsely** identifies someone without the illness as positive is $P(S|D') = 1 - P(S'|D') = 0.05$.
- When the test result is positive, the probability that you have the illness is $P(D|S)$:

$$
\begin{aligned}
& P(D|S) \\
&= P(D \cap S)/P(S) \\
&= P(D \cap S)/\left[P(S \cap D) + P(S \cap D')\right] \\
&= P(S|D)P(D)/\left[P(S|D)P(D) + P(S|D')P(D')\right] \\
&= 0.99 \cdot 0.0001/\left[0.99 \cdot 0.0001 + 0.05 \cdot 0.9999\right] \\
&= 0.002
\end{aligned}
$$

The probability of having the illness given a positive result from the test is only 0.002.

Reliability Analysis

Example

A system contains two components, A and B, connected in series as shown in the following diagram.

The system will function only if both components function. The probability that A functions is given by $P(A) = 0.98$, and the probability that B functions is given by $P(B) = 0.95$. Assume that A and B function independently. Find the probability that the system functions.

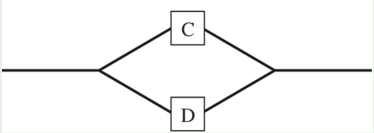
- Since the system will function only if both components function, it follows that

$$\begin{aligned} P(\text{system functions}) &= P(A \cap B) = P(A) \times P(B) \\ &= 0.95 \times 0.98 = 0.931 \end{aligned}$$

Reliability Analysis I

Example

A system contains two components, C and D, connected in parallel as shown in the following diagram.



The system will function if either C or D functions. The probability that C functions is 0.90, and the probability that D functions is 0.85. Assume C and D function independently. Find the probability that the system functions.

Reliability Analysis II

- Since the system will function so long as either of the two components functions, it follows that

$$
\begin{aligned}
&P(\text{system functions})\\
&\quad = \quad P(C \cup D)\\
&\quad = \quad P(C) + P(D) - P(C \cap D)\\
&\quad = \quad P(C) + P(D) - P(C) \times P(D)\\
&\quad = \quad 0.95 \times 0.98 = 0.931
\end{aligned}
$$