Importance Sampling Estimation of Joint Default Probability under Structural-Form Models with Stochastic Correlation

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Abstract

This paper aims to estimate joint default probabilities under the structural-form model with a random environment; namely stochastic correlation. By means of a singular perturbation method, we obtain an asymptotic expansion of a two-name joint default probability under a fast mean-reverting stochastic correlation model. The leading order term in the expansion is a joint default probability with an *effective* constant correlation. Then we incorporate an efficient importance sampling method used to solve a first passage time problem. This procedure constitutes a *homogenized* importance sampling to solve the full problem of estimating the joint default probability with stochastic correlation models.

1 Introduction

Estimation of a joint default probability under the structural-form model typically requires solving a first passage time problem. Black and Cox [1] and Zhou [17] provided financial motivations and technical details on the first passage time approach for one and two dimensional cases, respectively.

A high-dimensional setup of the first passage time problem is as follows. Assume that a credit portfolio includes n reference defaultable assets or names. Each asset value, S_{it} $1 \leq i \leq n$, is governed by

$$dS_{it} = \mu_i S_{it} dt + \sigma_i S_{it} dW_{it}, \tag{1}$$

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where μ_i denotes a constant drift rate, σ_i denotes a constant volatility and the driving innovation dW_{it} is an infinitesimal increment of a Brownian motion (Wiener process) W_i with the instantaneous constant correlation

$$d\left\langle W_i, W_j \right\rangle_t = \rho_{ij} dt.$$

Each name also has a barrier, B_i, $1 \leq i \leq n$, and default happens at the first time S_{it} falls below the barrier level. That is, the ith default time τ_i is defined by the first hitting time

$$\tau_i = \inf\{t \geq 0 : S_{it} \leq B_i\}. \quad (2)$$

Let the filtration $\mathscr{F}_{t \geq 0}$ be generated by all $S_{it}, i = 1, \cdots, n$ under a probability measure $\boldsymbol{P}$. At time 0, the joint default probability with a terminal time T is defined by

$$DP = \boldsymbol{E}\left\{\Pi_{i=1}^{n} \mathbf{I}(\tau_i \leq T) | \mathscr{F}_0\right\}. \quad (3)$$

Due to the high dimensional nature of this problem ($n = 125$ in a standard credit derivative [3], for example), Monte Carlo methods are very useful tools for computation. However, the basic Monte Carlo method converges slowly to the probability of multiple defaults defined in (3). We will review an efficient importance sampling scheme discussed in Han [10] to speed up the computation. This method is asymptotically optimal in reducing variance of the new estimator.

Engle [6] revealed the impact of correlation between multiple asset dynamics. A family of discrete-time correlation models called dynamic conditional correlation (DCC) has been widely applied in theory and practice. Hull et al. [15] examined the effect of random correlation in continuous time and suggested stochastic correlation for the structural-form model. This current paper studies the joint default probability estimation problem under the structural-form model with stochastic correlation. For simplicity, we consider a two-dimensional case, $n = 2$. This problem generalizes Zhou's study [17] with constant correlation.

Note that under stochastic correlation models, there exists no closed-form solution for the two-name joint default probability. A two-step approach is proposed to solve this estimation problem. First, we apply a singular perturbation technique and derive an asymptotic expansion of the joint default probability. Its leading order term is a default probability with an *effective* constant correlation so that the limiting problem becomes the standard setup of the first passage time problem. Second, given the accuracy of this asymptotic approximation, we develop a *homogenized* likelihood function for measure change. It allows that the efficient importance sampling method [10] can be applied for estimation of the two-name joint default probability under stochastic correlation models. Results of numerical simulation show that estimated joint default probabilities are sensitive to the change in correlation and our proposed method is efficient and robust even when the mean-reverting speed is not in a small regime.

The organization of this paper is as follows. Section 2 presents an asymptotic expansion of the joint default probability under a fast mean-reverting correlation by means of the singular perturbation analysis. Section 3 reviews the efficient importance sampling method to estimate joint default probabilities under the classical structural-form model with constant correlation. Section 4 constructs a homogenized importance sampling method to solve the full problem.

