Monte Carlo Calibration to Implied Volatility Surface under Volatility Models

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Abstract

The calibration problem of implied volatility surface under complex financial models can be formulated as a nonlinear high-dimensional optimization problem. To resolve this problem for genuine volatility models, we develop a sequential methodology termed two-stage Monte Carlo calibration method. It consists of the first stage-dimension separation for splitting parametric set into two subsets, and the second stage-standard error reduction for efficient evaluation of option prices. The first stage dimension separation aims to reduce dimensionality of the optimization problem by estimating some volatility model parameters *a priori* under the historical probability measure such that the total number of model parameters under an option pricing measure is significantly reduced. The second stage standard error reduction aims simultaneously to reduce variance of option payoffs by the martingale control variate algorithm, and to increase the total number of Monte Carlo simulation by the hardware graphics processing unit (GPU) for parallel computing. This two-stage Monte Carlo calibration method is capable of solving a variety of complex volatility models, including hybrid models and multifactor stochastic volatility models. Essentially, it provides a general framework to analyze backward information from the historical spot prices and the forward information from option prices.

Keywords: implied volatility surface, multi-factor stochastic volatility model, hybrid model, Fourier transform method, Monte Carlo simulation, standard error reduction, martingale control variate, GPU parallel computing.

Section 1: Introduction

Most current financial markets consist of the spot market such as stocks or a composite index and its derivative market such as futures or options. From the viewpoint of information content, the spot market contains backward information while the derivative market contains forward information. These distinguishable information can both be important for purposes of trading and risk management.

A model calibration problem of the implied volatility term structure is mainly about solving a nonlinear optimization problem for best fitting to an observed implied volatility surface. This kind of problem can be solved by many different techniques, including Fourier transform method [19,7], numerical methods for partial differential equations (PDEs), perturbation approximation method [10], etc. These methods are mostly based on the deterministic approach, which is often restricted to simpler volatility models, specific volatility model structures, or penalty terms. For example, one-factor and two-factor Heston stochastic volatility models admit closed-form solutions for European options, they can provide a good fit to implied volatility surfaces but they are highly model specific. Numerical methods for PDEs are often restricted by the curse of dimensionality. Perturbation approximation method is an alternative. It is computationally efficient for a wide range of stochastic volatility models, but the accuracy is less explicit. In most situations, solving the model calibration problem from an implied volatility surface by the deterministic approach implies that only forward information is retrieved.

Monte Carlo simulation for option pricing was first proposed by [3]. Unlike aforementioned deterministic methods, Monte Carlo simulation is a probabilistic approach suitable for computational problems in high dimension. Despite intensive studies on variance reduction to improve Monte Carlo simulation, its efficiency remains too low such that there exists very few results on applying Monte Carlo simulation for the model calibration problem in literature. Maruhn [26] utilized a GPU accelerated control variate method and for a smoothened optimization problem under one-factor Heston model. Han et al. [15] utilized (1) Markov chain Monte Carlo (McMC) estimation for one-factor stochastic volatility models under the historical probability measure to retrieve backward information, and (2) the martingale control variate method with randomized Sobol's low discrepancy sequence for implied volatility calibration to retrieve forward information.

One factor volatility model is not sufficient for model calibration to the whole term structure of implied volatilities. Christoffersen et al. [7] and Fouque et al. [12] confirmed that the dynamic of implied volatilities is better fitted under a multi-factor stochastic volatility model (in short, SVM) than a single-factor one. However, solving a complex model calibration problem, i.e. solving a high-dimensional nonlinear optimization problem, by Monte Carlo simulation is a challenge task. We manage to overcome this difficulty by a two-stage approach.

The first stage is about dimension separation. For those model parameters defined under the original historical probability measure, we try to estimate them from spot prices by means of an advanced econometric method such as McMC [15], Fourier transform method [25], etc. Fourier transform method is adopted in our approach for its advantage of fast computation. See [13] for detailed discussion.

The second stage is about standard error reduction. A standard error is defined by the square root of variance divided by the total number of simulation. To reduce a standard error, one can at best reduce the variance and enlarge the sample size of a Monte Carlo estimator simultaneously. The martingale control variate method [10] accelerated by GPU parallel computing was proposed in [14]. It is designed to significantly improve the computational efficiency by means of a software algorithm, i.e., variance reduction method, and a hardware device, i.e., GPU parallel computing. This standard error reduction method is then used for solving the low-dimensional nonlinear optimization problem in order to regress out some model parameters defined under a risk-neutral probability measure.

A natural consequence of the proposed two-stage procedure, termed two-stage Monte Carlo calibration method, is that both backward and forward information can be retrieved from spot prices and option prices, respectively. This method aims to tackle the model calibration problem in a systematic way. The literature has explored different volatility models other than multi-factor SVM. For example, a hybrid volatility model is defined by a combination of SVM and a local volatility model. Moreover, the proposed two-stage Monte Carlo calibration method can be effective for joint calibration to credit spreads or indices, equity option prices in American style, and a treasury bond yield. See [17] as an example to generalize the joint valuation framework by [4].

