Variance Reduction for MC/QMC Methods to Evaluate Option Prices

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Abstract

Several variance reduction techniques including importance sampling, (martingale) control variate, (randomized) Quasi Monte Carlo method, QMC in short, and some possible combinations are considered to evaluate option prices. By means of perturbation methods to derive some option price approximations, we find from numerical results in Monte Carlo simulations that the control variate method is more efficient than importance sampling to solve European option pricing problems under multifactor stochastic volatility models. As an alternative, QMC method also provides better convergence than basic Monte Carlo method . But we find an example where QMC method may produce erroneous solutions when estimating the low-biased solution of an American option. This drawback can be effectively fixed by adding a martingale control to the estimator adopting Quasi random sequences so that low-biased estimates obtained are more accurate than results from Monte Carlo method. Therefore by taking advantages of martingale control variate and randomized QMC, we

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find significant improvement on variance reduction for pricing derivatives and their sensitivities. This effect should be understood as that martingale control variate plays the role of a smoother under QMC method to permit better convergence.

Contents

1 Introduction

Monte Carlo method and Quasi Monte Carlo method (MC/QMC method in short) are important tools for integral problems in computational finance. They are popularly applied particularly in cases of solutions without closed form, for example American put option prices under the Black-Scholes model and European option prices under multi-factor stochastic volatility models, both which we will consider in the present paper. Our goal is to find an efficient variance reduction method to improve the convergence of MC/QMC methods.

The study of variance reduction methods for option pricing problems has been very fruitful during the last two decades [7]. They are important, just to name a few, for computing *greeks* (sensitivities of options prices with respect to model variables or parameters), risk management and model calibration. Due to the complexity of financial derivatives and pricing models, it is difficult to find one general approach to reduce variances of associated MC/QMC methods. However in [2] and [3], we can use (local) martingale control variate methods to evaluate European, Barrier and American options through Monte Carlo simulations. This paper concerns numerical comparisons with several variance reduction techniques such as control variate, importance sampling, Brownian bridge, randomized QMC and their possible combinations.

Our goal is to explore an efficient MC/QMC method to evaluate American put options under Black-Scholes model and European call options under multi-factor stochastic volatility models. Firstly, we are motivated by previous results in [1] and [2] where importance sampling method and martingale control variate methods are used respectively under Monte Carlo simulations for European option pricing problems. From many numerical comparisons between these two methods, we find that martingale control variate method performs much better than importance sampling in terms of variance reduction power. Secondly, we investigate the efficiency of QMC method for option pricing. As an integration method using quasi-random sequences (also called low-discrepancy sequences), QMC method [7] has better convergence rates than Monte Carlo method, see Section 4, under appropriate dimensionality and regularity of the integrand. However in many financial applications, these conditions are not satisfied. We give a counterexample in Section 4.2 that shows that using QMC method such as Niderreiter or Sobol sequences, gives erroneous estimates for low-biased solutions of American put options. To our best knowledge, this is the first counterexample showing the failure of applying QMC method in financial applications. However, when we combine a martingale control variate with QMC method, very accurate low-biased estimates are obtained compared to Monte Carlo method, see Table 7 for details. In brief, when the regularity of the estimator is not good enough, using QMC method can be problematic. The effect of martingale control variate thus plays the role of a smoother which improves the regularity of the controlled estimator.

Because of the mentioned benefits of martingale control variate and its combination with QMC method, we continue to explore in detail the effect of martingale control variate with randomized QMC method for the European option pricing problems. Under multifactor stochastic volatility models, the dimension of the randomized QMC method becomes high. Typically in our experiments the dimension goes up to 300. This can be an obstacle for QMC method to reduce variance in a significant way as its convergence rate depends on the dimension. Based on our experiments in Section 5, the effect of martingale control variate again tremendously improves the regularity of the controlled estimator. Compared to numerical results from the basic Monte Carlo method, randomized QMC method improves the variance only by a single digit, while martingale control variate under Monte Carlo simulations improves variance by 50 times. The combination of the martingale control variate with the randomized QMC method improves variance reduction ratios up to 700 times.

The organization of this paper is as follows. In Section 2, we introduce stochastic volatility models and European option price approximations obtained from [5] by means of singular and regular perturbation methods. Section 3 reviews two variance reduction methods, namely control variate and importance sampling, and compare their variance reduction performances. In Section 4, we introduce the QMC method and show a counterexample where the method fails. We then show how to combine this method with a correction by a martingale control variate. Section 5 tests several combinations of martingale control variate methods with and without randomized QMC method, including the Sobol sequence and L'Ecuyer type good lattice points together with the Brownian bridge sampling technique. We also consider option prices and their *deltas*, first-order partial derivative with respect to the underlying price.

2 Multi-factor Stochastic Volatility Models and Option Price Approximations

Under the physical probability measure, a family of multi-factor stochastic volatility models evolves as

$$\begin{aligned} dS_t &= \kappa S_t dt + \sigma_t S_t dW_t^{(0)}, \\ \sigma_t &= f(Y_t, Z_t), \\ dY_t &= \alpha c_1(Y_t)dt + \sqrt{\alpha}\, g_1(Y_t) dW_t^{(1)} \\ dZ_t &= \delta c_2(Z_t)dt + \sqrt{\delta}\, g_2(Z_t) dW_t^{(2)}, \end{aligned}$$

where S_t is the underlying asset price with a constant rate of return κ and the random volatility σ_t, Y_t and Z_t are driving volatility processes varying with time scales $1/\alpha$ and $1/\delta$ respectively. The standard Brownian motions (W_t^0, W_t^1, W_t^2) are possibly correlated as described below. The volatility function f is assumed bounded and bounded away from 0, and continuous with respect to its second variable z. The coefficient functions of Y_t, namely c_1 and g_1 are assumed to be chosen such that Y_t is an ergodic diffusion. The Ornstein-Uhlenbeck (OU) process is a typical example by defining α the rate of mean-reversion, $c_1(y) = m_1 - y$, and $g_1(y) = \nu_1\sqrt{2}$, where m_1 is the long-run mean and ν_1 is the long-run standard deviation, such that $\Phi = \mathcal{N}(m, \nu^2)$ is the invariant distribution. The coefficient functions of Z_t, namely c_2 and g_2 are assumed to satisfy the existence and uniqueness conditions of diffusions [18]. For simplicity, we set the process Z_t to be another OU process by choosing $c_2(z) = m_2 - z$, and $g_2(z) = \nu_2\sqrt{2}$, where m_2 is the long-run mean and ν_2 is the long-run standard deviation. Suppose (W_t^0, W_t^1, W_t^2) are correlated according to the following cross-variations:

$$\begin{aligned} d < W^{(0)}, W^{(1)} >_t &= \rho_1 dt, \\ d < W^{(0)}, W^{(2)} >_t &= \rho_2 dt, \\ d < W^{(1)}, W^{(2)} >_t &= \left(\rho_1\, \rho_2 + \sqrt{1-\rho_1^2}\, \rho_{12}\right) dt, \end{aligned}$$

where the instant correlations ρ_1, ρ_2, and ρ_{12} satisfy $\mid \rho_1 \mid < 1$ and $\mid \rho_2^2 + \rho_{12}^2 \mid < 1$ respectively.

