Systemic risk with jump diffusion processes

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Outline

- Systemic risk with jump diffusion processes
- Maximum principle of jump diffusion processes
- Systemic risk with a game feature

Motivation

- Flocking.
  S. Ha, K. Lee and D. Lévy. Emergence of time-asymptotic flocking in a stochastic Cucker-Smale system. Commun. Math. Sci. Vol.7,No.2, 453-469, (2009).
  F. Cucker, S. Smale. Emergent behavior in flocks, IEEE Trans. Automat. Control, 52, 852-862, (2007).
- Systemic risk.
  J.-P. Fouque, L. Sun. Systemic risk illustrated. (2012)
  J. Garnier, G. Papanicolaou and T. Yang. Diversification in financial networks may increase systemic risk. (2012)
  R. Carmona,J.-P. Fouque and L. Sun. Mean field games and systemic risk. (2014)

Stochastic Cucker-Smale system (SCS)

Let $\left(x_t^{(i)}, v_t^{(i)}\right) \in R^2$ be the position and veloctiy of particles for $i = 1, ..., N$ with the initial data $\left(x_0^{(i)}, v_0^{(i)}\right)$.

$$\begin{aligned} dx_t^{(i)} &= v_t^{(i)} dt, \\ dv_t^{(i)} &= \frac{\alpha}{N} \sum_{j=1}^{N} \psi\left(x_t^{(j)}, x_t^{(i)}\right)\left(v_t^{(j)} - v_t^{(i)}\right) dt + dL_t^{(i)}, 1 \leq i \leq N, \end{aligned}$$

where the noise term $L_t^{(i)}$ is a Lévy process on R generated by (a, σ, ν). We show that the SCS model driven by Lévy processes will satisfies the following criteria, for $1 \leq i, j \leq N$,

$$\begin{aligned} \text{(velocity alignment)} \quad \lim_{t \to \infty} \left| \mathbb{E}\left(v_t^{(i)}\right) - \mathbb{E}\left(v_t^{(j)}\right) \right| &= 0, \\ \text{(group formation)} \quad \sup_{0 \leq t < \infty} \left| \mathbb{E}\left(x_t^{(i)}\right) - \mathbb{E}\left(x_t^{(j)}\right) \right| &< \infty. \end{aligned}$$

Outline

- Systemic risk with jump diffusion processes

Systemic risk

- Systemic risk for jump diffusion processes with double-exponential jump size

We consider the log-monetary reserves of N banks possibly lending and borrowing to each other.

The model is

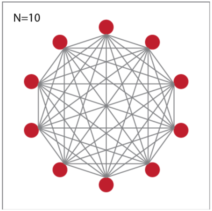
$$dX_t^i = \frac{a}{N} \sum_{j=1}^{N} \left(X_t^j - X_t^i \right) dt + dL_t^i, \quad 1 \leq i \leq N,$$

where $L_t^i = \sigma W_t^i + \sum_{j=1}^{N_t^i} \xi_j$, ξ_j has distribution $f(y;\theta) = \frac{\theta}{2} e^{-|y|\theta}, \theta > 0$ and N_t^i is a Poisson process with rate λ.

- Log-monetary reserves of bank i
  - Borrowing from the bank j if $X_t^j > X_t^i$
  - Lending to the bank j if $X_t^j < X_t^i$
  - rate of borrowing/lending

$$
\begin{aligned}
dX_t^i &= \frac{a}{N}\sum_{j=1}^{N}\left(X_t^j - X_t^i\right)dt + dL_t^i \\
&= a\left(\bar{X}_t - X_t^i\right)dt + dL_t^i .
\end{aligned}
$$

  - $a > 0$: borrowing if $\bar{X}_t > X_t^i$, lending if $\bar{X}_t < X_t^i$



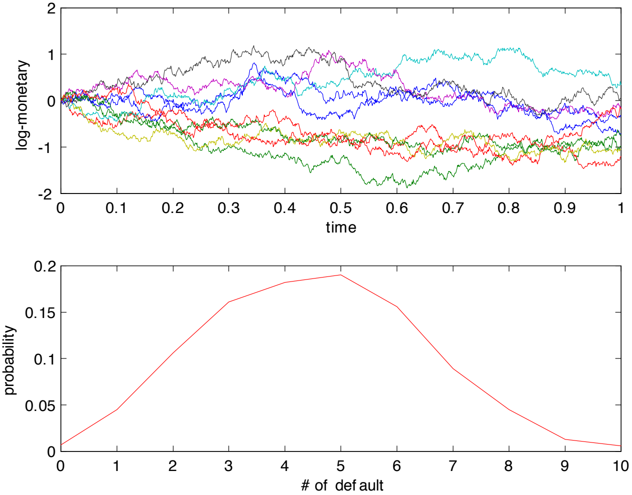
We now investigate how many banks have reached the default level $D < 0$ before $t = 1$.

Define $\{\text{default event}\} = \left\{\min_{0\leq t\leq 1} X_t^i \leq D, 1 \leq i \leq N\right\}$ and $K \equiv \{\# \text{ of default}\}$.