The structure of this paper is as follows. In Section 2, the two-stage Monte Carlo calibration method is introduced and how the backward and forward information content can be retrieved by this method throughout spot prices and option prices. Section 3 discusses modeling the multi-factor exponential Ornstein-Uhlenbeck (exp-OU) SVMs and the hybrid model can be associated with data frequencies. In Section 4, empirical studies are performed and various volatility models are compared. We conclude in Section 5.

Section II: Methodology: Two-Stage Monte Carlo Calibration Method

This section adopts the single-factor exp-OU SVM as the benchmark example to introduce the two-stage Monte Carlo calibration method. The first stage contains the Fourier transform method and the maximum likelihood method for estimating SVM parameters from spot prices. The second stage includes a speedup process for pricing European options. Then a low-dimensional nonlinear optimization problem can be solved by efficient Monte Carlo simulation.

2.1 Single-Factor Stochastic Volatility Model

Under the historical probability measure, assume the spot price S_t is governed by a single-factor exponential Ornstein-Uhlenbeck (exp-OU) SVM:

$$\frac{dS_t}{S_t} = \mu dt + \sigma_t dW_{0t} \tag{2.1}$$

$$\sigma_t = e^{\frac{Y_t}{2}} \tag{2.2}$$

$$dY_t = \boldsymbol{\alpha}(\boldsymbol{m} - Y_t)dt + \boldsymbol{\beta} dW_{1t} \tag{2.3}$$

$$\langle W_0 \, , W_1 \rangle_t = \boldsymbol{\rho} t \tag{2.4}$$

where W_{0t} and W_{1t} are two standard Brownian motions with the correlation coefficient ρ. The volatility process σ_t is defined by $e^{Y_t/2}$. The return rate of spot price S_t is denoted by μ. The initial state Y_0 is chosen as the long-run mean level m by default for simplicity, the mean-reverting rate is α, and the volatility of volatility is β.

Furthermore, under a risk-neutral probability measure, the underlying asset price process S_t is assumed to follow

$$\frac{dS_t}{S_t} = r dt + \sigma_t dW_{0t}^* \tag{2.5}$$

$$\sigma_t = e^{\frac{Y_t}{2}} \tag{2.6}$$

$$dY_t = \boldsymbol{\alpha}(\widetilde{\boldsymbol{m}} - Y_t)dt + \boldsymbol{\beta} dW_{1t}^* \tag{2.7}$$

$$\langle W_0^*, W_1^* \rangle_t = \boldsymbol{\rho} t \tag{2.8}$$

W_{0t}^* and W_{1t}^* represent two standard Brownian motions with the correlation coefficient ρ. The risk-free interest rate is denoted by r. Notice that the market price of volatility risk is assume to be constant. This assumption follows the convention to remain the structure of volatility dynamics [19], so that the option price can be defined under this dynamics (2.5-2.8).

2.2 Two-Stage Monte Carlo Calibration

According to this single-factor exp-OU model, parameters α, β, m and ρ need to be estimated from the observed spot price S_t, which consists of the backward information. On the first stage, only model calibration α, β, m defined in equation (2. 3) are estimated. Then in the second stage, $\widetilde{m}$ and ρ defined in (2. 7) and (2. 8) are estimated from observed option prices so that these two parameters contain the forward information. It is worth noting that the correlation parameter ρ is often chosen to be estimated under a risk-neutral probability measure because it is known that ρ is important to represent the curvature of the implied volatility curve [12].

2.2.1 Stage One: Parameter Estimation for Volatility Models

This stage concerns about the parameter estimation problem for stochastic volatility models under the historical probability measure. Han [13] resolved this estimation problem by means of (1) the Fourier transform method for observed stock prices or index prices, and (2) a maximum likelihood estimation (MLE) for estimated volatility time series.

Fourier Transform Method

Fourier transform method [25] is a nonparametric method to estimate multivariate volatility process. Its main idea is to reconstruct volatility as time series in terms of sine and cosine basis under the following continuous semi-martingale assumption. Let u_t be the log-price of an underlying asset price S at time t, so that $u_t = \ln(S_t)$, and follow a diffusion process

$$du_t = \mu_t dt + \sigma_t dW_t,$$

where μ_t is the instantaneous growth rate, σ_t is the instantaneous volatility, and W_t is a one-dimensional standard Brownian motion. Note that the original time interval [0,T] can always be rescaled to $[0,2\pi]$ so that the Fourier transform of $u(t)$ can be defined by $\forall\, k \in Z$,

$$\mathfrak{F}(u)(k) := \frac{1}{2\,\pi}\int_0^{2\pi} u(t)e^{-ikt}\,dt$$

$$= \frac{i}{k}\left[\frac{1}{2\pi}\big(u(2\pi) - u(0)\big) - \mathfrak{F}(du)(k)\right].$$