Under the risk-neutral probability measure $I\!P^\star$, a family of multi-factor SV models can be described as follows

$$dS_t = r S_t dt + \sigma_t S_t dW_t^{(0)\star}, \tag{1}$$

$$
\begin{aligned}
\sigma_t &= f(Y_t, Z_t), \\
dY_t &= \left(\alpha(m_1 - Y_t) - \nu_1\sqrt{2\alpha}\Lambda_1(Y_t, Z_t)\right) dt \\
&\quad + \nu_1\sqrt{2\alpha}\left(\rho_1 dW_t^{(0)\star} + \sqrt{1-\rho_1^2}\, dW_t^{(1)\star}\right), \\
dZ_t &= \left(\delta(m_2 - Z_t) - \nu_2\sqrt{2\delta}\Lambda_2(Y_t, Z_t)\right) dt \\
&\quad + \nu_2\sqrt{2\delta}\left(\rho_2 dW_t^{(0)\star} + \rho_{12} dW_t^{(1)\star} + \sqrt{1-\rho_2^2-\rho_{12}^2}\, dW_t^{(2)\star}\right),
\end{aligned}
$$

where $\left(W_t^{(0)\star}, W_t^{(1)\star}, W_t^{(2)\star}\right)$ are independent standard Brownian motions. The risk-free interest rate of return is denoted by r. The functions Λ_1 and Λ_2 are the combined market prices of risk and volatility risk, they are assumed to be bounded and dependent only on the variables y and z. The process (S_t, Y_t, Z_t) is Markovian.

The payoff of an European-style option is an integrable function, say H, of the stock price S_T at the maturity date T. The price of this option is defined as the expectation of the discounted payoff conditioned on the current stock price and driving volatility levels due to the Markov property of the joint dynamics (1). By introducing the notation $\varepsilon = 1/\alpha$, the European option price is given by

$$
P^{\varepsilon,\delta}(t,x,y,z) = I\!E^\star\left\{e^{-r(T-t)}H(S_T) \mid S_t = x, Y_t = y, Z_t = z\right\}. \tag{2}
$$

2.1 Vanilla European Option Price Approximations

By an application of Feynman-Kac formula, $P^{\varepsilon,\delta}(t,x,y,z)$ defined in (2) can also be represented by solving the three-dimensional partial differential equation

$$
\begin{aligned}
&\frac{\partial P^{\varepsilon,\delta}}{\partial t} + \mathcal{L}^{\varepsilon,\delta}_{(S,Y,Z)} P^{\varepsilon,\delta} - r\, P^{\varepsilon,\delta} = 0, \\
&P^{\varepsilon,\delta}(T,x,y,z) = H(x),
\end{aligned} \tag{3}
$$

where $\mathcal{L}^{\varepsilon,\delta}_{(S,Y,Z)}$ denotes the infinitesimal generator of the Markovian process (S_t, Y_t, Z_t) given by (1). Assuming that the parameters ε and δ are small, $0 < \varepsilon, \delta \ll 1$, Fouque *et al.* in [5] use a combination of regular and singular perturbation methods to derive the following pointwise option price approximation

$$
P^{\varepsilon,\delta}(t,x,y,z) \;\approx\; \tilde{P}(t,x,z),
$$

where

$$\begin{aligned}
\tilde{P} &= P_{BS} \qquad (4)\\
&+ (T-t)\left(V_0\frac{\partial}{\partial\sigma} + V_1 x\frac{\partial^2}{\partial x\partial\sigma} + V_2 x^2\frac{\partial^2}{\partial x^2} + V_3 x\frac{\partial}{\partial x}\left(x^2\frac{\partial^2}{\partial x^2}\right)\right)P_{BS},
\end{aligned}$$

with an accuracy of order $(\varepsilon|\log\varepsilon| + \delta)$ for call options. The leading order price $P_{BS}(t, x; \bar{\sigma}(z))$ is independent of the y variable and is the homogenized price which solves the Black-Scholes equation

$$\begin{aligned}
&\mathcal{L}_{BS}(\overline{\sigma}(z))P_{BS} = 0,\\
&P_{BS}(T, x; \bar{\sigma}(z)) = H(x).
\end{aligned}$$

Here the z-dependent **effective volatility** $\overline{\sigma}(z)$ is defined by

$$\overline{\sigma}^2(z) = \langle f^2(\cdot, z)\rangle, \qquad (5)$$

where the brackets denote the average with respect to the invariant distribution $\mathcal{N}(m_1, \nu_1^2)$ of the fast factor (Y_t). The parameters (V_0, V_1, V_2, V_3) are given by

$$V_0 = -\frac{\nu_2\sqrt{\delta}}{\sqrt{2}}\langle\Lambda_2\rangle\overline{\sigma}', \qquad (6)$$

$$V_1 = \frac{\rho_2\nu_2\sqrt{\delta}}{\sqrt{2}}\langle f\rangle\overline{\sigma}', \qquad (7)$$

$$V_2 = \frac{\nu_1\sqrt{\varepsilon}}{\sqrt{2}}\left\langle\Lambda_1\frac{\partial\phi}{\partial y}\right\rangle, \qquad (8)$$

$$V_3 = -\frac{\rho_1\nu_1\sqrt{\varepsilon}}{\sqrt{2}}\left\langle f\frac{\partial\phi}{\partial y}\right\rangle, \qquad (9)$$

where $\overline{\sigma}'$ denotes the derivative of $\bar{\sigma}$, and the function $\phi(y, z)$ is a solution of the Poisson equation

$$\mathcal{L}_0\phi(y, z) = f^2(y, z) - \overline{\sigma}^2(z).$$

The parameters V_0 and V_1 (resp. V_2 and V_3) are small of order $\sqrt{\varepsilon}$ (resp. $\sqrt{\delta}$). The parameters V_0 and V_2 reflects the effect of the market prices of volatility risk. The parameters V_1 and V_3 are proportional to the correlation coefficients ρ_2 and ρ_1 respectively. In [5], these parameters are calibrated using the observed implied volatilities. In the present work, the model (1) will be fully specified, and these parameters are computed using the formulas above.

3 Monte Carlo Simulations: Two Variance Reduction Methods

In this section, two variance reduction methods, namely importance sampling [1] and control variates [2], to evaluate European option prices by Monte Carlo simulations are compared under multi-factor stochastic volatility models. The technique of importance sampling has been introduced to evaluate European and Asian option prices in [1, 6]. We briefly review this methodology in Section 3.1. A control variate method based on [2] is reviewed in Section 3.2. This method has been applied to several option pricing problems including Barrier and American options [3]. In Section 3.3, test examples of one and two factor stochastic volatility models are demonstrated to show that the control variate method performs better than importance sampling in terms of variance reduction power.

