# The Normal Distribution

*(continued)*

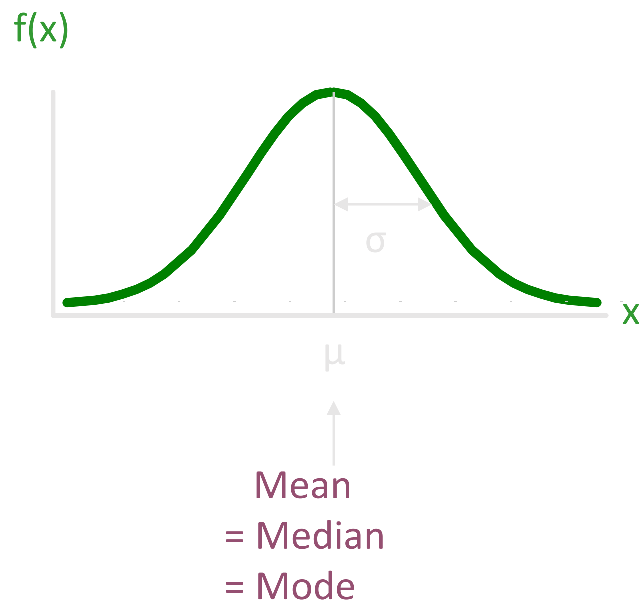
- Bell Shaped
- Symmetrical
- Mean, Median and Mode are Equal

**Location is determined by the mean, μ**

**Spread is determined by the standard deviation, σ**

**The random variable has an infinite theoretical range:**

**+ ∞ to − ∞**

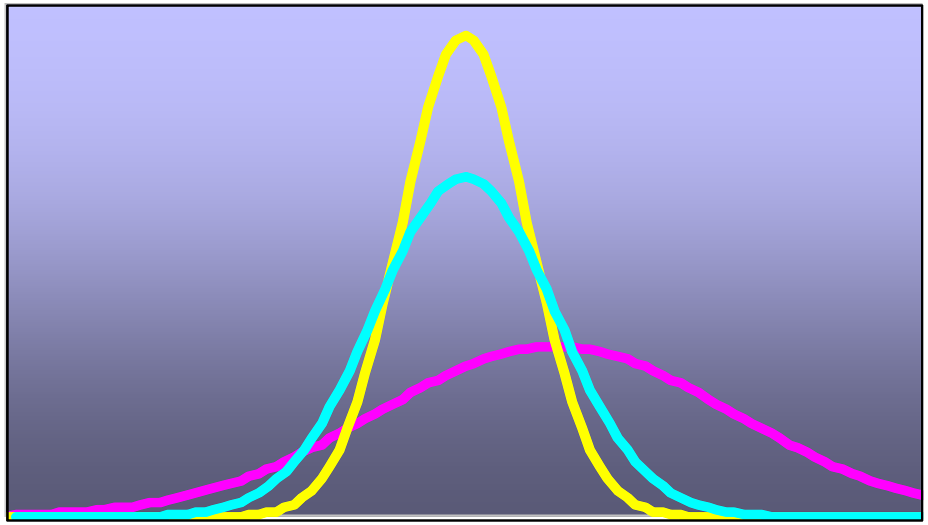


# The Normal Distribution

*(continued)*

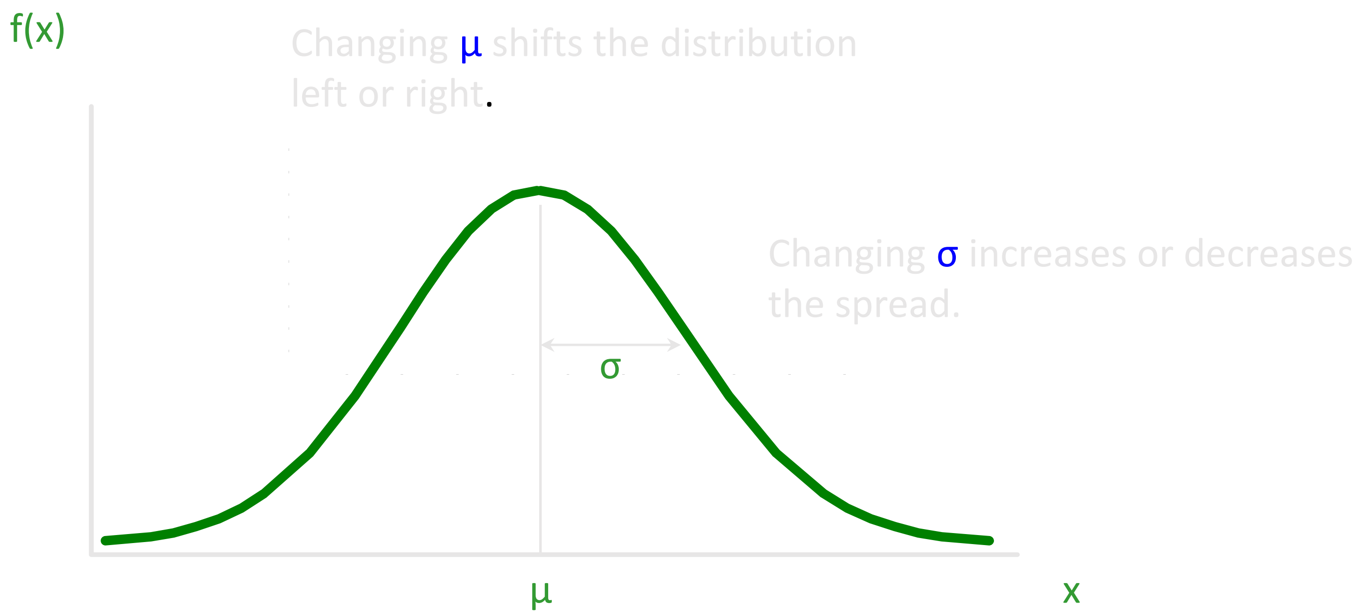
- The normal distribution closely approximates the probability distributions of a wide range of random variables
- Distributions of sample means approach a normal distribution given a “large” sample size
- Computations of probabilities are direct and elegant
- The normal probability distribution has led to good business decisions for a number of applications