Moreover, the Fourier coefficient of instantaneous log-return in frequency k, $\mathfrak{F}(du)(k)$, is defined as

$$\mathfrak{F}(du)(k) \equiv \frac{1}{2\pi}\int_0^{2\pi} \exp(-ikt)du_t.$$

Given two functions Φ and Ψ, the Bohr convolution product is defined as

$$(\Phi *_B \Psi)(k) \equiv \lim_{N\to\infty}\frac{1}{2N+1}\sum_{s=-N}^{N}\Phi(s)\Psi(k-s).$$

Malliavin and Mancino [25] proved that in frequency domain, the Fourier coefficient of the instantaneous variance is

$$\frac{1}{2\pi}\mathfrak{F}(\sigma^2)(k) = \big(\mathfrak{F}(du) *_B \mathfrak{F}(du)\big)(k), for\ all\ k \in Z. \tag{2.9}$$

The convergence of the equation is attained in probability. Therefore, the instantaneous variance $\sigma^2(t)$ can be calculated by the inverse Fourier transform

$$\sigma^2(t) = 2\pi\,\mathfrak{F}^{-1}\Big(\big(\mathfrak{F}(du) *_B \mathfrak{F}(du)\big)(k)\Big). \tag{2.10}$$

Equation (2.10) reveals that given a set of spot price data $\{S_t\}$ under the historical probability measure, its instantaneously volatility $\{\sigma_t\}$ can be estimated by the Fourier transform method. Next, one continues to apply MLE for parameters of a specific SVM. An instructed procedure can be found in [13].

MLE for EXP-OU Stochastic Volatility Model

The MLE estimators for exp-OU SVM parameters was derived in [16] when authors studies the VaR/CVaR estimation problem under some SVM for applications in risk management.

The notation Y_t below is abused to denote a discrete-time estimated volatility, i.e. $Y_t = 2\ln\hat{\sigma}_t$ and Δ_t denotes the equally time step size. MLE estimators include the mean-reverting rate α, volatility of volatility β, and long-run mean m in the following:

$$\alpha = \frac{1}{\Delta_t}\left[1 - \frac{(\sum_{t=2}^{N} Y_t)(\sum_{t=1}^{N-1} Y_t) - (N-1)(\sum_{t=1}^{N-1} Y_t Y_{t+1})}{(\sum_{t=1}^{N-1} Y_t)^2 - (N-1)(\sum_{t=1}^{N-1} Y_t^2)}\right] \tag{2. 11}$$

$$\beta = \sqrt{\frac{1}{(N-1)\Delta_t}\sum_{t=1}^{N-1}[Y_{t+1} - (\alpha m \Delta_t + (1-\alpha\Delta_t)Y_t)]^2} \tag{2.12}$$

$$m = \frac{-1}{\alpha\Delta_t}\left[\frac{(\sum_{t=2}^{N} Y_t)(\sum_{t=1}^{N-1} Y_t^2) - (\sum_{t=1}^{N-1} Y_t)(\sum_{t=1}^{N-1} Y_t Y_{t+1})}{(\sum_{t=1}^{N-1} Y_t)^2 - (N-1)(\sum_{t=1}^{N-1} Y_t^2)}\right] \tag{2.13}$$

Barucci and Mancino [1] proposed an alternative multivariate Fourier estimation method for the full single-factor stochastic volatility model. Since our interest only concerns the volatility dynamics itself, rather than the joint dynamics of the underlying asset prices and their volatilities, the Fourier transform method with MLE is sufficient for our estimation purpose.

2.2.2 Stage Two: Option Pricing by Standard Error Reduction

This stage concerns numerical and technical methods for fast option pricing by

Monte Carlo simulation. The goal is reduce standard errors by (1) reducing the variance and (2) increasing the total number of simulation. This standard error reduction problem has been studied in [14] for GPU accelerated martingale control variate (MCV).

Reduce Variance by Martingale Control Variate

Fouque and Han [11] proposed a numerical algorithm termed the martingale control variate for variance reduction. They constructed a synthetic delta-hedging portfolio as a martingale control to eliminate market risk associated with the realized discounted payoff. Given the pricing system (2.5-8), a European option price can be represented as:

$$P(0, S_0, Y_0) = E^*[e^{-rT} H(S_T) | S_0, Y_0]\,, \qquad (2.14)$$

where $H(\cdot)$ is the payoff function and T is the maturity. The martingale control variate estimator is given below:

$$\frac{1}{N}\sum_{i=1}^{N}\left[e^{-rT}H\left(S_T^{(i)}\right) - M_0^{(i)}(P_{BS})\right],$$

where the super script i denotes the i-th sample and N denotes the sample size of Monte Carlo simulations.

The martingale control is defined as a stochastic integral by

$$M_0(P_{BS}) = \int_0^T e^{-rs}\frac{\partial P_{BS}}{\partial x}(s, S_s; \overline{\sigma})\sigma_s S_s dW_{1s}^*,$$

where P_{BS} denotes the classical Black-Scholes formula given by a homogenized constant volatility $\overline{\sigma}$. See [10] for detailed derivation and its error analysis.