To simplify notations, we present the stochastic volatility model in (1) in the vector form

$$dV_t = b(t, V_t)dt + a(t, V_t)d\eta_t, \tag{10}$$

where we set

$$v = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad V_t = \begin{pmatrix} S_t \\ Y_t \\ Z_t \end{pmatrix}, \qquad \eta_t = \begin{pmatrix} W_t^{(0)\star} \\ W_t^{(1)\star} \\ W_t^{(2)\star} \end{pmatrix},$$

and we define the drift

$$b(t, v) = \begin{pmatrix} rx \\ \alpha(m_1 - y) - \nu_1\sqrt{2\alpha}\,\Lambda_1(y, z) \\ \delta(m_2 - z) - \nu_2\sqrt{2\delta}\,\Lambda_2(y, z) \end{pmatrix},$$

and the diffusion matrix

$$a(t, v) = \begin{pmatrix} f(y, z)x & 0 & 0 \\ \nu_1\sqrt{2\alpha}\,\rho_1 & \nu_1\sqrt{2\alpha}\,\sqrt{1-\rho_1^2} & 0 \\ \nu_2\sqrt{2\delta}\,\rho_2 & \nu_2\sqrt{2\delta}\,\rho_{12} & \nu_2\sqrt{2\delta}\,\sqrt{1-\rho_2^2-\rho_{12}^2} \end{pmatrix}.$$

The price $P(t, x, y, z)$ of an European option at time t is given by

$$P(t, v) = I\!\!E^\star\left\{e^{-r(T-t)}H(S_T) \mid V_t = v\right\}. \tag{11}$$

The basic Monte Carlo simulation estimates the option price $P(0, S_0, , Y_0, Z_0)$ at time 0 by the sample mean

$$\frac{1}{N}\sum_{k=1}^{N} e^{-rT} H(S_T^{(k)}) \tag{12}$$

where N is the total number of sample paths and $S_T^{(k)}$ denotes the k-th simulated stock price at time T.

3.1 Importance Sampling

A change of drift in the model dynamics (10) can be obtained by

$$dV_t = (b(t, V_t) - a(t, V_t)h(t, V_t))\, dt + a(t, V_t) d\tilde{\eta}_t, \tag{13}$$

where

$$\tilde{\eta}_t = \eta_t + \int_0^t h(s, V_s) ds.$$

The instantaneous shift $h(s, V_s)$ is assumed to satisfy the Novikov's condition

$$\mathbb{E}^\star \left\{ \exp\left(\frac{1}{2}\int_0^T h^2(s, V_s) ds \right) \right\} < \infty.$$

By Girsanov Theorem, one can construct the new probability measure $\tilde{\mathbb{P}}$ by

$$\frac{d\mathbb{P}^\star}{d\tilde{\mathbb{P}}} = Q_T,$$

where the Radon-Nikodyn derivative is defined as

$$Q_T = \exp\left(\left\{ \int_0^T h(s, V_s) d\tilde{\eta}_s - \frac{1}{2}\int_0^T ||h(s, V_s)||^2 ds \right\} \right), \tag{14}$$

such that $\tilde{\eta}_t$ is a Brownian motion under $\tilde{\mathbb{P}}$. The option price P can be written as

$$P(t, v) = \tilde{\mathbb{E}}\left\{ e^{-r(T-t)} H(S_T) Q_T \mid V_t = v \right\}. \tag{15}$$

By an application of Ito's formula to $P(t, V_t)\, Q_t$, one could obtain a zero variance of the discounted payoff $e^{-r(T-t)} H(S_T) Q_T$ by optimally choosing

$$h = -\frac{1}{P}\left(a^T\, \nabla P \right). \tag{16}$$

(See details in [1, 15].) The super script notation T denotes transpose and ∇ denotes the gradient. However, neither the price P nor its gradient ∇P were known in advance.

The idea of importance sampling techniques introduced in [1] is to approximate unknown option price $P^{\varepsilon,\delta}$ by $\tilde{P}$ as in (4). Then the Monte Carlo simulations are done under the new measure $\tilde{I\!P}$:

$$P(t,x,y,z) \approx \frac{1}{N}\sum_{k=1}^{N} e^{-r(T-t)} H(S_T^{(k)}) Q_T^{(k)}, \tag{17}$$

where N is the total number of simulations, and $S_T^{(k)}$ and $Q_T^{(k)}$ denote the final value of the k-th realized trajectory (13) and weight (14) respectively.

3.2 Control Variate Method

A control variate with m multiple controls is defined as:

$$P^{CV} \triangleq P^{MC} + \sum_{i=1}^{m} \lambda_i (\hat{P}_C^i - P_C^i). \tag{18}$$

We denote by P^{MC} the sample mean of outputs from an IID simulation procedure. Each $\hat{P}_C^i$ represents the sample mean of those outputs jointly distributed by the previous simulation procedure. In addition, we assume $\hat{P}_C^i$ has the mean P_C^i which at best has a closed-form expression in order to reduce computational cost. The control variate P^{CV} thus becomes an unbiased estimator of P^{MC}. Each control parameter λ_i needs to be chosen to minimize the variance of P^{CV} as the coefficients in least squares regression. A detailed discussion on control variates can be found in [7] and an application to Asian option option in [8].

A constructive way to build control variate estimators under diffusion models (1) is as follows. Based on Ito's formula, the discounted option price satisfies

$$de^{-rs}P(s,S_s,Y_s,Z_s) = e^{-rs}\left(\frac{\partial}{\partial t} + \mathcal{L}^{\varepsilon,\delta}_{(S,Y,Z)} - r\cdot\right) P\,ds + e^{-rs}\,\nabla P\cdot(a\cdot d\eta_s).$$

The first term on the right hand side is crossed out because of (3). Integrating above equation in time between the current time t and the expiry date T, and using the terminal condition $P(T,S_T,Y_T,Z_T) = H(S_T)$,

$$P(t,S_t,Y_t,Z_t) = e^{-r\,(T-t)}H(S_T) - \int_t^T e^{-r(s-t)}\nabla P\cdot(a\cdot d\eta_t) \tag{19}$$

is deduced.
However, the unknown price process $\{P(s, X_s, Y_s, Z_s)\}$ along trajectories between $\{t \leq s \leq T\}$ appears in each stochastic integral in (19). The use of price approximation (4)

$$P^{\varepsilon,\delta}(t,x,y,z) \quad \approx \quad \tilde{P}(t,x,z),$$

suggests a constructive way to build the control variate

$$\begin{aligned} P^{CV} &= e^{-r(T-t)}H(S_T) - \int_t^T e^{-r(s-t)}\frac{\partial \tilde{P}}{\partial x}\sigma_s S_s dW_s^{(0)\star} \\ &- \int_t^T e^{-r(s-t)}\frac{\partial \tilde{P}}{\partial z}\nu_2\sqrt{2\delta}\left(\rho_2 dW_s^{(0)\star} + \rho_{12} dW_s^{(1)\star} + \sqrt{1-\rho_2^2-\rho_{12}^2}dW_s^{(2)\star}\right) \end{aligned} \tag{20}$$

because $\tilde{P}$ is independent of the variable y. It is readily observed that P^{CV} is unbiased since by the martingale property of the stochastic integrals, the conditional expectation of the stochastic integrals are zero. In addition, it naturally suggests multiple estimators of optimal control parameters. To fit in the setup of the control variate with multiple controls (18), we have chosen for $i \in \{1,2\}$:

$$\begin{aligned} &P^{MC} = e^{-r(T-t)}H(S_T) \\ &\lambda_i = -1 \\ &\hat{P}_C^i = \int_t^T e^{-r(s-t)}\frac{\partial \tilde{P}}{\partial x}\sigma_s S_s dW_s^{(0)\star} I_{\{i=1\}} \\ &+ \int_t^T e^{-r(s-t)}\frac{\partial \tilde{P}}{\partial z}\nu_2\sqrt{2\delta}\left(\rho_2 dW_s^{(0)\star} + \rho_{12} dW_s^{(1)\star} + \sqrt{1-\rho_2^2-\rho_{12}^2}dW_s^{(2)\star}\right) I_{\{i=2\}} \\ &P_C^i = 0. \end{aligned}$$

3.3 Numerical Results

Two sets of numerical experiments are proposed in order to compare the variance reduction performances of importance sampling and control variate described previously. The first set of experiments is for one-factor SV models and the second set is for two-factor SV models. These experiments are done only for vanilla European call options.