# Many Normal Distributions



By varying the parameters μ and σ, we obtain different normal distributions

# The Normal Distribution Shape




Given the mean $\mu$ and variance $\sigma^2$ we define the normal distribution using the notation

$$X \sim N(\mu, \sigma^2)$$

# The Normal Probability Density Function

- The formula for the normal probability density function is

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

Where e = the mathematical constant approximated by 2.71828

π = the mathematical constant approximated by 3.14159

μ = the population mean

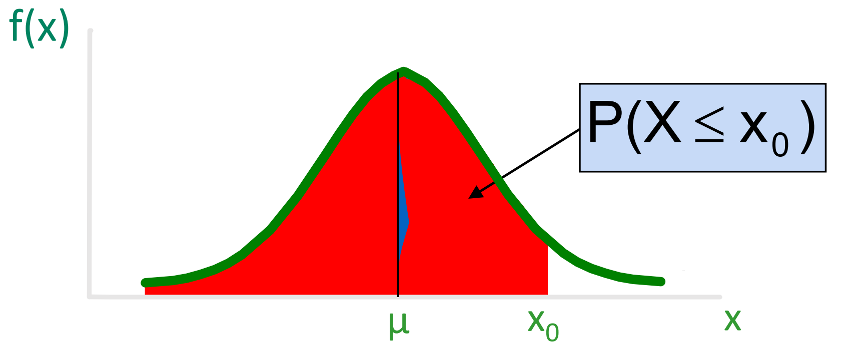
$\sigma^2$ = the population variance

x = any value of the continuous variable, $-\infty < x < \infty$

# Cumulative Normal Distribution

- For a normal random variable X with mean μ and variance $\sigma^2$ , i.e., $X \sim N(\mu, \sigma^2)$, the cumulative distribution function is