2 Stochastic Correlation Model: Two Dimensional Case

The closed-form solution of a two-name joint default probability under a constant correlation model is given in [2]. Assume that asset prices (S_{1t}, S_{2t}) driven by two geometric Brownian motions with a constant correlation $\rho, -1 \leq \rho \leq 1$ are governed by

$$\begin{aligned} dS_{1t} &= \mu_1 S_{1t} dt + \sigma_1 S_{1t} dW_{1t} \\ dS_{2t} &= \mu_2 S_{2t} dt + \sigma_2 S_{2t} (\rho dW_{1t} + \sqrt{1-\rho^2} dW_{2t}), \end{aligned}$$

following the usual setup in (1). When the default boundary is deterministic of an exponential type $Be^{\lambda t}$, each default time τ_i can be defined as

$$\tau_i = \inf\{t \geq 0; S_{it} \leq B_i e^{\lambda_i t}\} \tag{4}$$

for $i \in \{1,2\}$. This setup is slightly more general than our constant barriers (2) but it causes no extra difficulty when log-transformation is applied. No initial default, i.e., $S_{i0} > B_i$ for each i, is assumed to avoid the trivial case. The joint default probability defined by

$$P(0,x_1,x_2) = \boldsymbol{P}(\tau_1 \leq T, \tau_2 \leq T)$$

can be expressed as

$$P(0,x_1,x_2) = P_1(0,x_1) + P_2(0,x_2) - Q^{1,2}(0,x_1,x_2) \tag{5}$$

where $P_i := \boldsymbol{P}(\tau_i \leq T)$ denotes the ith marginal default probability and $Q^{1,2} := \boldsymbol{P}(\tau_1 \leq T \text{ or } \tau_2 \leq T)$ denotes the probability that at least one default happens. The closed-form formula for each P_i, $i \in \{1,2\}$, is

$$P_i = \mathcal{N}\left(-\frac{d_i}{\sqrt{T}} - \frac{\mu_i - \lambda_i}{\sigma_i}\sqrt{T}\right) + e^{\frac{2(\lambda_i - \mu_i)d_i}{\sigma_i}} \mathcal{N}\left(-\frac{d_i}{\sqrt{T}} + \frac{\mu_i - \lambda_i}{\sigma_i}\sqrt{T}\right),$$

where $d_i = \frac{\ln(S_0^i/K_i)}{\sigma_i}$. The last term $Q^{1,2}$ can be expressed as a series of modified Bessel functions (see [2] for details) and we skip it here.

Hull et al. (2005) proposed a mean-reverting stochastic correlation for the structural-form model, and they found empirically a better fit to spreads of credit derivatives.

We assume that the correlation process $\rho_t = \rho(Y_t)$ is driven by a mean-reverting process Y_t such as the Ornstein-Uhlenbeck process. A small time scale parameter ε is incorporated into the driving correlation process Y_t so that the correlation changes rapidly compared with the asset dynamics of S. The two-name dynamic system with a fast mean-reverting stochastic correlation is described by

$$\begin{aligned} dS_{1t} &= \mu_1 S_{1t} dt + \sigma_1 S_{1t} dW_{1t} \\ dS_{2t} &= \mu_2 S_{2t} dt + \sigma_2 S_{2t} \left(\rho(Y_t) dW_{1t} + \sqrt{1-\rho^2(Y_t)} dW_{2t} \right) \\ dY_t &= \frac{1}{\varepsilon}(m - Y_t) dt + \frac{\sqrt{2}\beta}{\sqrt{\varepsilon}} dZ_t, \end{aligned} \tag{6}$$

where the correlation function $\rho(\cdot)$ is assumed smooth and bounded in $[-1,1]$, and the driving Brownian motions W's and Z are assumed to be independent of each other. The joint default probability under a fast mean-reverting stochastic correlation model is defined as

$$P^{\varepsilon}(t,x_1,x_2,y) := \boldsymbol{E}\left\{ \Pi_{i=1}^{2} \mathbf{I}\{ \min_{t \le u \le T} S_{iu} \le B_i \} | S_{1t} = x_1, S_{2t} = x_2, Y_t = y \right\}, \tag{7}$$

provided no default before time t.