Table 1: Parameters used in the one-factor stochastic volatility model (1).

r	m_1	m_2	ν_1	ν_2	ρ_1	ρ_2	ρ_{12}	Λ_1	λ_2	$f(y,z)$
10%	-2.6	0	1	0	-0.3	0	0	0	0	$\exp(y)$

Table 2: Initial conditions and call option parameters.

$\$S_0$	Y_0	Z_0	$\$K$	T years
110	-2.32	0	100	1

3.3.1 One-Factor SV Models

Under the framework of the two-factor SV model (1), an one-factor SV model is obtained by setting all parameters as well as the initial condition used to describe the second factor Z_t in (1) to zero. Our test model is chosen as in Fouque and Tullie [6], in which they used an Euler scheme to discretize the diffusion process V_t to run the Monte Carlo simulations. The time step is 10^{-3} and the number of realizations is 10000.
The one-factor stochastic volatility model is specified in Tables 1 and 2. In [6] the authors proposed an importance sampling technique by using an approximate option price obtained by a fast mean-reversion expansion. This approach is described in Section 3.1. Since only the one-factor SV model is considered, the zero-order price approximation reduces to

$$P_{BS}(t,x;\bar{\sigma}) = x\mathcal{N}(d_1(x)) - Ke^{-r(T-t)}\mathcal{N}(d_2(x)), \tag{21}$$

where

$$\begin{aligned} d_1(x) &= \frac{\ln(x/K) + (r + \frac{1}{2}\bar{\sigma}^2)(T-t)}{\bar{\sigma}\sqrt{T-t}}, \\ d_2(x) &= d_1(x) - \bar{\sigma}\sqrt{T-t}, \\ \mathcal{N}(d) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-u^2/2}du, \end{aligned}$$

the constant effective volatility $\bar{\sigma} = \bar{\sigma}(0)$ is defined in (5), and the first-order price approximation reduces to

$$\tilde{P} = P_{BS} + (T-t)\left(V_2 x^2 \frac{\partial^2}{\partial x^2} + V_3 x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2}{\partial x^2}\right)\right) P_{BS}.$$

Table 3: Comparison of estimated variance reduction ratio for European call options with various α's. Notation VAR_{MC} is the sample variance from basic Monte Carlo simulation. $VAR^{IS}(\tilde{P})$ is the sample variance computed from the important sampling with $\tilde{P}$ defined in (4) as an approximate option price. $VAR^{CV}(P_{BS})$ is the sample variance computed from the control variate with P_{BS} defined in (21) as an approximate option price.

α	$VAR_{MC}/VAR^{IS}(\tilde{P})$	$VAR_{MC}/VAR^{CV}(P_{BS})$
0.5	7.8095	49.8761
1	15.7692	29.9266
5	19.3333	31.8946
10	29.6250	51.4791
25	36.7143	100.1556
50	48.0000	183.6522
100	106.3333	229.9224

In [6] it is found that the importance sampling technique performs best by employing the first-order price approximation $\tilde{P}$. According to different level of mean-reverting rate α, numerical results shown on Table 1 in [6] are copied to the second column of our Table 3, in which VAR_{MC} denotes the variance computed from basic Monte Carlo and $VAR^{IS}(\tilde{P})$ denotes the variance computed from importance sampling.
Our procedure to construct the control variate was described in Section 3.2. Since only one-factor model is considered, the control variate defined in (20) is reduced to

$$P^{CV} = e^{-r(T-t)}H(S_T) - \int_t^T e^{-r(s-t)}\frac{\partial P_{BS}}{\partial x}\sigma_s S_s dW_s^{(0)\star}. \tag{22}$$

Notice that we choose the zero-order option price approximation P_{BS} instead of the first-order price approximation $\tilde{P}$. The reason is that we have not found any major improvement by using $\tilde{P}$ instead of P_{BS} in our empirical results. In the third column of Table 3, we list the sample variance ratios obtained from the basic Monte Carlo and the Monte Carlo with our control variate, namely $VAR_{MC}/VAR^{CV}(P_{BS})$. From this test example, the control variate given in (22) apparently dominates the importance sampling.

Table 4: Parameters used in the two-factor stochastic volatility model (1).

r	m_1	m_2	ν_1	ν_2	ρ_1	ρ_2	ρ_{12}	Λ_1	λ_2	$f(y,z)$
10%	-0.8	-0.8	0.5	0.8	-0.2	-0.2	0	0	0	$\exp(y+z)$

Table 5: Initial conditions and call option parameters.

$\$S_0$	Y_0	Z_0	$\$K$	T years
55	-1	-1	50	1

3.3.2 Two-Factor SV models

We continue to investigate the performance of variance reduction for the two-factor SV model (1) defined in Table 4 and 5. Fouque and Han [1] present an importance sampling technique as described in Section 3.1 to evaluate European option prices. Their numerical results extracted from Table 3 in [1] are summarized as variance ratios in the third column of Table 6. According to different rates of mean-reversion α's and δ's for each factor, we illustrate ratios of sample variances computed from the basic Monte Carlo, denoted by VAR_{MC} and the Monte Carlo simulations with importance sampling, denoted by $VAR^{IS}(\tilde{P})$. Among these Monte Carlo simulations, there is a total of 5000 sample paths in (17) simulated based on the discretization of the diffusion process V_t using an Euler scheme with time step $\Delta t = 0.005$.
As in the case of one-factor SV models, we do not find apparent advantage of variance reduction by choosing the first-order approximate option price $\tilde{P}$ compared to using of the zero-order approximation P_{BS}. Hence the control variate implemented in this numerical experiment is given by

$$P^{CV} = e^{-r(T-t)}H(S_T) - \int_t^T e^{-r(s-t)} \frac{\partial P_{BS}}{\partial x} \sigma_s S_s dW_s^{(0)\star} \tag{23}$$
$$- \int_t^T e^{-r(s-t)} \frac{\partial P_{BS}}{\partial z} \nu_2 \sqrt{2\delta} \left(\rho_2 dW_s^{(0)\star} + \rho_{12} dW_s^{(1)\star} + \sqrt{1-\rho_2^2-\rho_{12}^2} dW_s^{(2)\star} \right),$$

where

$$\begin{aligned} P_{BS}(t,x;\bar{\sigma}(z)) &= x\mathcal{N}(d_1(x,z)) - Ke^{-r(T-t)}\mathcal{N}(d_2(x,z)), \\ d_1(x,z) &= \frac{\ln(x/K) + (r+\frac{1}{2}\bar{\sigma}^2(z))(T-t)}{\bar{\sigma}(z)\sqrt{T-t}}, \end{aligned} \tag{24}$$

Table 6: Comparison of sample variances for various values of α and δ. Notation VAR_{MC} is the sample variance from basic Monte Carlo simulation, $VAR^{IS}(\tilde{P})$ is the sample variance computed from the important sampling with $\tilde{P}$ defined in (4) as an approximate option price. $VAR^{CV}(P_{BS})$ is the sample variance computed from the control variate with P_{BS} defined in (24) as an approximate option price.