Enlarge Simulation Number by GPU Parallel Computing

First manufactured by Nvidia in 1999, a graphics processing unit (GPU) has a highly paralleled hardware structure, which is designed for computer graphics rendering. A modern GPU has been designed to accelerate computations for scientific, engineering, and financial applications. GPU appeals to Monte Carlo method because massive parallelism can be exploited. It provides an alternative to largely increase the total number of simulation so that the standard error of any Monte Carlo estimator can be effectively reduced by such hardware device-GPU. In [18], GPU computing with financial applications under the Matlab computational environment is conducted, while in [14] advanced examples on computational finance are explored under CUDA (Compute Unified Device Architecture), a low-end programming language. In this paper, we employed CUDA for programming implementation.

Numerical Performance of Standard Error Reduction

Table 2.1 is depicted from [14]. Numerical results exhibit that the standard error reduction method is able to improve the accuracy and speed up the process simultaneously.

Table 2.1 Results of option pricing by Monte Carlo methods under a single-factor stochastic volatility model

		GPU	CPU	Speed up
Basic Monte Carlo (BMC)	Expectation	17.1324	16.8473	
	Standard Error	0.0328	0.0305	
	Computing Time	0.162 (s)	40.47 (s)	**X249**
MCV	Expectation	16.8030	16.8072	
	Standard Error	0.0037	0.0037	
	Computing Time	0.454 (s)	71.54 (s)	**X157**
Accuracy	Variance Reduction Ratio	**X76**	**X65**	

When the total effect of variance reduction on GPU acceleration is defined as the execution time multiplied by the square of standard error, we shall see that the combination of GPU and MCV versus the combination of CPU and BMC results in a total reduction of 6057 times.

2.3 Model Calibration to Implied Volatility Surface: An Optimization Problem

Under a risk-neutral probability measure, the parameter set of the pricing dynamics includes $\alpha, \beta, \widetilde{m},$ and ρ. The first two parameters α and β are estimated under the historical probability measure from observed spot prices. Dimension separation is the key feature in the first stage. Other model parameters $\widetilde{m},$ and ρ need to be regressed out from solving a nonlinear optimization problem defined below in two dimensions, rather than four dimensions.

From practical viewpoint, one might be more interested in solving this nonlinear optimization problem maturity by maturity. Despite this action causes inconsistency on the original pricing model, one is able to find more information on how long-run mean level and the correlation evolve through time in the future. That is to say, for each maturity T, a set of time dependent parameters $\Theta = \{\widetilde{m}(\mathrm{T}),\ \rho(T)\}$ are obtained.

Hence for each maturity, we solve a nonlinear minimization problem by minimizing the mean square error (MSE) between the implied volatilities calculated from the option prices under SVM and those observed implied volatilities from the option market. A low-dimensional optimization (minimization) problem is defined by

$$\min_{\Theta} \frac{1}{N} \sum_{i=1}^{N} \left[Imp\_Volatiltiy_{Model}^{(i)}(\Theta) - Imp\_Volatiltiy_{Market}^{(i)} \right]^2 \qquad (2.15)$$

where $(Imp\_Volatiltiy_{Model}^{(i)}(\Theta))$ denotes the implied volatility calculated from a SVM model with parameter set Θ, and $(Imp\_Volatiltiy_{Market}^{(i)})$ denotes the implied volatility calculated from the option market data.

Section 3: Generalized Volatility Models

The two-stage Monte Carlo calibration method can be extended to various different kinds of volatility models. For illustration, multi-factor SVM and a hybrid model are discussed here. A potential identification problem, i.e. co-linearity between model parameters, exists on a class of multi-factor exp-OU type SVM. Empirical calibration results and comparisons are made in next section.

3.1 Multi-Factor Stochastic Volatility Model

The popularity of electronic trading drastically enriches the spectrum of data frequencies. A data scientist may need to model low, medium, and/or high frequencies of the spot price in a monthly, daily, and/or minutely basis, respectively. A multi-factor exp-OU SVM can take separated-frequency data into account. The time scale in each volatility factor corresponds to a data frequency. For example, a two-factor model may correspond to intraday and daily data. A three-factor model may take an additional weekly data into account.

3.1.1 Two-Factor Exp-OU SVM

Adding a new stochastic volatility process $\boldsymbol{Y_{2t}}$ from different time-scale data[2] into the one-factor exp-OU model defined in equations (2.5-8) under a risk-neutral probability measure induces the price dynamics ass

$$\frac{dS_t}{S_t} = rdt + \sigma_t dW_{0t}^* \tag{3.1}$$

$$\sigma_t = e^{\frac{Y_{1t}+Y_{2t}}{2}} \tag{3.2}$$

$$dY_{1t} = \boldsymbol{\alpha_1}(\widetilde{m_1} - Y_{1t})dt + \boldsymbol{\beta_1} dW_{1t}^* \tag{3.3}$$

$$dY_{2t} = \boldsymbol{\alpha_2}(\widetilde{m_2} - Y_{2t})dt + \boldsymbol{\beta_2} dW_{2t}^* \tag{3.4}$$

$$\langle W_0^*, W_1^* \rangle_t = \boldsymbol{\rho_1} t \tag{3.5}$$

$$\langle W_0^*, W_2^* \rangle_t = \boldsymbol{\rho_2} t \tag{3.6}$$

Note we assume that $\langle W_1^*, W_2^* \rangle_t = 0$ for simplicity. The driving volatility processes Y_{1t} and Y_{2t} can be modeled by two separated data frequencies.