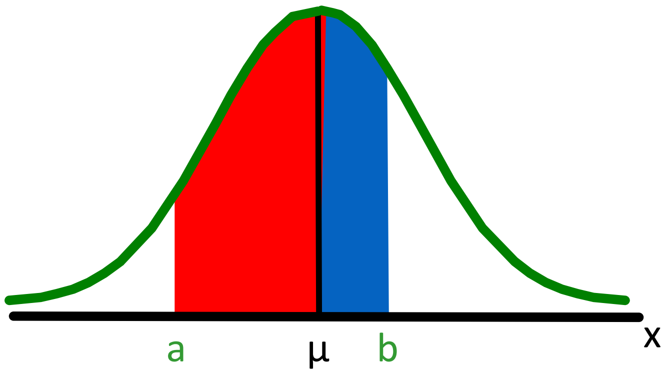
$$F(x_0) = P(X \le x_0)$$



# Finding Normal Probabilities

The probability for a range of values is measured by the area under the curve

$$P(a < X < b) = F(b) - F(a)$$



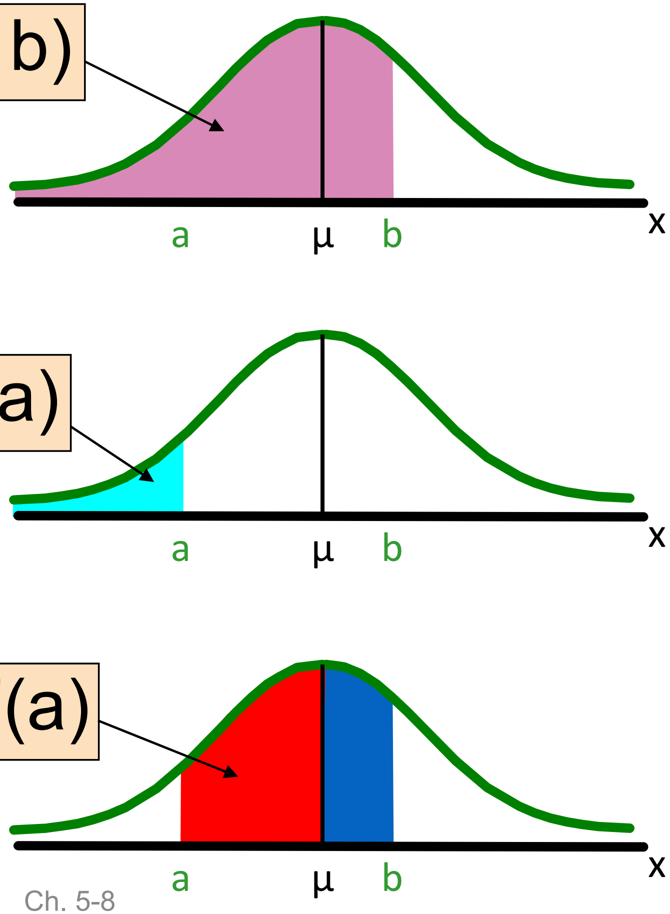
# Finding Normal Probabilities

*(continued)*

$$F(b) = P(X < b)$$

$$F(a) = P(X < a)$$

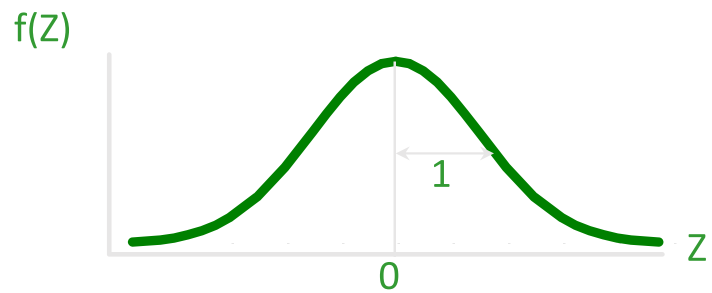
$$P(a < X < b) = F(b) - F(a)$$



# The Standard Normal Distribution

- Any normal distribution (with any mean and variance combination) can be transformed into the standardized normal distribution (Z), with mean 0 and variance 1

$$Z \sim N(0,1)$$




- Need to transform X units into Z units by subtracting the mean of X and dividing by its standard deviation

$$Z = \frac{X - \mu}{\sigma}$$

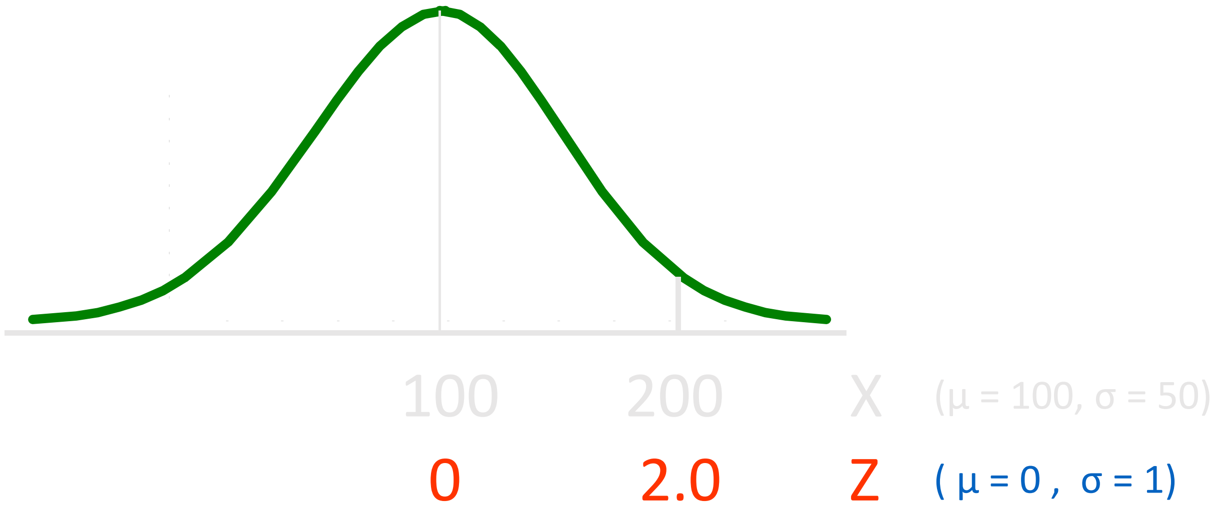
# Example

- If X is distributed normally with mean of 100 and standard deviation of 50, the Z value for X = 200 is

$$Z = \frac{X - \mu}{\sigma} = \frac{200 - 100}{50} = 2.0$$

- This says that X = 200 is two standard deviations (2 increments of 50 units) above the mean of 100.