α	δ	$VAR_{MC}/VAR^{IS}(\tilde{P})$	$VAR_{MC}/VAR^{CV}(P_{BS})$
5	1	13.4476	15.6226
20	0.1	17.5981	101.7913
50	0.05	32.9441	167.7985
100	0.01	24.4564	284.6705

$$d_2(x,z) \quad = \quad d_1(x,z) - \overline{\sigma}(z)\sqrt{T-t}.$$

In the fourth column of Table 6, we list the sample variance ratios obtained from the basic Monte Carlo and the Monte Carlo with our control variate, namely $VAR_{MC}/VAR^{CV}(P_{BS})$. Comparing the third and fourth columns in Table 6, a significant variance reduction is readily observed. From this test example and indeed from other extensive numerical experiments, the control variate given above is superior to the importance sampling.

In [2], a detail account for the accuracy of (martingale) control variate method is analyzed and a comment on the difficulty to analyze the variance associated with importance sampling is stated. For some option price P or its approximation, the martingale term $\mathcal{M}(P,T)$ defined by

$$\mathcal{M}(P,T) = \int_0^T e^{-rs}\frac{\partial P}{\partial x}(s,S_s)\sigma S_s dW_s^{(0)\star}$$

can be interpreted as the delta hedging portfolio accumulated up to time T from time 0. Thus the term $\mathcal{M}(P;T)$ is also called the hedging martingale by the price P and the estimator defined from (23), i.e.

$$\frac{1}{N}\sum_{i=1}^{N}\left[e^{-rT}H(S_T^{(i)}) - \mathcal{M}^{(i)}(P_{BS},T)\right], \tag{25}$$

is called the martingale control variate estimator. Intuitively the effectiveness of the martingale control variate $e^{-rT}H(S_T) - \mathcal{M}(P_{BS};T)$ is due to the fact that if delta trading $\frac{\partial \tilde{P}}{\partial x}(t,x)$ is closed to the actual hedging strategy, fluctuations of the replicating error will be small so that the variance of the estimator (25) should be small. Under OU-type processes to model (Y_t, Z_t) in (1) with $0 < \varepsilon, \delta \ll 1$, the variance of the martingale control variate for European options is small of order ε and δ. This asymptotic result is shown in [2]. Variance analysis to American options and Asian options can be found in [3] and [8] respectively.

4 Quasi Monte Carlo Method and A Counterexample

All Monte Carlo methods studied so far are fundamentally related to pseudo random sequences that generate random samples in our simulations. As an alternative integration methods, the use of the quasi-random sequences (also called low-discrepancy sequences) to generate random samples needed in simulations is called Quasi-Monte Carlo method. QMC method has drawn a lot of attention in financial applications, for example see [12] and [20], because they are able to provide better convergence rates.

4.1 Introduction to Quasi Monte Carlo Method

There are two classes of low-discrepancy sequences (LDS in short) as explained extensively in [10], [16] and [23]. One is called the digital net sequences, such as Halton sequence, Sobol sequence, Faure sequence, and Niederreiter $(t,s)-$sequence, etc. To estimate an integral with a smooth integrand over a hypercube space, this kind of LDS has convergence rate $\mathcal{O}(\frac{(\log N)^s}{N})$, where s is the dimension of the problem and N denotes the number of quasi-random sequence. The other class is the integration lattice rule points. This type of LDS is especially efficient for estimating multivariate integrals with periodic and smooth integrands, and it has convergence rate $\mathcal{O}(\frac{(\log N)^{\alpha s}}{N^\alpha})$, where $\alpha > 1$ is a parameter related to the smoothness of the integrand. L'Ecuyer [14] also made contributions to lattice rules based on linear congruential generator. One important feature of this type of lattice rule points (referred to L'Ecuyer's type lattice rule points, LTLRP, thereafter) is that it is easy to generate high dimensional LTLRP point sets with convergence rate comparable to digital net sequences. We will apply the LTLRP as well since our test examples are high dimensional. Besides the above LDS, we also apply the Brownian bridge (BB) sampling technique to our test problems. Detailed information about Brownian bridge sampling can be found in [7].

It is possible to measure the QMC error through a confidence interval while preserving much of the accuracy of the QMC method. Owen [19] showed that for smooth integrands, the root mean square error of the integration over the hypercube space using a class of randomize nets is $\mathcal{O}\left(1/N^{1.5-\varepsilon}\right)$ for all $\varepsilon > 0$. This accuracy result promotes the use of *randomized* QMC methods. See for example [11] and [14] for the use of randomization schemes.

Despite that regularity of the integrand function corresponding to the payoff $H(S_T)$ is generally poor [7], there are still many applications of using QMC or randomized QMC as a computational tool. In Section 4.2 we give a counterexample of using QMC method for pricing lower bound solutions of American put options. The error of QMC methods applying to the basic Monte Carlo estimator can be very sensitive to the choice of quasi random sequences.

As shown in Table 7, low-biased prices calculated from either Niederreiter or Sobol sequence are found greater than the benchmark true American option prices in all cases of intial stock price $S(0)$, though Sobol sequence does generate smaller estimates than Niederreiter. Surprisingly by adding a hedging martingale as a control to construct the new estimator of control variate, the accuracy of the low-biased America option price estimates are found to be significantly better for

(1) Monte Carlo Simulations shown on Columns 2 and 3 in Table 7

(2) two quasi-random sequences shown on Columns 4,5 (Niererreiter) and 6,7(Sobol) respectively.

In particular we observe that the low-biased estimates obtained from Sobol sequence corrected by martingale control (Sobol +CV shown on Column 7) are even more accurate than estimates obtained from martingale control variate (MC + CV) shown on Column 3. This effect documents that the hedging martingale improve the smoothness of the original American option pricing problem for QMC method.