3.1.2 Two-Stage Monte Carlo Calibration

An identification problem in this two-factor model can be easily seen below. By linear transformation, the centered driving volatility processes $\widetilde{Y_{1t}} = Y_{1t} - \widetilde{m_1}$ and $\widetilde{Y_{2t}} = Y_{2t} - \widetilde{m_2}$ are governed by

$$d\widetilde{Y_{1t}} = -\boldsymbol{\alpha_1}\widetilde{Y_{1t}}dt + \boldsymbol{\beta_1} dW_{1t}{}^* \tag{3.7}$$

$$d\widetilde{Y_{2t}} = -\boldsymbol{\alpha_2}\widetilde{Y_{2t}}dt + \boldsymbol{\beta_2} dW_{2t}{}^* \tag{3.8}$$

so that the volatility σ_t can be expressed by

$$\begin{aligned} \sigma_t &= e^{\frac{Y_{1t}+Y_{2t}}{2}} \\ &= exp\left(\frac{\widetilde{Y_{1t}} + \widetilde{Y_{2t}} + (\widetilde{m_1 + m_2})}{2}\right) \end{aligned} \tag{3.9}$$

The sum $\widetilde{m_1} + \widetilde{m_2}$ can be identified throughout the calibration but not for each individual unless more information, for example VIX term structure, is given. For our purpose of model calibration to implied volatility surfaces, this identification problem does help to drop one parameter off in the pricing dynamics such that the optimization problem is reformatted by:

[2] For example, Y_{1t} stands for the stochastic volatility process from intraday data, comparing with Y_{2t} that stands for the stochastic volatility process from daily data.

$$\min_{(\widetilde{m}_1+\widetilde{m}_2),\rho_1,\rho_2} \frac{1}{N}\sum_{i=1}^{N}\begin{bmatrix} Imp\_Volatiltiy_{Model}^{(i)}((\widetilde{m}_1+\widetilde{m}_2),\rho_1,\rho_2) \\ -Imp\_Volatiltiy_{Market}^{(i)} \end{bmatrix}^2 \tag{3. 10}$$

3.1.3 Three-Factor Exp-OU SVM

It is sensible to build up a three-factor exp-OU model by incorporating the identification effect as in the following

$$\frac{dS_t}{S_t} = rdt + \sigma_t dW_{0t}^* \tag{3. 11}$$

$$\sigma_t = exp\left(\frac{\widetilde{Y_{1t}} + \widetilde{Y_{2t}} + \widetilde{Y_{3t}} + (\widetilde{m}_1 + \widetilde{m_2} + \widetilde{m_3})}{2}\right) \tag{3. 12}$$

$$d\widetilde{Y_{1t}} = -\boldsymbol{\alpha_1}\widetilde{Y_{1t}}dt + \boldsymbol{\beta_1}dW_{1t}{}^* \tag{3. 13}$$

$$d\widetilde{Y_{2t}} = -\boldsymbol{\alpha_2}\widetilde{Y_{2t}}dt + \boldsymbol{\beta_2}dW_{2t}{}^* \tag{3. 14}$$

$$d\widetilde{Y_{3t}} = -\boldsymbol{\alpha_3}\widetilde{Y_{3t}}dt + \boldsymbol{\beta_3}dW_{3t}{}^* \tag{3. 15}$$

$$\langle W_0^*, W_1^*\rangle_t = \boldsymbol{\rho_1}t \tag{3. 16}$$

$$\langle W_0^*, W_2^*\rangle_t = \boldsymbol{\rho_2}t \tag{3. 17}$$

$$\langle W_0^*, W_3^*\rangle_t = \boldsymbol{\rho_3}t \tag{3. 18}$$

Note we assume that $\langle W_1^*, W_2^*\rangle_t = \langle W_1^*, W_3^*\rangle_t = \langle W_2^*, W_3^*\rangle_t = 0$ for simplicity. One can think of that the additional driving volatility process $\widetilde{Y_{3t}}$ takes a low-frequency data (weekly data) into account.