# Comparing X and Z units

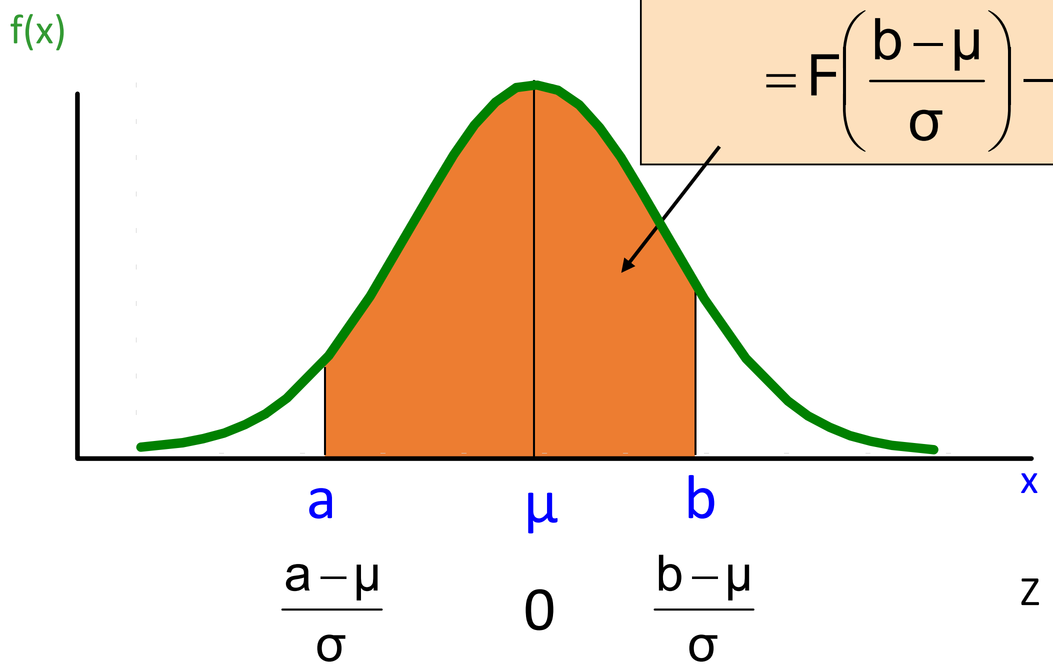



Note that the distribution is the same, only the scale has changed. We can express the problem in original units (X) or in standardized units (Z)

# Finding Normal Probabilities

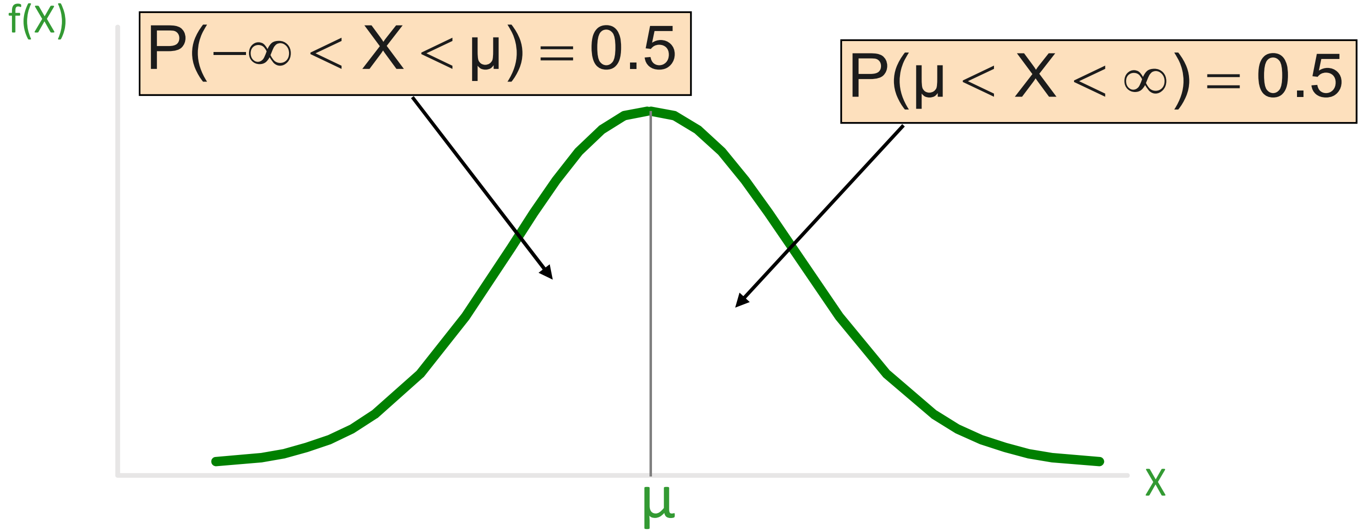
$$P(a < X < b) = P\left(\frac{a-\mu}{\sigma} < Z < \frac{b-\mu}{\sigma}\right)$$