4.2 Low-Biased Estimate of American Put Option Price

The right to early exercise a contingent claim is an important feature for derivative trading. An American option offers its holder, not the seller, the right but not the obligation to exercise the contract any time prior to maturity during its contract life time. Based on the no arbitrage argument, the American option price at time 0, denoted by P_0, with maturity $T < \infty$ is considered as an optimal stopping time problem [3, 22] defined by

$$P_0 = \sup_{0 \le \tau \le T} I\!\!E^\star \left\{ e^{-r\tau} H\left(S_\tau\right) \right\}, \tag{26}$$

where τ denotes a bounded stopping time less than or equal to the maturity T. We shall assume in this section that the underlying dynamics S_t follows Black-Scholes model so that $dS_t = rS_t dt + \sigma S_t dW_t^\star$.
Longstaff and Schwartz [17] took a dynamic programming approach and proposed a least-square regression to estimate the *continuation value* at each in-the-money asset price state. By comparing the continuation value and the instant exercise payoff, their method exploits a decision rule, denoted by $\underline{\tau}$, for early exercise along each sample path generated. As the fact that $\underline{\tau}$ being a suboptimal stopping rule, Longstaff-Schwartz' method induces a low-biased American option price estimate

$$I\!\!E^\star \left\{ e^{-r\underline{\tau}} H\left(S_{\underline{\tau}}\right) \right\}. \tag{27}$$

It is shown in [2] and [3] that we can use a locally hedging martingale to preserve the low-biased estimate (27) by

$$I\!\!E^\star \left\{ e^{-r\underline{\tau}} H\left(S_{\underline{\tau}}\right) - \int_0^{\underline{\tau}} e^{-rs} \frac{\partial P_E}{\partial x}(s, S_s) \sigma S_s dW_s^\star \right\} \tag{28}$$

where $\tilde{P}$ is an approximation of the American option price. By the spirit of hedging martingale discussed in Section 3.2, we consider $\tilde{P} = P_E$ the counterpart European option price. In the case of the American put option, P_0 is unknown but its approximation P_E admits a closed-form solution, known as the Black-Scholes formula. Its delta, used in (28), is given by

$$\frac{\partial P_E}{\partial x}(t, x; T, K, r, \sigma) = \mathcal{N}\left(\frac{\ln(x/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \right) - 1,$$

where $\mathcal{N}(x)$ denotes the cumulative normal integral function.
As an example we consider a pricing problem at time 0 for the American put option with parameters $K = 100, r = 0.06, T = 0.5$, and $\sigma = 0.4$. Numerical results of the low-biased estimates by MC/QMC with or without hedging martingales are demonstrated in Table 7. The first column illustrates a set of

different initial asset price S_0. The true American option prices corresponding to S_0 are given in Column 6, depicted from from Table 1 of [22]. Monte Carlo simulations are implemented by sample size $N = 5000$ and time step size (Euler discretization) $\Delta t = 0.01$. Column 2 and Column 3 illustrate low-biased estimates and their standard errors (in parenthesis) obtained from MC estimator related to Equation (27) and MC+CV estimator related to Equation (28) respectively.
Foe Monte Carlo method, we observe that (1) almost all estimates obtained from martingale control variate are below the true prices (2) the standard errors are significantly reduced after adding the martingale control $\mathcal{M}(P_E; \underline{\tau})$. A variance analysis for the applications of Monte Carlo methods to estimate high and low-biased American option prices can be found in [3].
For QMC methods we use 5000 Niederreiter and Sobol sequences of dimension 100. In column 4 we see clearly that in all cases of S_0, low-biased QMC estimates are unreasonably greater than the true American prices. The striking part is that after adding hedging martingales, low-biased estimates related to Nied+CV shown on Column 5 and Sobol+CV shown on Column 7 are indeed below the benchmark true prices. These results strongly indicate that the hedging martingale plays the role of a smoother for MC/QMC methods. Because the complexity of American option pricing problems is high, we explore the smooth effect of hedging martingales by considering European option pricing problems under multifactor stochastic volatility models in next section.

5 A Smooth Estimator: Control Variate for MC/QMC Methods

We have seen in Section 3 the (martingale) control variate method performs better in variance reduction than importance sampling for pricing European option prices under multi-factor stochastic volatility models, and in Section 4 that a possibility of misusing QMC method and a way to fix it by (local martingale) control variate for pricing American options. In this section we give a comprehensive studies in efficiencies, including variance reduction ratios and computing time, of using control variate technique developed in the previous section, combined with Monte Carlo and quasi-Monte Carlo methods. Our test examples are European option prices and its *delta*, the

Table 7: Comparisons of low-biased estimates (Column 2-5) and the actual American option prices (Column 6). MC denotes the basic Monte Carlo estimates. MC+CV denotes the control variate estimates with the hedging martingale $\mathcal{M}(P_E; \underline{\tau})$ being the additive control. Standard errors are shown in the parenthesis. QMC and QMC+CV denote calculations of Equation (27) and (28) using quasi sequences respectively.

S_0	MC	MC+CV	Nied	Nied+CV	Sobol	Sobol+CV	P_0(true)
80	20.7368 (0.2353)	20.6876 (0.0124)	22.6082	**21.3201**	21.7576	**21.6013**	21.6059
85	17.3596 (0.2244)	17.3586 (0.0134)	19.4384	**17.9261**	18.3297	**18.0350**	18.0374
90	14.3871 (0.2125)	14.3930 (0.0139)	16.6127	**14.8104**	15.1183	**14.8961**	14.9187
95	11.8719 (0.1995)	11.8434 (0.0148)	13.8932	**12.1300**	12.3594	**12.1673**	12.2314
100	9.8529 (0.1881)	9.6898 (0.0157)	11.6586	**9.8260**	10.0792	**9.8754**	9.946
105	7.9586 (0.1684)	7.8029 (0.0154)	9.6797	**7.8798**	8.1467	**7.9813**	8.0281
110	6.2166 (0.1518)	6.2606 (0.0150)	7.9423	**6.2726**	6.4757	**6.3924**	6.4352
115	5.0815 (0.1367)	5.0081 (0.0144)	6.5097	**4.9778**	5.1302	**5.0681**	5.1265
120	4.0885 (0.1245)	3.9389 (0.0146)	5.3008	**3.8998**	3.9772	**3.9784**	4.0611

first partial derivative of option price with respect to its initial stock price, under random volatility environment.

5.1 European Call Option Estimation

We assume that the underlying asset S is given by (1). In our computations, we use C++ on Unix as our programming language. The pseudo random number generator we used is $ran2()$ in [21]. In our comparisons, the sample sizes for MC method are 10240, 20480, 40960, 81920, 163840, and 327680, respectively; and those for Sobol sequence related methods are 1024, 2048, 4096, 8192, 16384, and 32768, respectively, each with 10 random shifts; and the sample sizes for L'Ecuyer's type lattice rule points (LTLRP for short) related methods are 1021, 2039, 4093, 8191, 16381, and 32749, respectively, and again, each with 10 random shifts.
In the following examples, we divide the time interval $[0, T]$ into $m = 128$ subintervals. In Table 8, the first column labeled as N indicates the number of Monte Carlo simulations or the Quasi-Monte Carlo points. The second column labeled as MC indicates the option price estimates (standard errors in the parenthesis) based on the basic MC estimator (12). All rest columns record variance reduction ratios between many specific MC/QMC methods and the basic MC estimates. For example, the third column labeled as MC+CV indicates the variance reduction ratios as the squares of the standard errors in the second column versus the standard errors obtained from the martingale control variate estimation (25). The fourth column labeled as Sobol indicates the variance reduction rations as the squares of the standard errors in the second column versus the standard errors obtained from the estimation (12) by randomized Sobol sequence.
Model parameters and initial setup of the European call option pricing problems under two-factor stochastic volatility models are chosen the same in Table 4 and Table 5 respectively. For mean-reverting rates α and δ in volatility processes, we take $\alpha = 50$, $\delta = 0.5$. Numerical results are listed in Tables 8 and 9, where MC+CV stands for Monte Carlo method using control variate technique, Sobol+BB means the quasi-Monte Carlo method using Sobol sequence with Brownian bridge sampling technique, LTLRP for QMC method using L'Ecuyer type lattice rule points, etc.