3.2 Hybrid Model

The idea of the hybrid model came from FX options dealing with the "sticky-delta rule" and "sticky-strike rule," see [5]. The hybrid model can be viewed as an extension as a combination of the local volatility model [9,8] and the stochastic volatility model. Under each hybrid model, its calibration procedure can be different from others. Jex et al. [21] combined the Heston model and the local volatility model and implemented a two-dimensional trinomial tree for calibrating the FX American binary options. Another popular procedure is to solve the forward Kolmogorov PDE, see [23,27,28], by finite difference methods. On the other hand, Lee et al. [22] applied Monte Carlo algorithm with control variate to obtain a solution to the Kolmogorov PDE. Choi et al. [6] applied the perturbation method to approximate option prices under a hybrid model – the stochastic elasticity of variance. In this paper, we apply the two-stage Monte Carlo calibration method proposed in Section 2 for a class of hybrid models, which combine the local volatility model and a one-factor exp-OU

SVM as follows:

$$\frac{dS_t}{S_t} = \mu dt + \sigma_t dW_{0t}, \ \ S_0 = x \qquad (3.19)$$

$$\sigma_t = {S_t}^{\gamma-1} e^{Y_t/2} \qquad (3.20)$$

$$dY_t = \alpha(m - Y_t)dt + \beta dW_{1t}, \ \ Y_0 = m \qquad (3.21)$$

where $d\langle W_0, W_1\rangle_t = \rho dt$, ρ is the correlation coefficient, Y_t is the driving volatility process, ${S_t}^{\gamma-1}$ represents the local volatility component of the model. By setting $\gamma = 1$, the hybrid model degenerates to an one-factor exp-OU SVM. One can observe that there are more parameters in our model, including β, γ, α, β_2, and ρ.

When the two-stage Monte Carlo calibration method is employed, model parameters (α, β) on the first stage one don't admit closed-form solutions, but they can be solved numerically. On the second stage, we additionally estimate implied elasticity parameter γ under a risk-neutral probability measure.

$$\frac{dS_t}{S_t} = rdt + \sigma_t dW_{0t}^*, \ \ S_0 = x \qquad (3.25)$$

$$\sigma_t = {S_t}^{\gamma-1} e^{Y_t/2} \qquad (3.26)$$

$$dY_t = \alpha(\tilde{m} - Y_t)dt + \beta_2 dW_{1t}^*, \ \ Y_0 = \tilde{m} \qquad (3.27)$$

$$d\langle W_0^*, W_1^*\rangle_t = \rho dt \qquad (3.28)$$

So that the optimization problem that need to be solved in the second stage is formulated as below:

$$\min_{\tilde{m},\rho,\gamma} \frac{1}{N}\sum_{i=1}^{N}\left[\frac{Price_{Model}^{(i)}(T, K_i, \tilde{m}, \rho, \gamma) - Price_{Market}^{(i)}(T, K_i)}{Price_{Market}^{(i)}(T, K_i)}\right]^2 \qquad (3.29)$$

where N is the number of strike prices, $Price_{Model}^{(i)}(T, K_i, \tilde{m}, \rho, \gamma)$ is the model price at the i-th strike price and at time T, $Price_{Market}^{(i)}(T, K_i)$ is the real market price at the i-th strike price and at time T.

Section 4: Empirical Studies

We apply the proposed two-stage Monte Carlo calibration method to a dataset and examine fitting performance from various volatility models. The dataset consists of S&P 500 index prices and its option prices. Data resource is from Bloomberg. The index data contains three different frequencies including one-minute intraday data, daily data for two years, and weekly data for ten years, while options are daily observed. Tested data period is from 2013/01/03 to 2013/02/15; the dataset is summarized below.

Table 4.1 Dataset Summary

Frequency	Period	Data Number of a Period	Abbreviated Code
1 Minute	1 Day	405	1m_1d
1 Day	2 Years	500	1d_2y
1 Week	10 Years	520	1w_10y

4.1 Empirical Results from Stage One Estimation

On the first stage, some volatility model parameters can be estimated from index prices under different data frequencies by means of the Fourier transform method and MLE. To be more precise, on any tested date, historical data of one day, two years, and ten years are analyzed. Three corresponding instantaneous volatilities are estimated from the Fourier transform estimator (2.10), then exp-OU model parameters are estimated from MLE estimators (2.11-13). Han [13] applied this method to analyze TAIEX (Taiwan Stock Exchange Capitalization Weighted Stock Index) under similar data frequencies. Notice that for the hybrid model, we need to employ numerical methods to solve the MLE by using daily data because its MLE doesn't admit a closed-form solution.

Take the one-factor exp-OU SVM and the hybrid volatility model as benchmarks. Since model parameters of the mean-reverting rate and the volatility of volatility are crucial for dimensional separation of the full parameter set. Table 4.1 illustrates estimation results by the first stage method. It is observed that both mean-reverting rate α and vol-vol β are proportional to data sampling frequency. That is, higher the data frequency, larger the parameter value. Note that those parameters are well separated, especially the rate of mean reversion. These estimations show a strong evidence of multiple time scales on volatilities.

