Joint Probability of Discrete RVs

Joint Probability of Continuous RVs

Covariance and Correlation

Bivariant Normal Distribution

Linear Functions of Random Variables

Lecture #6

Joint Probability Distributions

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Joint Probability Distribution

Definition

The **joint probability mass function** of the discrete random variables X and Y, denoted as $f_{XY}(x, y)$, satisfies

- $f_{XY}(x, y) \geq 0$
- $\sum_x \sum_Y f_{XY}(x, y) = 1$
- $f_{XY}(x, y) = P(X = x, Y = y)$

Definition

Let A be any set consisting pairs of (x, y) values. Then the probability $P[(x, y) \in A]$ is obtained by summing the joint pmf over pairs in A:

$$P[(x, y) \in A] = \sum_{(x,y) \in A} P(x, y)$$

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Example: Joint Probability Distribution

Example

A large insurance agency services a number of customers who have purchased both a homeowner's policy and an automobile policy. For each type of policy, a deductible amount must be specified. For automobile policy, the choices are $\$100$ and $\$200$, whereas for homeowner's policy, the choices are 0, $\$100$, and $\$200$. Let $X =$ the deductible amount of the auto policy and $Y =$ the deductible amount of the homeowner's policy. The joint pmf for possible (X, Y) are summarized in the following table:

$P(x, y)$	$y = 0$	$y = 100$	$y = 200$
$x = 100$	0.20	0.10	0.20
$x = 200$	0.05	0.15	0.30

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Example: Joint Probability Distribution

- The total probability

$$\sum_x \sum_y P(x, y) = 1$$

- The probability of $\$100$ deductible on both policies is

$$P(X = 100, Y = 100) = 0.1$$

- The probability of $Y \geq 100$ is

$$P(Y \geq 100) = 0.1 + 0.2 + 0.15 + 0.3 = 0.75$$

which corresponds to the (X, Y) pairs $(100, 100)$, $(100, 200)$, $(200, 100)$, and $(200, 200)$.

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Marginal Probability Distribution

Definition

If X and Y are discrete random variables with joint probability mass function $f_{XY}(x,y)$, then the **marginal probability mass function** of X and Y are

$$f_X(x) = P(X = x) = \sum_y f_{XY}(x,y) \tag{1}$$

and

$$f_Y(y) = P(Y = y) = \sum_x f_{XY}(x,y) \tag{2}$$

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Example: Joint Probability Distribution (Con't)

- The probability of $X = 100$ and $X = 200$, are computing row totals in the joint probability table:

$$P(X = 100) = 0.2 + 0.1 + 0.2 = 0.5$$

and

$$P(X = 200) = 0.05 + 0.15 + 0.3 = 0.5$$

$P(x, y)$	$y = 0$	$y = 100$	$y = 200$	
$x = 100$	0.20	0.10	0.20	0.5
$x = 200$	0.05	0.15	0.30	0.5
	0.25	0.25	0.5	

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Conditional Probability

Definition

Given discrete random variables X and Y with joint probability mass function $f_{XY}(x,y)$, the **conditional probability mass functions** of Y given $X = x$ is

$$f_{Y|x}(y) = f_{XY}(x,y)/f_X(x), \quad f_X(x) > 0 \tag{3}$$

Note that a conditional probability function $f_{Y|x}(y)$ is also a probability mass function. The following properties are satisfied:

- $f_{Y|x}(y) \geq 0$
- $\sum_y f_{Y|x}(y) = 1$
- $P(Y = y|X = x) = f_{Y|x}(y)$

Example: Joint Probability Distribution (Con't)

- Let us fix the random variable X and find the conditional probability mass function of Y:

$f_Y(y\|x)$	$y = 0$	$y = 100$	$y = 200$
$x = 100$	$\frac{0.2}{0.5} = 0.4$	$\frac{0.1}{0.5} = 0.2$	$\frac{0.2}{0.5} = 0.4$
$x = 200$	$\frac{0.05}{0.5} = 0.1$	$\frac{0.15}{0.5} = 0.3$	$\frac{0.3}{0.5} = 0.6$

- Let us fix the random variable Y and find the conditional probability mass function of X:

$f_X(x\|y)$	$y = 0$	$y = 100$	$y = 200$
$x = 100$	$\frac{0.2}{0.25} = 0.8$	$\frac{0.1}{0.25} = 0.4$	$\frac{0.2}{0.5} = 0.4$
$x = 200$	$\frac{0.05}{0.25} = 0.2$	$\frac{0.15}{0.25} = 0.6$	$\frac{0.3}{0.5} = 0.6$

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Conditional Mean and Variance