From Table 8, we observed the following facts. Using the control variate technique, the variance reduction ratios are around 48 for pseudo-random sequences. Without control variate, both Sobol sequence and L'Ecuyer type

Table 8: Comparison of simulated European call option values and variance reduction ratios for $\alpha = 50$, $\delta = 0.5$

N	MC	MC+CV	Sobol	Sobol+CV	Sobol+BB	Sobol+CV+BB
1024	11.839(0.126)	45.8	5.0	339.3	2.6	129.3
2048	11.837(0.090)	48.0	2.3	304.8	4.0	138.4
4096	11.862(0.064)	48.4	1.8	124.8	2.5	158.4
8192	11.804(0.045)	47.4	2.3	124.0	2.9	148.4
16384	11.816(0.032)	47.1	1.4	176.1	7.7	115.5
32768	11.857(0.022)	48.1	1.7	235.9	4.5	479.9

N	LTLRP	LTLRP+CV	LTLRP+BB	LTLRP+CV+BB
1021	2.0	75.5	7.3	687.9
2039	3.1	135.1	7.0	298.5
4093	3.1	143.9	2.2	140.1
8191	4.2	347.8	4.9	286.0
16381	3.1	227.9	7.8	94.8
32749	6.4	728.7	15.1	741.6

Table 9: Comparison of time (in seconds) used in the simulation of the above European option

N	MC	MC+CV	Sobol	Sobol+CV	Sobol+BB	Sobol+CV+BB
1024	7	10	7	9	7	12
2048	13	19	13	17	13	18
4096	26	40	26	35	28	39
8192	54	82	56	70	56	78
16384	109	167	107	139	107	157
32768	225	316	222	301	218	318

N	LGLP	LGLP+CV	LGLP+BB	LGLP+CV+BB
1021	5	9	7	11
2039	11	17	13	21
4093	24	34	27	40
8191	45	72	56	78
16381	90	167	106	160
32749	184	293	219	311

lattice rule points, even combined with Brownian bridge sampling technique, the variance reduction ratios are only a few times better than the MC sampling. However, when combined with control variate, the variance reduction ratios for the Sobol sequence vary from about 124 to 339 for Sobol+CV and from 115 to 480 for Sobol+CV+BB; and the variance reduction ratios for the L'Ecuyer type lattice rule points range from about 75 to 729 for LTLRP+CV and from 94 to 742 for LTLRP+CV+BB. This implicitly indicates that the new controlled payoff $e^{-rT}\left(S_T-K\right)^+ - \mathcal{M}(P_{BS})$ is smoother than the original call payoff $e^{-rT}\left(S_T-K\right)^+$. It can be easily seen that under the Black-Scholes model with the constant volatility σ, the controlled payoff is exactly equal to the Black-Scholes option price $P_{BS}(0, S_0; \sigma)$, which is a constant so as a smooth function; while the original call payoff function is only continuous and even not differentiable.
Another interesting observation is that the variance reduction ratios do not always increase when the two low-discrepancy sequences are combined with control variate and Brownian bridge sampling, compared with when they are combined with control variate without Brownian bridge sampling.
Regarding time used in simulations, from Table 9 we observed that the time differences among methods without control variates are not significant, but the time differences between methods with and without control variates are not ignorable. Similar conclusions are true regarding time used in simulations for other cases.

5.2 Accuracy Results

To see the smooth effect of a martingale control, Theorem 1 [2] shows that $Var\left(e^{-rT}H\left(S_T\right) - \mathcal{M}_0(P_{BS})\right) \leq C \max\{\varepsilon, \delta\}$ for smooth payoff function H when ε and δ are small enough. That is, the original variance $Var\left(e^{-rT}H\left(S_T\right)\right)$ is reduced from order 1 to small order of ε and δ using the martingale control. Then by Proinov bound [16] it is easy to show that the error of QMC method is of order $\sqrt{\varepsilon}$ and $\sqrt{\delta}$. Thus it implied that the variance of randomized QMC is of small order ε and δ.

5.3 Delta Estimation

Estimating the sensitivity of option prices over state variables and model parameters are important for risk management. In this section we consider

only the partial derivative of option price with respect to the underlying risk asset price, namely *delta*. To compute Delta, we adopt (1) pathwise differentiation (2) central difference approximation to formula our problems. Then as in previous section we use martingale control variate in Monte Carlo simulations and a combination of martingale control variate with Sobol sequence in randomized QMC method.
By pathwise differentiation (see [7] for instance), the chain rule can be applied to Equation (2) so that

$$\frac{\partial P^{\varepsilon,\delta}}{\partial S_0}(0, S_0, Y_0, Z_0) = \mathbb{E}^\star \left\{ e^{-rT} \mathbf{I}_{\{S_T > K\}} \frac{\partial S_T}{\partial S_0} \mid S_0, Y_0, Z_0 \right\}$$

is obtained. Since

$$e^{rT} \frac{\partial S_T}{\partial S_0} = e^{\int_0^T \sigma_t dW_t^{(0)*} - \frac{1}{2}\int_0^T \sigma_t^2 dt} \tag{29}$$

is an exponential martingale, one can construct a $\mathbb{P}^\star$-equivalent probability measure $\tilde{P}$ by Girsanov Theorem. As a result, under the new measure $\tilde{P}$ the delta $\frac{\partial P^{\varepsilon,\delta}}{\partial S_0}(0, S_0, Y_0, Z_0)$ has a probabilistic representation of the digital-type option

$$P_D^{\varepsilon,\delta}(0, S_0, Y_0, Z_0) := \frac{\partial P^{\varepsilon,\delta}}{\partial S_0}(0, S_0, Y_0, Z_0) = \tilde{E}\left\{ \mathbf{I}_{\{S_T > K\}} \mid S_0, Y_0, Z_0 \right\}, \tag{30}$$

where the dynamics of S_t must follow

$$dS_t = \left(r + f^2(Y_t, Z_t) \right) S_t dt + \sigma_t S_t d\tilde{W}_t^{(0)}, \tag{31}$$

with $\tilde{W}^{(0)}$ being a standard Brownian motion under $\tilde{P}$. The dynamics of Y_t and Z_t will change according to the drift change of $W_t^{(0)}$.
Following the same argument of option price approximation, or see Appendix in [2], the digital call option $P_D^{\varepsilon,\delta}(0, S_0, Y_0, Z_0)$ admits the *homogenized* approximation $\bar{P}_D(\bar{S}_0, Z_0) := \bar{E}\left\{ \mathbf{I}_{\{\bar{S}_T > K\}} \mid \bar{S}_0 = S_0, Z_0 \right\}$, where the "homogenized" stock price $\bar{S}_t$ satisfies

$$d\bar{S}_t = \left(r + \bar{\sigma}^2(Z_t) \right) \bar{S}_t dt + \bar{\sigma}(Z_t) \bar{S}_t d\bar{W}_t^{(0)}$$

with $\bar{W}_t^{(0)}$ being a standard Brownian motion [4]. In fact, the homogenized approximation $\bar{E}\left\{ \mathbf{I}_{\{\bar{S}_T > K\}} \mid \bar{S}_t, Z_t \right\}$ is a probabilistic representation of the