$$= F\left(\frac{b-\mu}{\sigma}\right) - F\left(\frac{a-\mu}{\sigma}\right)$$



# Probability as Area Under the Curve

The **total area under the curve is 1.0**, and the curve is symmetric, so half is above the mean, half is below




$$P(-\infty < X < \infty) = 1.0$$

# The Standard Normal Table

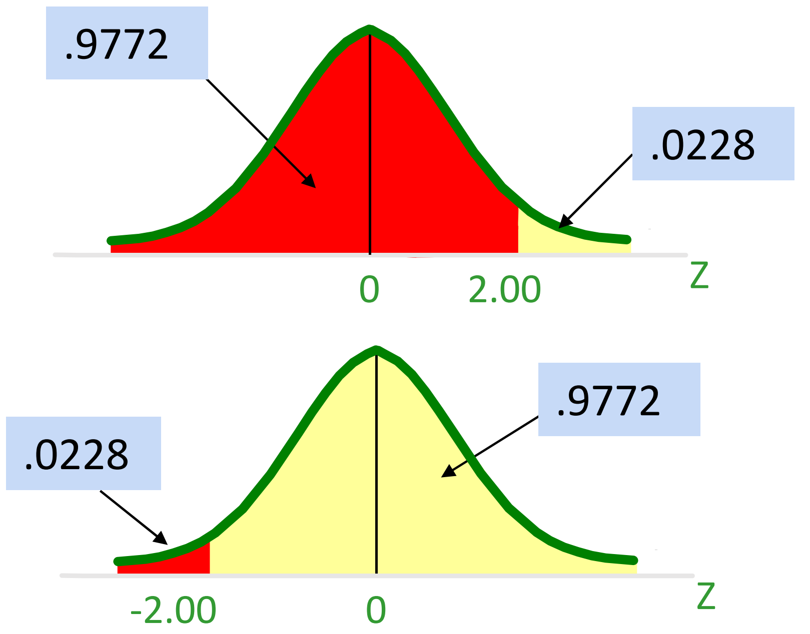
*(continued)*

- For negative Z-values, use the fact that the distribution is symmetric to find the needed probability:

Example:

$P(Z < -2.00) = 1 – 0.9772$

$= 0.0228$



# General Procedure for Finding Probabilities

To find $P(a < X < b)$ when $X$ is distributed normally:

- Draw the normal curve for the problem in terms of X

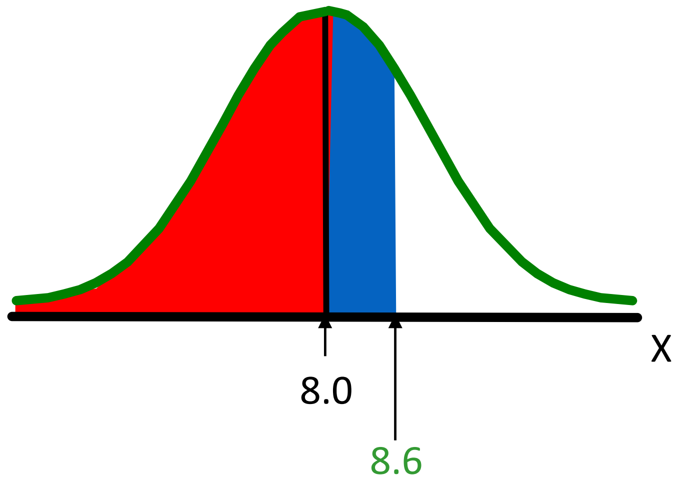
- Translate X-values to Z-values

- Use the Cumulative Normal Table

# Finding Normal Probabilities

- Suppose X is normal with mean 8.0 and standard deviation 5.0
- Find P(X < 8.6)