From the modeling point of view, the assumption of a mean-reverting correlation is consistent with DCC model, see Engle [6], in which a quasi-correlation is often assumed mean-reverting. From the statistical point of view, a Fourier transform method developed by Malliavin and Mancino [16] provides a nonparametric way to estimate dynamic volatility matrix in the context of a continuous semi-martingale. Our setup of the stochastic correlation model (6) satisfies assumptions in [16]. This implies that model parameters of volatility and correlation defined in (6) can be estimated via the Fourier transform method. Moreover, from the computational point of view, stochastic correlation introduces a random environment into the classical first passage time problem in dynamic models. This situation is similar to Student-t distribution over the Gaussian distribution in static copula models [5] arising from reduced-form models in credit risk. Han and Wu [13] have recently solved this static Gaussian copula problem with a random environment; namely, Student-t copula. In contrast, the stochastic correlation estimation problem considered in this paper fills a gap of research work for a random environment in dynamic models.

2.1 Formal Expansion of The Perturbed Joint Default Probability

By an application of Feynman-Kac formula, $P^{\varepsilon}(t,x_1,x_2,y)$ solves a three-dimensional partial differential equation (PDE)

$$\left(\frac{1}{\varepsilon}\mathscr{L}_0 + \mathscr{L}_1 \right) P^{\varepsilon}(t,x_1,x_2,y) = 0, \tag{8}$$

where partial differential operators are

$$\begin{aligned}
\mathscr{L}_0 &= \beta^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y} \\
\mathscr{L}_1(\rho(y)) &= \mathscr{L}_{1,0} + \rho(y)\mathscr{L}_{1,1} \\
\mathscr{L}_{1,0} &= \frac{\partial}{\partial t} + \sum_{i=1}^{2} \frac{\sigma_i^2 x_i^2}{2}\frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{2} \mu_i x_i \frac{\partial}{\partial x_i} \\
\mathscr{L}_{1,1} &= \sigma_1 \sigma_2 x_1 x_2 \frac{\partial^2}{\partial x_1 \partial x_2}.
\end{aligned}$$

The terminal condition is $P^{\varepsilon}(T,x_1,x_2,y) = I_{\{x_1 \le B_1\}} I_{\{x_2 \le B_2\}}$ and two boundary conditions are $P^{\varepsilon}(t,B_1,x_2,y) = P^{\varepsilon}(t,x_1,B_2,y) = 0$.

Suppose that the perturbed joint default probability admits the following expansion

$$P^{\varepsilon}(t,x_1,x_2,y) = \sum_{i=0}^{\infty} \varepsilon^i P_i(t,x_1,x_2,y).$$

Substituting this into (8),

$$\begin{aligned}
0 &= \left(\frac{1}{\varepsilon}\mathscr{L}_0 + \mathscr{L}_1\right)\left(P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \cdots\right) \\
&= \frac{1}{\varepsilon}(\mathscr{L}_0 P_0) + (\mathscr{L}_0 P_1 + \mathscr{L}_1 P_0) + \varepsilon(\mathscr{L}_0 P_2 + \mathscr{L}_1 P_1) \\
&\quad + \varepsilon^2(\mathscr{L}_0 P_3 + \mathscr{L}_1 P_2) + \cdots
\end{aligned}$$

is obtained. By equating each term in order of ε to zero, a sequence of PDEs must be solved.

For the $\mathscr{O}(\frac{1}{\varepsilon})$ term, $\mathscr{L}_0 P_0(t,x_1,x_2,y) = 0$. One can choose P_0 as variable $y-$independent. For the $\mathscr{O}(1)$ term, $(\mathscr{L}_0 P_1 + \mathscr{L}_1 P_0)(t,x_1,x_2,y) = 0$, which is a Poisson equation. Because $\mathscr{L}_0$ is the generator of an ergodic process Y_t, by centering condition we can obtain $< \mathscr{L}_1 > P_0 = 0$. The notation $< \cdot >$ means the averaging with respect to the invariance measure of the ergodic process Y. Thus the leading order term P_0 solves the *homogenized* PDE:

$$(\mathscr{L}_{1,0} + \bar{\rho}\mathscr{L}_{1,1})P_0(t,x_1,x_2) = 0,$$

where $\bar{\rho} =< \rho(y) >_{OU}= \int \rho(y)\frac{1}{\sqrt{2\pi}\nu} e^{-\frac{(y-m)^2}{2\nu^2}} dy$ with the terminal condition is $P_0(T,x_1,x_2) = I_{\{x_1 \le B_1\}} I_{\{x_2 \le B_2\}}$ and two boundary conditions are $P_0(t,B_1,x_2) = P_0(t,x_1,B_2) = 0$. The closed-form solution of $P_0(t,x_1,x_2)$ exists with a similar formulation presented in (5).