Table 10: Comparison of variance reduction ratios to estimate the Δ of an European call option by martingale control variate method

N	MC	MC+CV	Sobol	Sobol+CV
1024	0.8320(0.00513)	13.3	2.4	17.5
2048	0.8413(0.00358)	13.2	2.2	8.4
4096	0.8405(0.00254)	13.8	2.9	21.4
8192	0.8371(0.00180)	13.5	3.8	11.3
16384	0.8397(0.00127)	13.7	3.0	21.4
32768	0.8397(0.00090)	13.7	1.3	18.8

homogenized "delta", $\frac{\partial P_{BS}}{\partial x}$, where P_{BS} defined in Section 2.1.
The martingale control for the digital call option price (30) can be constructed as in Section 3.2 so that similar martingale control variate estimator is obtained as

$$\frac{1}{N}\sum_{k=1}^{N}\left[e^{-rT}\mathbf{I}_{\left\{S_T^{(k)}>K\right\}} - \mathcal{M}^{(k)}(\bar{P}_D, T)\right].$$

Numerical results of variance reduction by MC/QMC to estimate *delta* can be found in Table 10. All model parameters, initial conditions and mean-reverting rates are chosen the same in previous section.

Another way to approximate the *delta* is by central difference. A small increment $\Delta_S > 0$ is chosen to discretize the partial derivative by

$$P_D^{\varepsilon,\delta} = \frac{\partial P^{\varepsilon,\delta}}{\partial S_0} \approx \frac{P^{\varepsilon,\delta}(0, S_0 + \Delta_S/2, Y_0, Z_0) - P^{\varepsilon,\delta}(0, S_0 - \Delta_S/2, Y_0, Z_0)}{\Delta_S}.$$

Each European option price corresponding to different initial stock price $S_0 + \Delta_S/2$ and $S_0 - \Delta_S/2$ respectively is computed by the martingale control variate method with MC/QMC. Numerical results of variance reduction by MC/QMC to estimate *delta* can be found in Table 11.

In contrast to the European call option cases, QMC method doesn't make a great benefit in variance reduction in both pathwise differentiation and central difference approximation. This is because the regularity of the *delta* function is worse than the call function.

Table 11: Comparison of variance reduction ratios to approximate the Δ of an European call option by Central Difference Scheme

N	MC	MC+CV	Sobol	Sobol+CV
1024	0.8490(0.00507)	15.8	2.6	25.7
2048	0.8354(0.00357)	14.7	1.8	7.9
4096	0.8378(0.00253)	14.7	2.8	21.6
8192	0.8355(0.00179)	14.9	5.5	13.6
16384	0.8381(0.00126)	14.7	4.8	19.4
32768	0.8384(0.00090)	14.7	2.8	15.2

6 Conclusion

Using (randomized) QMC methods for irregular or high dimensional problems in computational finance may not be efficient as shown in pricing American option under Black-Scholes model and European option under multifactor stochastic volatility models, respectively. Based on the delta hedging strategy in trading financial derivatives, the value process of a hedging portfolio is considered as a martingale control in order to reduce the risk (replication error) of traded derivatives. For the martingale control, its role as a smoother for MC/QMC methods becomes clear when significant variance reduction ratios are obtained. An explanation of the effect of the smoother under perturbed volatility models can be found in [8].

References

[1] J.-P. Fouque and C.-H. Han, Variance Reduction for Monte Carlo Methods to Evaluate Option Prices under Multi-factor Stochastic Volatility Models, *Quantitative Finance*, Volume 4, number 5, (1-10), 2004.

[2] J.-P. Fouque and C.-H. Han, A Martingale Control Variate Method for Option Pricing with Stochastic Volatility, *ESAIM Probability & Statistics*, 11, (40-54), 2007.

[3] J.-P. Fouque and C.-H. Han, Asymmetric Variance Reduction for Pricing American Options, to appear on *Mathematical Modeling and Numerical Methods in Finance*, Edited by A. Bensoussan and Q. Zhang. To appear 2008.

[4] J.-P. Fouque, G. Papanicolaou, R. Sircar, *Derivatives in Financial Markets with Stochastic Volatility,* Cambridge University Press, 2000.

[5] J.P. Fouque, G. Papanicolaou, R. Sircar and K. Solna, Multiscale Stochastic Volatility Asymptotics, *SIAM Journal on Multiscale Modeling and Simulation*, 2(1), (22-42), 2003.

[6] J.-P. Fouque and T. Tullie, Variance Reduction for Monte Carlo Simulation in a Stochastic Volatility Environment, *Quantitative Finance*, 2002.

[7] Paul Glasserman, *Monte Carlo Methods in Financial Engineering,* Springer Verlag, 2003.

[8] C.-H. Han and Y. Lai, Generalized Control Variate Methods to Price Asian Options, submitted.

[9] S. Heston, A Closed-form solution for options with stochastic volatility with applications to bond and currency options, *Review of Financial Studies,* 6, 2, 1993.

[10] L. Hua and Y. Wang, *Applications of Number Theory in Numerical Analysis,* Springer-Verlag, 1980.

[11] P. Jackel, *Monte Carlo Methods in Finance*, John Wiley & Sons Ltd. 2002.

[12] C. Joy, P. Boyle and K. S. Tan, Quasi-Monte Carlo Methods in Numerical Finance, *Management Science*, 42(6), 926-938, 1996.

[13] I. Karatzas, S. E. Shreve, *Brownian Motion and Stochastic Calculus,* 2/e, Springer, 2000.

[14] P. L'Ecuyer and C. Lemieux, Variance Reduction via Lattice Rules, *Management Science*, 46(9), 1214-1235, 2000.

[15] B. Lapeyre, E. Pardoux, R. Sentis, *Introduction to Monte Carlo Methods for Transport and Diffusion Equations,* Oxford University Press, 2003.

[16] H. Niederreiter, *Random Number Generation and Quasi-Monte Carlo Methods*, SIAM, Philadelphia, 1992.

[17] F. Longstaff and E. Schwartz, Valuing American Options by Simulation: A Simple Least-Squares Approach, *Review of Financial Studies.* 14: 113-147, 2001.

[18] Bernt Oksendal, *Stochastic Differential Equations,* Springer, 1998.

[19] A.B. Owen, Scrambled net variance for integrals of smooth functions, *Annals of Statistics,* 25:1541-1562, 1997.

[20] S. Paskov and J. Traub, Faster Valuation of Financial Derivatives, *Journal of Portfolio Management*, 22(1), 113-120, 1995.

[21] W. H. Press, S. A. Teukolsky, W. T. Vetterling and B. P. Flannery, *Numerical recipes in C: The Art of Scientific Computing*, New York : Cambridge University Press, 1992.

[22] L.C.G. Rogers, Monte Carlo valuation of American Options, *Mathematical Finance,* 12, 271-286, 2002.

[23] I. Sloan and S. Joe, *Lattice Methods for Multiple Integration*, Clarendon Press, Oxford, 1994.