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Definition

The **conditional mean** of Y given $X = x$, denoted as $E(Y|x)$ or $\mu_{Y|x}$, is

$$E(Y|x) = \sum_y y f_{Y|x}(y) \tag{4}$$

and the **conditional variance** of Y given $X = x$, denoted as $V(Y|x)$ or $\sigma_{Y|x}$, is

$$V(Y|x) = \sum_y \left(y - \mu_{Y|x}\right)^2 f_{Y|x}(y) = \sum_y y^2 f_{Y|x}(y) - \mu_{Y|x}^2 \tag{5}$$

Example: Joint Probability Distribution (Con't)

- Find the conditional mean and variance of Y given $X = 100$:

$$E(Y|100) = 0 \times 0.4 + 100 \times 0.2 + 200 \times 0.4 = 100$$

and

$$V(Y|100) = 0 \times 0.4 + 100^2 \times 0.2 + 200^2 \times 0.4 - 100^2 = 8000$$

- Find the conditional mean and variance of X given $Y = 200$:

$$E(X|200) = 100 \times 0.4 + 200 \times 0.6 = 160$$

and

$$V(X|200) = 100^2 \times 0.4 + 200^2 \times 0.6 - 160^2 = 2400$$

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Independence

Definition

For discrete random variables X and Y, if any one of the following properties is true, the others are also true. X and Y are **independent**.

1. $f_{XY}(x,y) = f_X(x) f_Y(y)$ for all x and y
2. $f_{Y|x}(y) = f_Y(y)$ for all x and y with $f_X(x) > 0$
3. $f_{X|y}(x) = f_X(x)$ for all x and y with $f_Y(y) > 0$
4. $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ for any sets A and B in the range of X and Y, respectively

In previous insurance example, consider

$$P(100,100) = 0.1 \neq 0.5 \times 0.25 = P_X(X = 100)P_Y(Y = 100),$$

so X and Y are not independent. Independence of X and Y requires every entry in the joint probability table be the product of the corresponding row and column marginal probabilities.

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Multiple Discrete Random Variables

Definition (Joint Probability Mass Function)

The **joint probability mass function** of $X_1, X_2, \ldots, X_p$ is

$$f_{X_1,X_2,\ldots,X_p}(x_1, x_2, \ldots, x_p) = P(X_1 = x_1, X_2 = x_2, \ldots, X_p = x_p) \quad (6)$$

for all points $(x_1, x_2, \ldots, x_p)$ in the range of $X_1, X_2, \ldots, X_p$.

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Definition (Marginal Probability Mass Function)

If $X_1, X_2, \ldots, X_p$ are discrete random variables with joint probability mass function $f_{X_1,X_2,\ldots,X_p}(x_1, x_2, \ldots, x_p)$, the **marginal probability mass function** of any X_i is

$$f_{X_i}(x_i) = P(X_i = x_i) = \sum_{x_j, j \neq i} f_{X_1,X_2,\ldots,X_p}(x_1, x_2, \ldots, x_p) \tag{7}$$

where the sum is over all points in the range of $X_1, X_2, \ldots, X_p$ for which $X_i = x_i$.

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Definition (Mean and Variance)

The mean $E(X_i)$ and variance $V(X_i)$ for $i = 1, 2, \ldots, p$ can be determined as follows:

$$E(X_i) = \sum_{x_1, x_2, \ldots, x_p} x_i f_{X_1, X_2, \ldots, X_p}(x_1, x_2, \ldots, x_p) \quad (8)$$

$$V(X_i) = \sum_{x_1, x_2, \ldots, x_p} (x_i - \mu_{X_i})^2 f_{X_1, X_2, \ldots, X_p}(x_1, x_2, \ldots, x_p) \quad (9)$$

where the sum is over all points in the range of $X_1, X_2, \ldots, X_p$.

Multiple Discrete Random Variables

Definition (Independence)

Discrete random variables $X_1, X_2, \ldots, X_p$ are independent if and only if

$$f_{X_1,X_2,\ldots,X_k}(x_1, x_2, \ldots, x_k) = f_{X_1}(x_1) f_{X_2}(x_k) \cdots f_{X_k}(x_k) \qquad (10)$$

for any $2 \leq k \leq p$.

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Definition (Distribution of a Subset of RVs)

If $X_1, X_2, \ldots, X_p$ are discrete random variables with joint probability mass function $f_{X_1,X_2,\ldots,X_p}(x_1, x_2, \ldots, x_p)$, the **joint probability mass function** of $X_1, X_2, \ldots, X_k$, $k < p$, is

$$\begin{aligned} & f_{X_1,X_2,\ldots,X_k}(x_1, x_2, \ldots, x_k) \\ &= P(X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k) \\ &= \sum P(X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k) \qquad (11) \end{aligned}$$

where the sum is over all points in the range $X_1, X_2, \ldots, X_p$ for which $X_1 = x_1, X_2 = x_2, \ldots, X_k = x_k$.