Combining $\mathscr{L}_0 P_1 + \mathscr{L}_1 P_0 = 0$ with $< \mathscr{L}_1 > P_0 = 0$, we obtain $\mathscr{L}_0 P_1 = -(\mathscr{L}_1 P_0 - < \mathscr{L}_1 > P_0)$ such that

$$\begin{aligned}P_1(t,x_1,x_2,y) &= -\mathscr{L}_0^{-1}\left(\mathscr{L}_1 - <\mathscr{L}_1>\right)P_0(t,x_1,x_2)\\ &= -\mathscr{L}_0^{-1}\left(\rho(y)-\bar{\rho}\right)\mathscr{L}_{1,1}P_0(t,x_1,x_2)\\ &= -\varphi(y)\sigma_1\sigma_2 x_1 x_2\frac{\partial^2}{\partial x_1\partial x_2}P_0(t,x_1,x_2),\end{aligned}$$

where $\varphi(y)$ is assumed to solve the Poisson equation $\mathscr{L}_0\,\varphi(y) = \rho(y) - \bar{\rho}$.

Similar argument goes through successive expansion terms. We skip the lengthy derivation but simply summarize each successive term for $n \geq 0$

$$P_{n+1}(t,x_1,x_2,y) = \sum_{i\geq 0, j\geq 1}^{i+j=n+1}\varphi_{i,j}^{(n+1)}(y)\,\mathscr{L}_{1,0}^{i}\,\mathscr{L}_{1,1}^{j}\,P_n,$$

where a sequence of Poisson equations must be solved from

$$\begin{aligned}\mathscr{L}_0\,\varphi_{i+1,j}^{(n+1)}(y) &= \left(\varphi_{i,j}^{(n)}(y) - <\varphi_{i,j}^{(n)}(y)>\right)\\ \mathscr{L}_0\,\varphi_{i,j+1}^{(n+1)}(y) &= \left(\rho(y)\,\varphi_{i,j}^{(n)}(y) - <\rho\,\varphi_{i,j}^{(n)}>\right).\end{aligned}$$

Hence, a recursive formula for calculating the joint default probability $P^{\varepsilon} = P_0 + \varepsilon P_1 + \varepsilon^2 P_2 + \cdots$ is derived.

In summary, we have formally derived that

$$P^{\varepsilon}(t,x_1,x_2,y) = P_0(t,x_1,x_2;\bar{\rho}) + \mathscr{O}(\varepsilon), \tag{9}$$

where the accuracy result can be obtained by a regularization technique presented in [14].

Remark: The asymptotic expansion presented in this section can be generalized to multi-dimensional cases.

3 Efficient Importance Sampling for the First Passage Time Problem

In this section, we review the efficient importance sampling scheme proposed in [10] for the first passage time problem (3) in order to improve the convergence of Monte Carlo simulation. The basic Monte Carlo simulation approximates the joint default probability defined in (3) by the following estimator

$$DP \approx \frac{1}{N}\sum_{k=1}^{N}\Pi_{i=1}^{n}\mathbf{I}\left(\tau_i^{(k)} \leq T\right), \tag{10}$$

where $\tau_i^{(k)}$ denotes the kth i.i.d. sample of the ith default time defined in (4) and N denotes the total number of simulations.