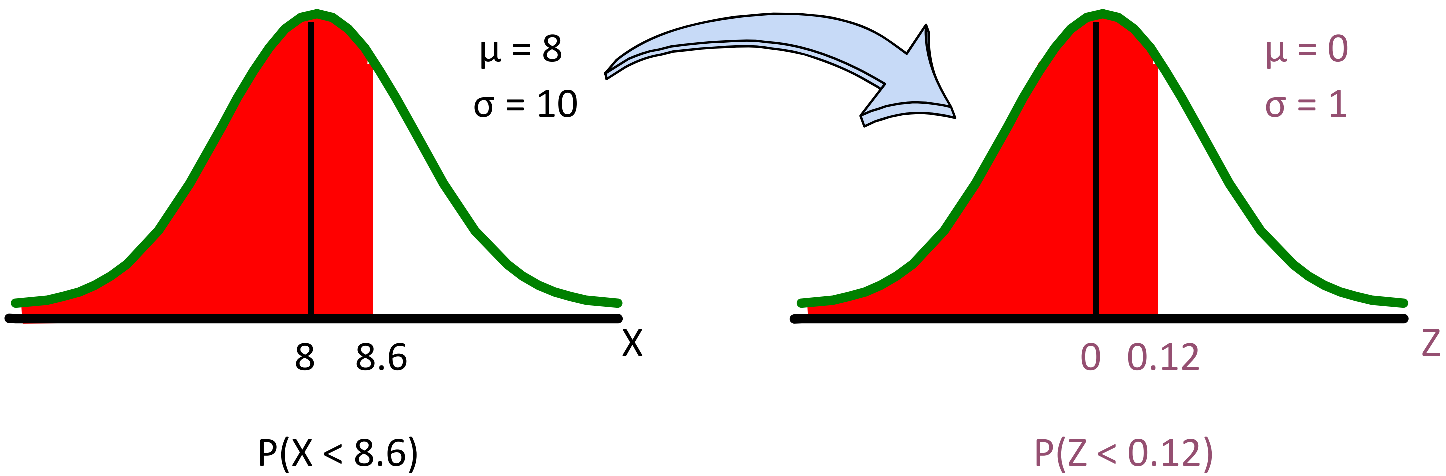


# Finding Normal Probabilities

*(continued)*

- Suppose X is normal with mean 8.0 and standard deviation 5.0. Find P(X < 8.6)

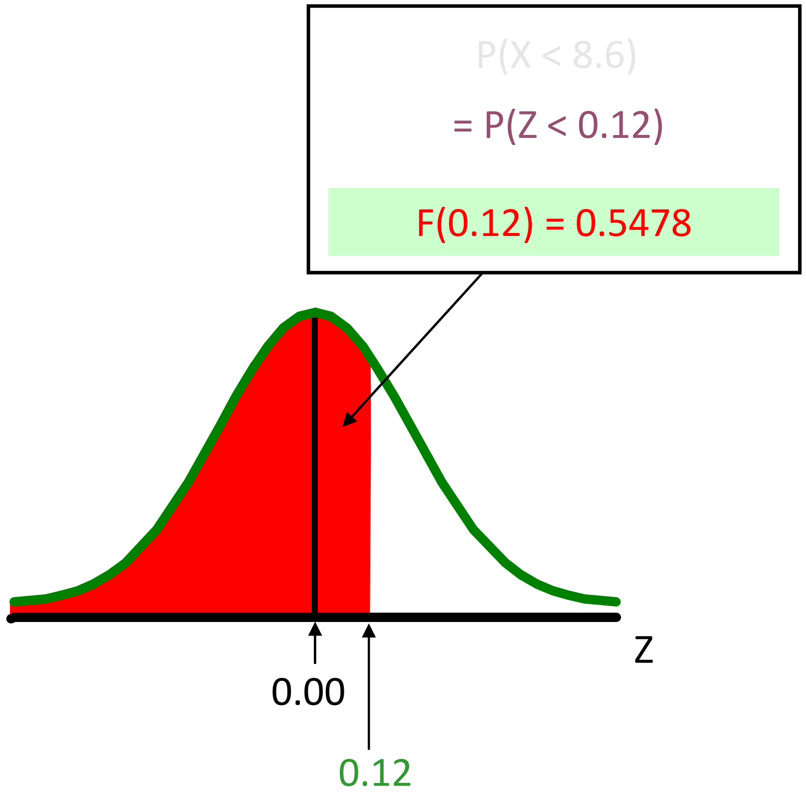
$$Z = \frac{X - \mu}{\sigma} = \frac{8.6 - 8.0}{5.0} = 0.12$$



# Solution: Finding P(Z < 0.12)

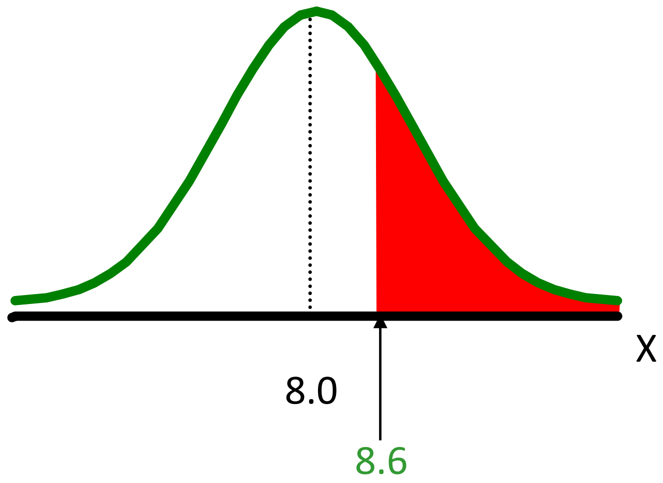
Standardized Normal Probability Table (Portion)

| z | F(z) |
|---|---|
| .10 | .5398 |
| .11 | .5438 |
| .12 | .5478 |
| .13 | .5517 |