Joint Probability Distribution

Definition

A **joint probability density function** for the continuous random variables X and Y, denoted as $f_{XY}(x,y)$, satisfies the following properties

- $f_{XY}(x,y) \geq 0, \quad \forall x, y$
- $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dxdy = 1$
- For any region R of two-dimensional space

$$P((X,Y) \in R) = \iint_R f_{XY}(x,y)dxdy$$

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Example: Continuous Joint Probability Distribution

Example

Let the random variable X denote the time until a computer server connects to your machines, and let Y denote the time until the server authorizes you as a valid user. Each of these random variables measures the wait from a common starting time and $X < Y$. Assume that the joint probability density function for X and Y is

$$f_{XY}(x, y) = 6 \times 10^{-6} \exp(-0.001x - 0.002y), \quad 0 < X < Y$$

Determine the probability that $X < 1000$ and $Y < 2000$.

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Example: Continuous Joint Prob. Distribution (Con't)

The probability that $X < 1000$ and $Y < 2000$ is determined by

$$P(X < 1000, Y < 2000) = \int_0^{1000} \int_x^{2000} f_{XY}(x,y)\,dy\,dx$$

$$
\begin{aligned}
&= \int_0^{1000} \int_x^{2000} 6 \times 10^{-6} \exp(-0.001x - 0.002y)\,dy\,dx \\
&= 6 \times 10^{-6} \int_0^{1000} \left(\frac{e^{-0.002x} - e^{-4}}{0.002} \right) e^{-0.001x}\,dx \\
&= 0.003 \int_0^{1000} \left(e^{-0.003x} - e^{-4} e^{-0.001x} \right) dx \\
&= 0.003 \left[\left(\frac{1 - e^{-3}}{0.003} \right) - e^{-4} \left(\frac{1 - e^{-1}}{0.001} \right) \right] \\
&= 0.915
\end{aligned}
$$

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Definition (Marginal Probability Density Function)

If the joint probability density function of continuous random variables X and Y is $f_{XY}(x,y)$, the **marginal probability density functions** of X and Y, are

$$f_X(x) = \int_y f_{XY}(x,y)dy \tag{12}$$

and

$$f_Y(y) = \int_x f_{XY}(x,y)dx \tag{13}$$

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Example: Continuous Joint Prob. Distribution (Con't)

- Find the marginal probability density function of Y:

$$
\begin{aligned}
f_Y(y) &= \int_0^y f_{XY}(x,y)dx \\
&= \int_0^y 6 \times 10^{-6} e^{(-0.001x-0.002y)} dx \\
&= 6 \times 10^{-6} e^{-0.002y} \int_0^y e^{-0.001x} dx \\
&= 6 \times 10^{-6} e^{-0.002y} \left(\frac{1 - e^{-0.001y}}{0.001} \right) \\
&= 6 \times 10^{-3} \left(e^{-0.002y} - e^{-0.003y} \right)
\end{aligned}
$$

- Determine the marginal distribution of X.
- Calculate the probability that Y exceeds 2000 milliseconds.

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Definition (Conditional Probability Density Function)

Given continuous random variables X and Y is with joint probability density function $f_{XY}(x, y)$, the **conditional probability density functions** of Y given $X = x$ is

$$f_{Y|x}(y) = \frac{f_{XY}(x, y)}{f_X(x)}, \quad f_X(x) > 0 \tag{14}$$

The conditional probability density function $f_{Y|x}(y)$ is a probability density function for all and satisfies the following properties:

- $f_{Y|x}(y) \geq 0$
- $\int f_{Y|x}(y)dy = 1$
- $P(Y \in B|X = x) = \int_B f_{Y|x}(y)dy$

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Example: Continuous Joint Prob. Distribution (Con't)

- Determine the marginal distribution of X.

$$\begin{aligned}
f_X(x) &= \int_x^{\infty} f_{XY}(x,y)dy \\
&= \int_x^{\infty} 6 \times 10^{-6} e^{(-0.001x-0.002y)} dy \\
&= 6 \times 10^{-6} e^{-0.001x} \int_x^{\infty} e^{-0.002y} dy \\
&= 6 \times 10^{-6} e^{-0.001x} \left(\frac{e^{-0.002x}}{0.002} \right) \\
&= 0.003 e^{-0.003x}, \quad x > 0
\end{aligned}$$

This is an exponential distribution with $\lambda = 0.003$.