By Girsanov theorem, one can construct an equivalent probability measure $\tilde{P}$ defined by the following Radon-Nikodym derivative

$$\frac{dP}{d\tilde{P}} = Q_T(h.) = \exp\left(\int_0^T h(s,S_s)\cdot d\tilde{W}_s - \frac{1}{2}\int_0^T ||h(s,S_s)||^2 ds\right), \tag{11}$$

where we denote by $S_s = (S_{1s},\cdots,S_{ns})$ the state variable (asset value process) vector and $\tilde{W}_s = \left(\tilde{W}_{1s},\cdots,\tilde{W}_{ns}\right)$ the vector of standard Brownian motions, respectively. The function $h(s,S_s)$ is assumed to satisfy Novikov's condition such that $\tilde{W}_t = W_t + \int_0^t h(s,S_s)ds$ is a vector of Brownian motions under $\tilde{P}$.

The importance sampling scheme proposed in [10] selects a **constant** vector $h = (h_1,\cdots,h_n)$ which satisfies the following n conditions

$$\tilde{E}\left\{S_{iT}|\mathscr{F}_0\right\} = B_i, i = 1,\cdots,n. \tag{12}$$

These equations can be simplified by using the explicit log-normal density of S_{iT}, so the following sequence of linear equations for h_i's:

$$\Sigma_{j=1}^{i}\rho_{ij}h_j = \frac{\mu_i}{\sigma_i} - \frac{\ln B_i/S_{i0}}{\sigma_i T}, i = 1,\cdots,n, \tag{13}$$

can be considered. If the covariance matrix $\Sigma = (\rho_{ij})_{1\le i,j,\le n}$ is non-singular, the vector h exists uniquely and the equivalent probability measure $\tilde{P}$ is uniquely determined. The joint default probability defined from the first passage time problem (see (3)) can be estimated from

$$DP = \tilde{E}\left\{\Pi_{i=1}^{n}\mathbf{I}\left(\tau_i \le T\right) Q_T(h)|\mathscr{F}_0\right\} \tag{14}$$

by simulation.

4 Homogenized Importance Sampling under Stochastic Correlation

The objective of this paper is to estimate the joint default probability defined in (7) under a class of stochastic correlation models. A direct application of the efficient importance sampling described in Section 3 is impossible because it requires a constant correlation ρ to solve for the unique h in (13). Fortunately, this hurdle can be overcome by the asymptotic approximation of the joint default probability (see Equation (9)) because its leading-order approximation term has a constant correlation $\bar{\rho}$. As a result, our methodology to estimate the two-name joint default probability with stochastic correlation is simply to apply the efficient importance sampling scheme associated with the *effective* correlation, derived from the singular perturbation analysis. Detailed variance analysis for this methodology is left as a future work. A recent large deviation theory derived in Feng et al. [8] can be a valuable source to provide a guideline for solving this theoretical problem.

Table 1 Two-name joint default probability estimations under a stochastic correlation model are calculated by the basic Monte Carlo (BMC) and the homogenized importance sampling (HIS), respectively. Several time scales ε are given to compare the effect of stochastic correlation. The total number of simulations is 10^4 and an Euler discretization scheme is used by taking time step size $T/400$, where T is one year. Other parameters include $S_{10} = S_{20} = 100, \sigma_1 = 0.4, \sigma_2 = 0.4, B_1 = 50, B_2 = 40, Y_0 = m = \pi/4, \beta = 0.5, \rho(y) = |sin(y)|$. Standard errors are shown in parenthesis.

$\alpha = \frac{1}{\varepsilon}$	BMC	HIS
0.1	$0.0037(6*10^{-4})$	$0.0032(1*10^{-4})$
1	$0.0074(9*10^{-4})$	$0.0065(2*10^{-4})$
10	$0.011(1*10^{-3})$	$0.0116(4*10^{-4})$
50	$0.016(1*10^{-3})$	$0.0137(5*10^{-4})$
100	$0.016(1*10^{-3})$	$0.0132(4*10^{-4})$

Table 1 illustrates estimations of default probabilities of two names under stochastic correlation models by means of the basic Monte Carlo method and the homogenized importance sampling method. It is observed that the two-name joint default probabilities are of order 10^{-2} or 10^{-3}. Though these estimated probabilities are not considered very small, the homogenized importance sampling can still improve the variance reduction ration by 6.25 times at least. Note also that the performance of homogenized importance sampling is very robust to the time scale ε, even it is not in a small regime (for example $\varepsilon = 10$) as the singular perturbation method required.