We are now interested in the loss distribution

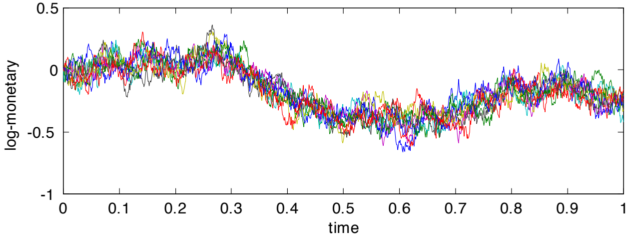
$$p = \mathbb{P}\left(K = k\right), k = 0, 1..., N.$$

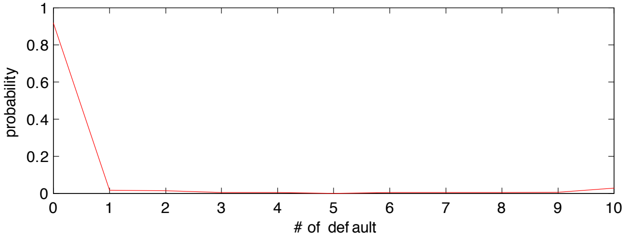
Simulations for **small** rate of borrowing/lending



$a = 1, N = 10, D = -0.7, \sigma = 1$ *and* $t = 1$.

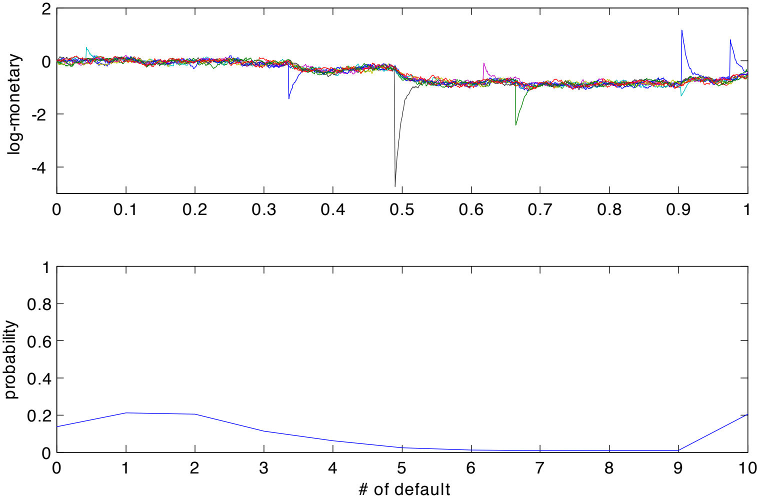
Simulations for **large** rate of borrowing/lending





$a = 100$, $N = 10$, $D = -0.7$, $\sigma = 1$ *and* $t = 1$.

large rate of borrowing/lending with jumps



$a = 100, N = 10, D = -0.7, \sigma = 1, \theta = 1, \lambda = 1$ and $t = 1$.

First passage time for the ensemble average

Systemic event :

$$\left\{ \min_{0 \le t \le T} \bar{X}_t \le D \right\}$$

We now focus on the event where the ensemble average $\bar{X}_t = \frac{1}{N} \sum_{i=1}^{N} L_t^i$ reaches the default level $D < 0$.
If $L_t^i = \sigma W_t^i$, we have

$$\begin{aligned}
\mathbb{P}\left(\min_{0 \le t \le T} \bar{X}_t \le D \right) &= \mathbb{P}\left(\min_{0 \le t \le T} \frac{1}{N} \sum_{i=1}^{N} W_t^i \le D \right) \\
&= \mathbb{P}\left(\min_{0 \le t \le T} \tilde{W}_t \le \frac{D\sqrt{N}}{\sigma} \right) \\
&= 2\Phi\left(\frac{D\sqrt{N}}{\sigma T} \right),
\end{aligned}$$

where $\tilde{W}_t$ is a standard Brownian motion.

First passage time for the ensemble average

If $L_t^i = \sigma W_t^i + \sum_{j=1}^{N_t^i} \xi_j$ and ξ_i has distribution $f(y;\theta) = \frac{\theta}{2} e^{-|y|\theta}, \theta > 0$, then $\mathbb{P}\left(\min_{0 \le t \le T} \frac{1}{N} \sum_{i=1}^{N} L_t^i \le D\right)$ can be evaluated by inversion Laplace transform.

The moment generating function of L_t is given by

$$\mathbb{E}\left[e^{zL_t}\right] = \exp\left\{G(z)\, t\right\},$$

where $G(x) \equiv \frac{1}{2}\sigma^2 x^2 + \lambda\left(\frac{\theta^2}{\theta^2 - x^2} - 1\right)$.

Let $\tau_D = \inf\left\{t \ge 0; L_t \le D\right\}$, where $D < 0$.