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Example: Continuous Joint Prob. Distribution (Con't)

- The conditional probability density function of Y given $X = x$ is

$$
\begin{aligned}
f_{Y|x}(y) &= \frac{f_{XY}(x,y)}{f_X(x)} \\
&= \frac{6 \times 10^{-6} e^{(-0.001x - 0.002y)}}{0.003 e^{-0.003x}} \\
&= 0.002 e^{(0.002x - 0.002y)}, \quad 0 < x < y
\end{aligned}
$$

- Calculate the probability that Y exceeds 2000 milliseconds given $x = 1500$.

$$
\begin{aligned}
P(y > 2000 | x = 1500) &= \int_{2000}^{\infty} f_{Y|1500}(y) dy \\
&= \int_{2000}^{\infty} 0.002 e^{(0.002(1500) - 0.002y)} dy \\
&= 0.002 e^{3} \frac{e^{-4}}{0.002} = 0.368
\end{aligned}
$$

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Definition (Conditional Mean and Variance)

The **conditional mean** of Y given $X = x$, denoted as $E(Y|x)$ or $\mu_{Y|x}$, is

$$E(Y|x) = \int y f_{Y|x}(y) dy \quad (15)$$

and the **conditional variance** of Y given $X = x$, denoted as $V(Y|x)$ or $\sigma_{Y|x}$, is

$$V(Y|x) = \int (y - \mu_{Y|x})^2 f_{Y|x}(y) dy = \int y^2 f_{Y|x}(y) dy - \mu_{Y|x}^2 \quad (16)$$

Example: Continuous Joint Prob. Distribution (Con't)

- The conditional mean of Y given $x = 1500$ is

$$\begin{aligned} E(Y|X = 1500) &= \int_{1500}^{\infty} y f_{Y|1500}(y) dy \\ &= \int_{1500}^{\infty} y 0.002 e^{(0.002(1500) - 0.002y)} dy \\ &= 0.002 e^{3} \int_{1500}^{\infty} y e^{-0.002y} dy \\ &= 2000 \end{aligned}$$

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Definition (Independence)

For continuous random variables X and Y, if one of the following properties is true, the others are also true, and X and Y are said to be **independent**:

1. $f_{XY}(x,y) = f_X(x)f_Y(y)$ for all x and y
2. $f_{Y|x}(y) = f_Y(y)$ for all x and y with $f_X(x) > 0$
3. $f_{X|y}(x) = f_X(x)$ for all x and y with $f_Y(y) > 0$
4. $P(X \in A, Y \in B) = P(X \in A)P(Y \in B)$ for any sets A and B in the range of X and Y, respectively

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Example: Independence

- Assume that the joint probability density function for X and Y is

$$f_{XY}(x,y) = 6\times 10^{-6}\exp(-0.001x-0.002y), \quad 0 < X < Y$$

- The marginal density functions of X and Y are

$$\begin{aligned} f_X(x) &= 0.003e^{-0.003x} \\ f_Y(y) &= 6 \times 10^{-3}\left(e^{-0.002y} - e^{-0.003y}\right) \end{aligned}$$

- X and Y are NOT independent, because

$$f_{XY}(x,y) \neq f_X(x) f_Y(y)$$

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Example: Independence

- Assume that the joint probability density function for X and Y is

$$f_{XY}(x,y) = 2 \times 10^{-6} \exp(-0.001x - 0.002y)$$

where $0 < X$ and $0 < Y$.

- The marginal density functions of X and Y are

$$\begin{aligned} f_X(x) &= \int_0^\infty 2 \times 10^{-6} e^{-0.001x-0.002y} dy \\ &= 0.001 e^{-0.001x} \\ f_Y(y) &= \int_0^\infty 2 \times 10^{-6} e^{-0.001x-0.002y} dx \\ &= 0.002 e^{-0.002y} \end{aligned}$$

- X and Y are independent, because

$$f_{XY}(x,y) = f_X(x) f_Y(y)$$

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Expected Value of a Function

Definition (Expected Value of a Function)

The expected value of a function $h(X,Y)$ of two random variables

$$E[h(X,Y)] = \begin{cases} \sum\sum h(x,y) f_{XY}(x,y) & \text{discrete} \\ \iint h(x,y) f_{XY}(x,y)\,dx\,dy & \text{continuous} \end{cases} \tag{17}$$

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Example: Concert Tickets

Example

Five friends have purchased tickets to a certain concert. If the tickets are for seats $1-5$ and the tickets are randomly distributed among the five. What is the expected number of seats separating any particular two of the five?