# Upper Tail Probabilities

- Suppose X is normal with mean 8.0 and standard deviation 5.0.
- Now Find P(X > 8.6)



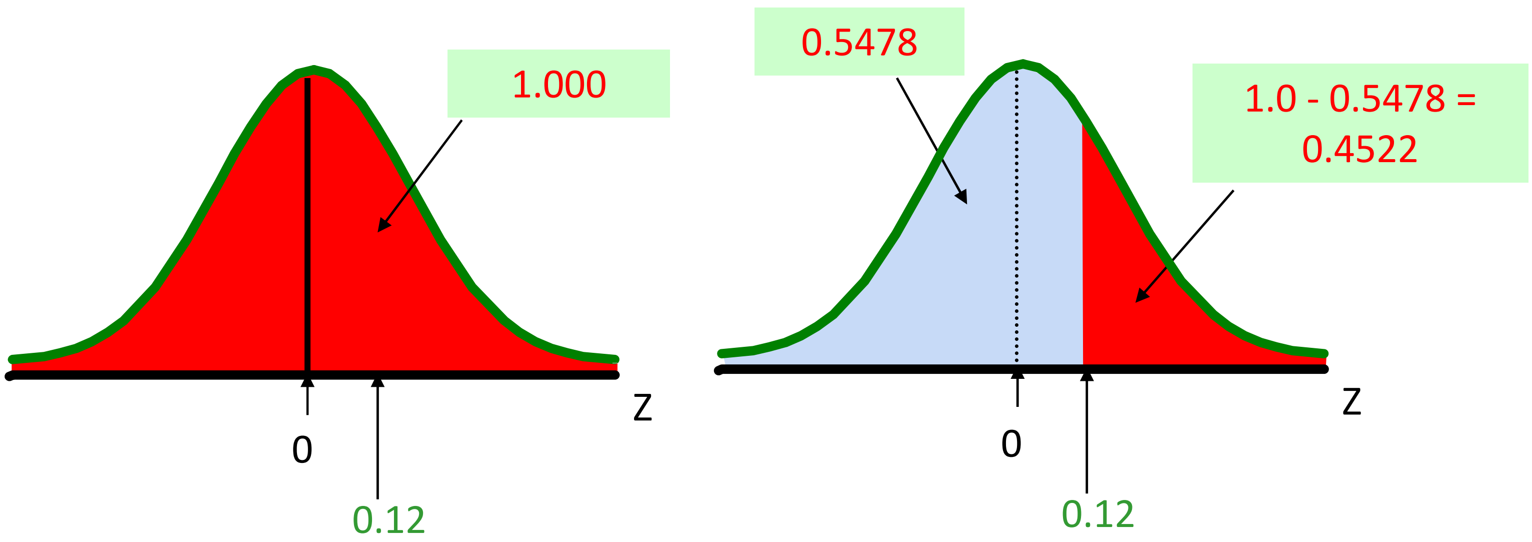
# Upper Tail Probabilities

*(continued)*

- Now Find P(X > 8.6)…

P(X > 8.6) = P(Z > 0.12) = 1.0 - P(Z ≤ 0.12)

= 1.0 - 0.5478 = 0.4522



# Finding the X value for a Known Probability

- Steps to find the X value for a known probability:
  1. Find the Z value for the known probability
  2. Convert to X units using the formula:

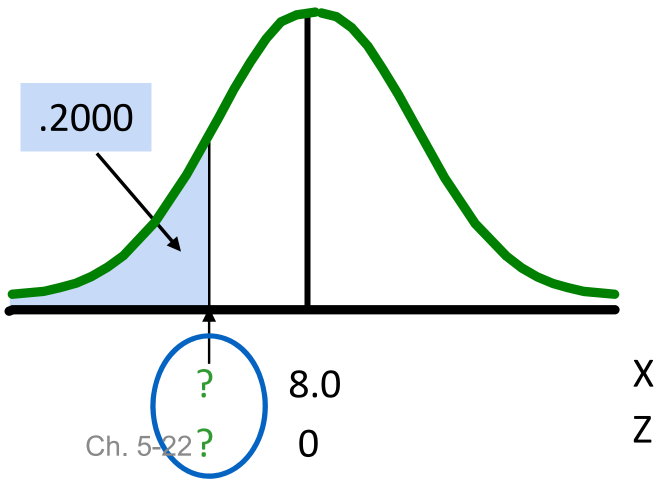
$$X = \mu + Z\sigma$$

# Finding the X value for a Known Probability

*(continued)*

Example:

- Suppose X is normal with mean 8.0 and standard deviation 5.0.
- Now find the X value so that only 20% of all values are below this X



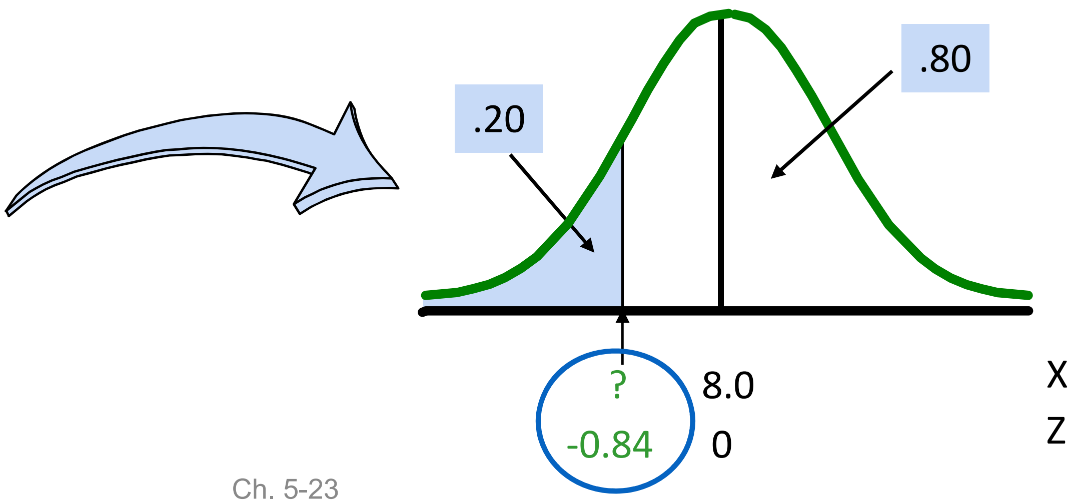
# Find the Z value for 20% in the Lower Tail

**1. Find the Z value for the known probability**

Standardized Normal Probability Table (Portion)

| z | F(z) |
|---|---|
| .82 | .7939 |
| .83 | .7967 |
| .84 | .7995 |
| .85 | .8023 |

- 20% area in the lower tail is consistent with a Z value of -0.84



# Finding the X value

2. Convert to X units using the formula:

$$
\begin{aligned}
X &= \mu + Z\sigma \\
&= 8.0 + (-0.84)5.0 \\
&= 3.80
\end{aligned}
$$

So 20% of the values from a distribution with mean 8.0 and standard deviation 5.0 are less than 3.80

# 5.4 Normal Distribution Approximation for Binomial Distribution

- Recall the binomial distribution:
  - n independent trials
  - probability of success on any given trial = P

- Random variable X:
  - $X_i$ =1 if the $i^{th}$ trial is "success"
  - $X_i$ =0 if the $i^{th}$ trial is "failure"

$$E[X] = \mu = nP$$

$$Var(X) = \sigma^2 = nP(1-P)$$

# Normal Distribution Approximation for Binomial Distribution

*(continued)*

- The shape of the binomial distribution is approximately normal if n is large
- The normal is a good approximation to the binomial when $nP(1 - P) > 5$
- Standardize to Z from a binomial distribution:

$$Z = \frac{X - E[X]}{\sqrt{Var(X)}} = \frac{X - np}{\sqrt{nP(1 - P)}}$$

# Normal Distribution Approximation for Binomial Distribution

*(continued)*

- Let X be the number of successes from n independent trials, each with probability of success P.

- If nP(1 - P) > 5,

$$P(a < X < b) = P\left( \frac{a - nP}{\sqrt{nP(1-P)}} \le Z \le \frac{b - nP}{\sqrt{nP(1-P)}} \right)$$

# Binomial Approximation Example

- 40% of all voters support ballot proposition A. What is the probability that between 76 and 80 voters indicate support in a sample of n = 200 ?
  - $E[X] = \mu = nP = 200(0.40) = 80$
  - $Var(X) = \sigma^2 = nP(1 - P) = 200(0.40)(1 - 0.40) = 48$

    ( note: $nP(1 - P) = 48 > 5$ )

$$
\begin{aligned}
P(76 < X < 80) &= P\left(\frac{76-80}{\sqrt{200(0.4)(1-0.4)}} \le Z \le \frac{80-80}{\sqrt{200(0.4)(1-0.4)}}\right) \\
&= P(-0.58 < Z < 0) \\
&= F(0) - F(-0.58) \\
&= 0.5000 - 0.2810 = 0.2190
\end{aligned}
$$