Table 4.1 Mean and Standard Deviation of $\boldsymbol{\alpha}$ and β

	$\overline{\alpha}$	σ_α	$\overline{\beta}$	σ_β
1m_1d	10173.49	2359.521	138.6426	12.61022
1d_2y	10.67259	0.521576	4.905778	0.123721
1w_10y	2.379328	0.101517	2.135863	0.04913
Hybrid (1d_2y)	10.70784	0.55554	4.910771	0.127643

4.2 Empirical Results from Stage Two Calibration

On this second stage, five different volatility models are compared. They include one-factor exp-OU SVM (1d_2y), two-factor exp-OU SVM (1m_1d, 1d_2y), two-factor exp-OU SVM (1d_2y, 1w_10y), three-factor exp-OU SVM (1m_1d, 1d_2y, 1w_10y), hybrid volatility model (1d_2y), abbreviated by 1 factor, 2 factors, 2factors_Low, 3 factors, and Hybrid in Table 4.2. On any date in the tested period, the total MSE associated with each model is used to compare model-fitting performance. In addition, regression methods derived from time-separated SVMs by perturbation approximation in [12] are incorporated for comparisons. Their regression equations are listed in Equations (4.1-3) and each regression method is abbreviated as LMMR_Fast, LM_Slow, and LMMR_Combine in Table 4.2, respectivelty.

$$I_{fast}^{surface} = A_{fast} + B_{fast}\times(LMMR), \tag{4.1}$$

$$I_{slow}^{surface} = A_{slow} + B_{slow}\times(LM) + C_{slow}\times(T-t), \tag{4.2}$$

$$I_{combine}^{surface} = A + B\times(LM) + C\times(LMMR) + D\times(T-t), \tag{4.3}$$

LMMR denotes "Log-Moneyness to Maturity Ratio" defined by $\log(\mathrm{K}/\mathrm{S})/(\mathrm{T}-\mathrm{t})$ and LM denotes "Log-Moneyness" defined by $\log(\mathrm{K}/\mathrm{S})$.

From Table 4.2, it is observed that perturbation methods induce larger total MSEs in general, but their computational cost is negligible compared to other volatility models, although occasionally 3 factors model is more accurate. Secondly, 2 factors model, i.e. two-factor exp-OU SVM (1m_1d, 1d_2y) performs superiorly to other volatility models. In comparison with two-factor exp-OU SVM (1m_1d, 1d_2y) and two-factor exp-OU SVM (1m_1d, 1w_10y), we document that the high frequency data - one minute intraday is more significant than low frequency data – one week for ten years on calibration. Note also that although the hybrid model performs just slightly worse than 2 factors model, it can be computed much efficiently than 2 factors model.

Table 4.2 Comparisons of Total Mean Square Error of implied volatility surface

	1 factor	2 factors	3 factors	2factors_ Low	Hybrid	LMMR_ Fast	LM_ Slow	LMMR_ Combine
20130104	2.36E-03	7.28E-04	7.26E-04	4.72E-02	7.49E-04	9.06E-04	7.80E-04	7.71E-04
20130107	2.26E-03	1.17E-03	1.23E-03	1.03E-01	1.38E-03	8.59E-03	9.54E-03	9.68E-03
20130108	1.25E-03	6.95E-04	7.62E-04	1.12E-01	9.22E-04	1.61E-02	1.90E-02	1.87E-02
20130109	1.89E-03	1.48E-03	1.64E-03	1.65E-01	2.21E-03	1.03E-02	1.39E-02	1.39E-02
20130110	1.46E-03	6.27E-04	6.64E-04	8.79E-02	6.29E-04	7.83E-03	1.06E-02	1.05E-02
20130111	6.90E-03	1.02E-03	8.28E-04	9.97E-02	1.21E-03	9.91E-02	8.88E-02	1.21E-01
20130114	4.26E-03	1.57E-03	6.19E-04	1.25E-01	1.77E-03	6.80E-02	6.33E-02	9.17E-02
20130115	3.19E-03	6.40E-04	6.16E-04	1.50E-01	1.46E-03	1.05E-03	2.54E-03	2.79E-03
20130116	3.03E-02	1.08E-03	3.08E-03	1.50E-01	5.89E-02	2.94E+00	3.03E+00	2.84E+00
20130117	6.81E-02	1.08E-04	1.18E-04	1.50E-01	4.85E-04	8.99E+00	8.86E+00	8.85E+00
20130118	5.77E-04	2.72E-04	3.23E-04	1.24E-01	2.74E-04	2.26E-03	8.66E-03	5.32E-04
20130122	4.49E-04	1.50E-04	3.79E-04	1.17E-01	3.50E-04	2.19E-03	6.65E-03	6.62E-03
20130123	1.35E-03	2.88E-04	4.73E-04	1.10E-01	5.13E-04	8.44E-04	3.13E-03	3.24E-03
20130124	1.55E-03	3.14E-04	4.78E-04	1.38E-01	1.57E-03	1.08E-01	9.89E-02	9.96E-02
20130125	1.03E-03	6.70E-04	7.56E-04	1.08E-01	6.61E-03	1.23E-03	2.28E-03	2.28E-03
20130128	8.17E-04	4.43E-04	4.53E-04	1.27E-01	4.13E-03	9.82E-02	8.99E-02	8.95E-02
20130129	1.10E-03	4.32E-04	4.75E-04	1.11E-01	4.78E-04	1.56E-03	3.24E-03	3.48E-03
20130130	1.19E-02	2.06E-03	2.13E-03	1.53E-01	2.58E-03	2.86E-01	2.70E-01	2.63E-01
20130131	5.37E-04	3.62E-04	3.78E-04	1.43E-01	3.60E-03	4.81E-02	4.49E-02	4.25E-02
20130201	4.41E-04	9.90E-05	9.91E-05	1.08E-01	2.35E-04	4.02E-03	7.27E-03	7.09E-03
20130204	6.08E-03	1.04E-03	2.25E-03	1.60E-01	1.24E-03	1.53E-01	1.41E-01	1.42E-01
20130205	1.22E-03	7.35E-04	4.63E-04	1.26E-01	9.67E-04	6.35E-02	5.68E-02	5.62E-02
20130206	2.80E-04	1.29E-04	1.91E-04	1.17E-01	6.76E-04	8.99E-03	1.34E-02	1.30E-02
20130207	1.64E-03	1.75E-04	3.34E-04	1.17E-01	1.16E-03	4.76E-03	1.08E-02	1.09E-02
20130208	2.84E-03	2.83E-04	4.24E-04	9.47E-02	3.13E-04	2.31E-02	3.03E-02	2.90E-02
20130211	3.08E-03	6.73E-04	2.84E-04	1.08E-01	7.07E-04	6.55E-02	5.94E-02	5.76E-02
20130212	1.31E-02	1.28E-02	1.72E-03	1.23E-01	1.87E-02	1.76E+00	1.74E+00	1.68E+00
20130213	1.99E-01	4.93E-02	6.39E-03	1.47E-01	1.01E-01	3.43E+00	3.44E+00	3.37E+00
20130214	3.62E-03	6.45E-04	7.73E-04	1.17E-01	6.22E-03	6.69E+00	6.66E+00	6.58E+00
20130215	4.42E-04	1.84E-04	5.72E-04	9.72E-02	6.31E-04	2.70E-02	2.21E-02	2.27E-02