- Let X and Y denote the seat numbers of the first and second individuals, respectively. Possible (X, Y) pairs are:

$h(x,y)$		x = 1	2	3	4	5
	1	–	0	1	2	3
	2	0	–	0	1	2
y	3	1	0	–	0	1
	4	2	1	0	–	0
	5	3	2	1	0	–

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Example: Concert Tickets

- The joint pmf of (X, Y) is

$$f_{XY}(x,y) = \begin{cases} \frac{1}{20}, & x, y = 1, 2, 3, 4, 5; x \neq y \\ 0 & \text{otherwise} \end{cases}$$

- The number of seats separating the two individuals is

$$h(X, Y) = |X - Y| - 1$$

- The expected value of $h(X, Y)$ is

$$\begin{aligned} E[h(X,Y)] &= \sum_{(x,y)} h(x,y) f_{XY}(x,y) \\ &= \sum_{x=1}^{5} \sum_{y=1, y\neq x}^{5} (|x-y| - 1) \cdot \frac{1}{20} \\ &= 1 \end{aligned}$$

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Covariance

Definition (Covariance)

The **covariance** between two random variables X and Y, denoted as $cov(X, Y)$ or σ_{XY}, is

$$\sigma_{XY} = E[(X - \mu_X)(Y - \mu_Y)] = E(XY) - \mu_X\mu_Y \tag{18}$$

- The covariance between X and Y describes the variation between the two random variables.
- Covariance is a measure of linear relationship between the random variables.

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Covariance

$$
\begin{aligned}
& E[(X-\mu_X)(Y-\mu_Y)] \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(x-\mu_X)(y-\mu_Y)f_{XY}(x,y)dxdy \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}[xy-\mu_X y - x\mu_Y + \mu_X\mu_Y]f_{XY}(x,y)dxdy \\
= & \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}xyf_{XY}(x,y)dxdy - \mu_X\mu_Y - \mu_X\mu_Y + \mu_X\mu_Y \\
= & E(XY) - \mu_X\mu_Y
\end{aligned}
$$

$$
\begin{aligned}
\mu_X\mu_Y &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}x\mu_Y f_{XY}(x,y)dxdy \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}y\mu_X f_{XY}(x,y)dxdy
\end{aligned}
$$

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Covariance between X and Y

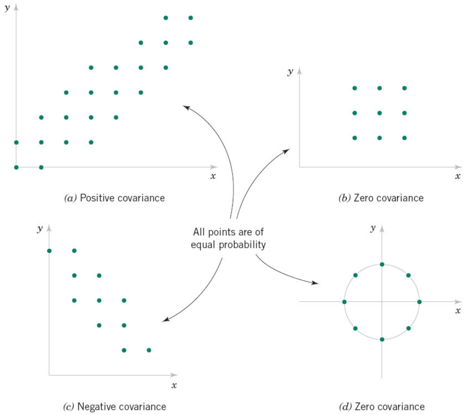

Figure 1: Joint probability distributions and the sign of covariance between X and Y.

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Correlation Coefficient

Definition (Correlation Coefficient)

The **correlation coefficient** between two random variables X and Y, denoted as ρ_{XY}, is

$$\rho_{XY} = \frac{cov(X,Y)}{\sqrt{V(X)V(Y)}} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} \tag{19}$$

- For any two random variables X and Y $-1 \leq \rho_{XY} \leq 1$.
- Correlation is dimensionless.
- If the points in the joint probability distribution of X and Y that receives positive prbability tend to fall along a line of positive (or negative) slope, ρ_{XY} is near $+1$ (or -1).
- If X and Y are independent random variables, $\sigma_{XY} = \rho_{XY} = 0$. $\sigma_{XY} = \rho_{XY} = 0$ does not imply that X and Y are independent.

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Example: Covariance of Two Discrete RVs

Example

Suppose that the random variable X has the following distribution: $P(X = 1) = 0.2$, $P(X = 2) = 0.6$, $P(X = 3) = 0.2$. Let $Y = 2X + 5$, i.e., $P(Y = 7) = 0.2$, $P(Y = 9) = 0.6$, $P(Y = 11) = 0.2$. Determine ρ_{XY}.