Next, small probability estimations are illustrated in Table 2. The homogenized importance sampling method provides fairly accurate estimations, say in the 95% confidence interval. The variance reduction rations can raise up to 2500 times for these small probability estimations. In addition, we observe again the robustness of this importance sampling to time scale parameter ε.

Table 2 Two-name joint default probability estimations under a stochastic correlation model are calculated by the basic Monte Carlo (BMC) and the homogenized importance sampling (HIS), respectively. Several time scales ε are given to compare the effect of stochastic correlation. The total number of simulations is 10^4 and an Euler discretization scheme is used by taking time step size $T/400$, where T is one year. Other parameters include $S_{10} = S_{20} = 100, \sigma_1 = 0.4, \sigma_2 = 0.4, B_1 = 30, B_2 = 20, Y_0 = m = \pi/4, \beta = 0.5, \rho(y) = |sin(y)|$. Standard errors are shown in parenthesis.

$\alpha = \frac{1}{\varepsilon}$	BMC	HIS
0.1	$-(-)$	$9.1*10^{-7}(7*10^{-8})$
1	$-(-)$	$7.5*10^{-6}(6*10^{-7})$
10	$-(-)$	$2.4*10^{-5}(2*10^{-6})$
50	$1*10^{-4}(1*10^{-4})$	$2.9*10^{-5}(3*10^{-6})$
100	$1*10^{-4}(1*10^{-4})$	$2.7*10^{-5}(2*10^{-6})$

It is also interesting to observe the effect of time scale from these numerical estimation results. When the stochastic correlation is more volatile (small ε), the probability of joint default increases as well. This is consistent with what observed under stochastic volatility models for option pricing [12]. It shows that these estimations from variance reduction methods are sensitive to changes in correlation and volatility. Hence, it is possible to develop a Monte Carlo calibration method [11] allowing model parameters to fit the implied volatility surface [9] or spreads of credit derivatives [3].

Model parameters within Tables 1 and 2 are homogeneous. That is, dynamics (6) of these two firms are indistinguishable because their model parameters are chosen as the same. Here we consider an inhomogeneous case in a higher dimension, say 4, to illustrate the efficiency of our proposed importance sampling method in Table 3. For simplicity, we fix the time scale ε but use varying firm specific model parameters. A factor structure that generalizes dynamics (6) is chosen as $d < S_i, S_j >_t = \sigma_i \sigma_j S_{it} S_{jt} \rho(Y_t)\, dt$ for $i \neq j \in \{1,2,3,4\}$.

Table 3 Four-name joint default probability estimations under a stochastic correlation model are calculated by the basic Monte Carlo (BMC) and the homogenized importance sampling (HIS), respectively. The time scale ε appearing in the stochastic correlation process is fixed as 10. Other parameters are $S_{i0} = 100, i \in \{1,2,3,4\}, \sigma_1 = 0.5, \sigma_2 = 0.4, \sigma_3 = 0.3, \sigma_4 = 0.2, Y_0 = m = 0, \beta = 0.5, \rho(y) = sin(y)$. Standard errors are shown in parenthesis. Two sets of default thresholds B's are chosen to reflect a bigger and a smaller probability of joint defaults, respectively. The total number of simulations is 10^4 and an Euler discretization scheme is used by taking time step size $T/400$, where T is one year.

Default Thresholds	BMC	HIS
$B_1 = B_2 = B_3 = B_4 = 70$	$0.0019(4*10^{-4})$	$0.0021(1*10^{-4})$
$B_1 = 30, B_2 = 40, B_3 = 50, B_4 = 60$	$-(-)$	$1.1*10^{-7}(2*10^{-8})$

5 Conclusion

Estimation of joint default probabilities under the structural-form model with stochastic correlation is considered as a variance reduction problem under a random environment. We resolve this problem by proposing a homogenized importance sampling method. It comprises (1) derivation of an asymptotic result by means of the singular perturbation analysis given a fast mean-reverting correlation assumption, and (2) incorporating the efficient importance sampling method from solving the classical first passage time problem. Numerical results show the efficiency and robustness of this homogenized importance sampling method even when the time scale parameter is not in a small regime.

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