In Figure 4.1, maturity-by-maturity fitting results are demonstrated on each subplot for comparison with one-factor exp-OU SVM (1d_2y) shown in green cross, two-factor exp-OU SVM(1m_1d, 1d_2y) shown in red cross, and actual market implied volatilities shown in blue circle.

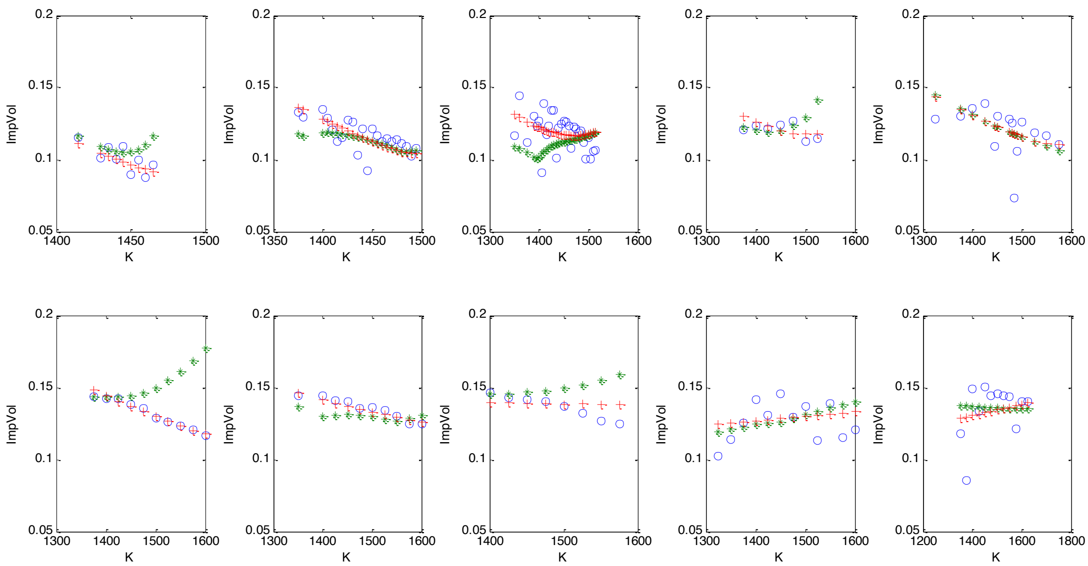

Figure 4.1 The fitting of implied volatility curves

Section 6: Conclusion

This paper shed a light on the use of Monte Carlo simulation to solve for the calibration problem of implied volatility surface. Dimension separation and standard error reduction constitute the two-stage procedure. The first stage aims to reduce dimensions in the optimization problem by utilizing the Fourier transform representation of the volatility dynamics. The second stage provides a high performance computing paradigm for option pricing by standard error reduction, which consists of a variance reduction algorithm and GPU parallel computing.

This two-stage Monte Carlo calibration method is applied to empirically validate various volatility models such as hybrid models and multiscale stochastic volatility models. Using the total MSE as a criterion, our empirical studies convey that two-factor exp-OU SVM is stable and superior to other volatility models, and the high frequency data representing a short time scale is documented as a key facor from the modeling perspective.

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