- $E(X) = 1 \times 0.2 + 2 \times 0.6 + 3 \times 0.2 = 2$
- $E(X^2) = 1 \times 0.2 + 2^2 \times 0.6 + 3^2 \times 0.2 = 4.4$
- $V(X) = E(X^2) - (E(X))^2 = 4.4 - 2^2 = 0.4$
- $E(Y) = 7 \times 0.2 + 9 \times 0.6 + 11 \times 0.2 = 9$
- $E(Y^2) = 7^2 \times 0.2 + 9^2 \times 0.6 + 11^2 \times 0.2 = 82.6$
- $V(Y) = E(Y^2) - (E(Y))^2 = 82.6 - 9^2 = 1.6$
- $E(XY) = 1 \times 7 \times 0.2 + 2 \times 9 \times .6 + 3 \times 11 \times 0.2 = 18.8$
- $\sigma_{XY} = E(XY) - \mu_X \mu_Y = 18.8 - 2 \times 9 = 0.8$
- $\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y} = \frac{0.8}{\sqrt{0.4 \times 1.6}} = \frac{0.8}{0.8} = 1$

Joint Probability of Discrete RVs

Joint Probability of Continuous RVs

Covariance and Correlation

Bivariant Normal Distribution

Linear Functions of Random Variables

Bivariant Normal Distribution

Definition (Bivariant Normal Distribution)

The probability density function of a **bivariant normal distribution** random variables

$$f_{XY}(x, y; \mu_X, \mu_Y, \sigma_X, \sigma_Y, \rho_{XY}) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}^2}} \exp\left\{\frac{-1}{2(1-\rho_{XY}^2)} \left[\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{2\rho_{XY}(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right]\right\} \quad (20)$$

for $-\infty < x < \infty$ and $-\infty < y < \infty$, with parameters $-\infty < \mu_x < \infty$, $-\infty < \mu_Y < \infty$, $\sigma_X > 0$, $\sigma_Y > 0$, and $-1 < \rho_{XY} < 1$.

Bivariant Normal Distribution: $\rho_{XY} = 0$

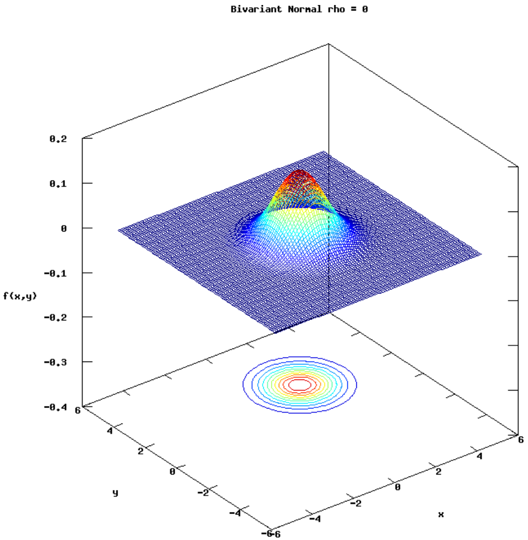

Figure 2: Bivariate Normal Distribution with $\rho_{XY} = 0$.

Joint Probability of Discrete RVs

Joint Probability of Continuous RVs

Covariance and Correlation

Bivariant Normal Distribution

Linear Functions of Random Variables

Bivariant Normal Distribution: $\rho_{XY} = 0.5$

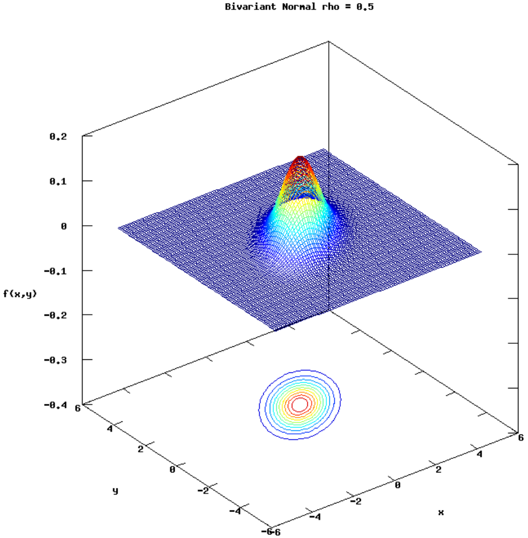

Figure 3: Bivariate Normal Distribution with $\rho_{XY} = 0.5$.

Joint Probability of Discrete RVs

Joint Probability of Continuous RVs

Covariance and Correlation

Bivariant Normal Distribution

Linear Functions of Random Variables

Bivariant Normal Distribution: $\rho_{XY} = 0.95$

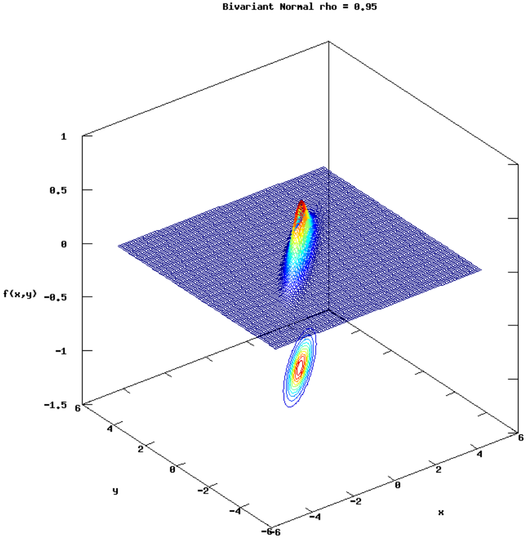

Figure 4: Bivariate Normal Distribution with $\rho_{XY} = 0.95$.

Joint Probability of Discrete RVs

Joint Probability of Continuous RVs

Covariance and Correlation

Bivariant Normal Distribution

Linear Functions of Random Variables

Bivariant Normal Distribution: $\rho_{XY} = -0.95$

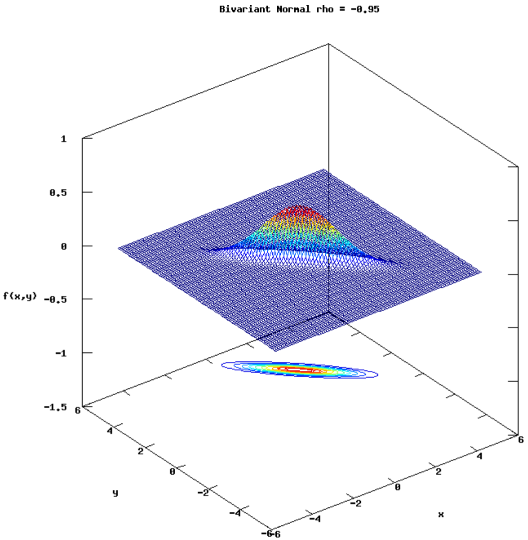

Figure 5: Bivariate Normal Distribution with $\rho_{XY} = -0.95$.

Joint Probability of Discrete RVs

Joint Probability of Continuous RVs

Covariance and Correlation

Bivariant Normal Distribution

Linear Functions of Random Variables

Bivariant Normal Distribution

If X and Y are bivariant normal distribution with joint probability density $f_{XY}(x, y; \mu_X, \mu_Y, \sigma_X, \sigma_Y, \rho_{XY})$,

- the marginal probability distribution of X and Y are $X \sim N(\mu_X, \sigma_X^2)$ and $Y \sim N(\mu_Y, \sigma_Y^2)$, respectively;
- the conditional probability distribution of Y given $X = x$ is normal with mean

$$\mu_{Y|x} = \mu_Y - \mu_X \rho \frac{\sigma_Y}{\sigma_X} + \frac{\sigma_Y}{\sigma_X} \rho x$$

  and variance

$$\sigma_{Y|x}^2 = \sigma_Y^2 (1 - \rho^2);$$

- the correlation between X and Y is ρ;
- with $\rho = 0$, X and Y are independent.

Joint Probability of Discrete RVs

Joint Probability of Continuous RVs

Covariance and Correlation

Bivariant Normal Distribution

Linear Functions of Random Variables

Bivariant Normal Distribution: Independence and Uncorrelation

$$
\begin{aligned}
f_{XY}(x, y; \mu_X, \mu_Y, \sigma_X, \sigma_Y, \rho_{XY} = 0) = & \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho_{XY}^2}} \\
& \exp\left\{ \frac{-1}{2(1-\rho_{XY}^2)} \cdot \right. \\
& \left. \left[\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{2\rho_{XY}(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} \right] \right\} \\
= & \frac{1}{2\pi\sigma_X\sigma_Y} \exp\left\{ \frac{-1}{2} \left[\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} \right] \right\} \\
= & \frac{1}{\sigma_X\sqrt{2\pi}} \exp\left\{ -\frac{(x-\mu_X)^2}{2\sigma_X^2} \right\} \\
\cdot & \frac{1}{\sigma_Y\sqrt{2\pi}} \exp\left\{ -\frac{(y-\mu_Y)^2}{2\sigma_Y^2} \right\} \\
= & f_X(x; \mu_X, \sigma_X) \cdot f_Y(y; \mu_Y, \sigma_Y)
\end{aligned}
$$

Joint Probability of Discrete RVs

Joint Probability of Continuous RVs

Covariance and Correlation

Bivariant Normal Distribution

Linear Functions of Random Variables

Linear Combination

Definition (Linear Combination)

Given random variables $X_1, X_2, \ldots X_p$ and constants $c_1, c_2, \ldots c_p$,

$$Y = c_1X_1 + c_2X_2 + \cdots + c_pX_p \tag{21}$$

is a **linear combination** of $X_1, X_2, \ldots X_p$.

Joint Probability of Discrete RVs

Joint Probability of Continuous RVs

Covariance and Correlation

Bivariant Normal Distribution

Linear Functions of Random Variables

Mean of a Linear Combination

Theorem

If $Y = c_1X_1 + c_2X_2 + \cdots + c_pX_p$, *then*

$$E(Y) = c_1E(X_1) + c_2E(X_2) + \ldots + c_pE(X_p) \qquad (22)$$

$$\begin{aligned}
E(Y) &= E(c_1X_1 + c_2X_2 + \cdots + c_pX_p) \\
&= \sum \cdots \sum [(c_1x_1 + c_2x_2 + \ldots + c_px_p) \cdot \\
&\quad f_{X_1,X_2,\ldots X_p}(x_1, x_2, \ldots, x_p)] \\
&= c_1 \left[\sum \cdots \sum x_1 \cdot f_{X_1,X_2,\ldots X_p}(x_1, x_2, \ldots, x_p) \right] \\
&+ c_2 \left[\sum \cdots \sum x_2 \cdot f_{X_1,X_2,\ldots X_p}(x_1, x_2, \ldots, x_p) \right] \\
&\quad \ldots \\
&+ c_p \left[\sum \cdots \sum x_p \cdot f_{X_1,X_2,\ldots X_p}(x_1, x_2, \ldots, x_p) \right] \\
&= c_1E(X_1) + c_2E(X_2) + \ldots + c_pE(X_p)
\end{aligned}$$

Joint Probability of Discrete RVs

Joint Probability of Continuous RVs

Covariance and Correlation

Bivariant Normal Distribution

Linear Functions of Random Variables

Variance of a Linear Combination

Theorem

If $X_1, X_2, \ldots X_p$ are random variables, and $Y = c_1X_1 + c_2X_2 + \cdots + c_pX_p$, then in general

$$\begin{aligned} V(Y) &= c_1^2V(X_1) + c_2^2V(X_2) + \cdots + c_p^2V(X_p) \\ &+ 2\sum_{i<j}\sum c_ic_j\text{cov}(X_i, X_j). \end{aligned} \quad (23)$$

If $X_1, X_2, \ldots X_p$ are **independent***,*

$$V(Y) = c_1^2V(X_1) + c_2^2V(X_2) + \cdots + c_p^2V(X_p) \quad (24)$$

Joint Probability of Discrete RVs

Joint Probability of Continuous RVs

Covariance and Correlation

Bivariant Normal Distribution

Linear Functions of Random Variables

Example: Error Propagation

Example

A semiconduction production consists of three layers. If the variances in thickness of the first, sencond and third layers are 25, 40, and 30 nanometers squared, what is the vairance of the thickness of the final product?

- Let X_1, X_2, X_3 and X are random variables denoting the thickness of the respective layers, and the final product.
- Then $X = X_1 + X_2 + X_3$.
- The variance of X according to the Eq. is

$$\begin{aligned} V(X) &= V(X_1) + V(X_2) + V(X_3) \\ &= 25 + 40 + 30 = 95 \text{ nm}^2 \end{aligned}$$

Joint Probability of Discrete RVs

Joint Probability of Continuous RVs

Covariance and Correlation

Bivariant Normal Distribution

Linear Functions of Random Variables

Mean and Variance of an Average

Theorem

If $X_1, X_2, \ldots X_p$ *are random variables with* $E(X_i) = \mu, i = 1, \ldots, p$, *then the random variable*

$$\bar{X} = (X_1 + X_2 + \cdots + X_p)/p \tag{25}$$

has the mean

$$E(\bar{X}) = \mu \tag{26}$$

If $X_1, X_2, \ldots X_p$ *are also independent with variance* $V(X_i) = \sigma^2, i = 1, \ldots, p$, *then*

$$V(\bar{X}) = \frac{\sigma^2}{p} \tag{27}$$

Joint Probability of Discrete RVs

Joint Probability of Continuous RVs

Covariance and Correlation

Bivariant Normal Distribution

Linear Functions of Random Variables

Reproductive Property of Normal Dist.

Theorem

If $X_1, X_2, \ldots X_p$ *are independent, normal random variables with* $E(X_i) = \mu_i, V(X_i) = \sigma_i^2, i = 1, \ldots, p$, *then*

$$Y = c_1 X_1 + c_2 X_2 + \cdots + c_p X_p \quad (28)$$

is a normal random variable with

$$E(Y) = c_1 \mu_1 + c_2 \mu_2 + \cdots + c_p \mu_p \quad (29)$$

and

$$V(Y) = c_1^2 \sigma_1^2 + c_2^2 \sigma_2^2 + \cdots + c_p^2 \sigma_p^2 \quad (30)$$

Joint Probability of Discrete RVs

Joint Probability of Continuous RVs

Covariance and Correlation

Bivariant Normal Distribution

Linear Functions of Random Variables