Theorem (S. Kou and H. Wang, 2003)

For any $s \in (0, \infty)$. Let β_1 and β_2 be the only two positive roots of the equation

$$s = G(\beta),$$

where $0 < \beta_1 < \theta < \beta_2 < \infty$. Then the Laplace transform of τ_D is given by

$$\mathbb{E}\left[e^{-s\tau_D}\right] = \frac{\theta - \beta_1}{\theta} \frac{\beta_2}{\beta_2 - \beta_1} e^{D\beta_1} + \frac{\beta_2 - \theta}{\theta} \frac{\beta_1}{\beta_2 - \beta_1} e^{D\beta_2}.$$

By the numerical inversion, we can now compute the probability

$$\begin{aligned} & \mathbb{P}\left(\min_{0\le t\le T} \frac{1}{N}\sum_{i=1}^{N} L_t^i \le D\right) \\ = \; & \mathbb{P}\left(\min_{0\le t\le T}\left(\frac{\sigma}{\sqrt{N}}\tilde{W}_t + \sum_{j=1}^{\tilde{N}_t}\tilde{\xi}_j\right) \le D\right), \end{aligned}$$

where $\tilde{W}_t$ is a standard Brownian motion, $\tilde{N}_t$ is a Poisson processes with rate $N\lambda$ and $\tilde{\xi}_j$ has distribution $f(y;\theta) = \frac{\theta/N}{2} e^{-|y|\theta/N}, \theta > 0$.

Outline

- Maximum principle of jump diffusion processes

Maximum principle for jump diffusion processes

- State process

$$\begin{aligned} dX_t &= b(t, X_t, \alpha_t)\, dt + \sigma(t, X_t, \alpha_t)\, dW_t \\ &\quad + \int_R \gamma(t, X_{t^-}, \alpha_{t^-}, z)\, \tilde{N}(dt, dz), \end{aligned}$$

  where $\tilde{N}(dt, dz) = N(dt, dz) - \nu(dz)\, dt$.

- The performance criterion

$$J(\alpha) = E\left[\int_0^T f(t, X_t, \alpha_t)\, dt + g(X_T)\right],$$

  where $T < \infty$ is deterministic, f is continuous and g is concave.

- Consider the problem to find an admissible $\alpha^* \in A$ such that

$$J(\alpha^*) = \sup_{\alpha \in A} J(\alpha)$$

Hamiltonian

Define the Hamiltonian $H : [0, T] \times R \times U \times R \times R \times R \to R$ by

$$H(t,x,\alpha,p,q,r) = f(t,x,\alpha) + b(t,x,\alpha)\,p + \sigma(t,x,\alpha)\,q + \int_R \gamma(t,x,\alpha,z)\,r(t,z)\,\nu(dz),$$

where R is the set of functions $r : [0, T] \times R \to R$ such that the integrals converge. p, q and r satisfy the BSDE

$$\begin{aligned} dp_t &= -H_x(t,x,\alpha,p,q,r)\,dt + q\,dW_t + \int_R r\left(t^-, z\right) \tilde{N}(dt,dz), \\ p_T &= g'(X_T) \end{aligned}$$

A sufficient maximum principle

Theorem

Let $\alpha \in A$ with corresponding solution $X^* = X^{(\alpha^*)}$ and suppose there exists a solution $(p_t^*, q_t^*, r^*(t,z))$ of the corresponding adjoint equation. Moreover, suppose that

$$H(t, X_t^*, \alpha_t^*, p_t^*, q_t^*, r^*(t,\cdot)) = \sup_{u \in U} H(t, X_t^*, \alpha, p_t^*, q_t^*, r^*(t,\cdot))$$

and

$$H(x) := \max_{\alpha \in U} H(t, x, \alpha, p_t^*, q_t^*, r^*(t,\cdot))$$

exists and is a concave function of x, for all $t \in [0, T]$.
Then α^* is a optimal control.

Example : The stochastic linear regulator problem

Solve the stochastic control problem

$$J(x) = \inf_{\alpha} E^x \left[\int_0^T \left(X_t^2 + \theta \alpha_t^2 \right) dt + \lambda X_T^2 \right],$$

where

$$dX_t = \alpha_t dt + \sigma dW_t + \int_R z\tilde{N}(dt, dz), X_0 = x, \text{ and } T > 0 \text{ is a constant.}$$

We can solve this problem by using the stochastic maximum principle. Define the Hamiltonian

$$H(t, x, \alpha, p, q, r) = x^2 + \theta\alpha^2 + \alpha p + \sigma q + \int_R zr\left(t^-, z\right) \nu(dz)$$

The adjoint equation is

$$\begin{aligned} dp_t &= -2X_t dt + q_t dW_t + \int_R r\left(t^-, z\right) \tilde{N}(dt, dz); \quad t < T \\ p_T &= 2\lambda X_T. \end{aligned}$$

By imposing the first and second-order conditions, we see that $H\left(t,x,\alpha,p,q,r\right)$ is minimal for

$$\alpha=\alpha_t=\hat{\alpha}_t=-\frac{1}{2\theta}p_t.$$

To find a solution of the adjoint euqation, we consider the ansatz

$$p_t=h_tX_t,$$

where $h_t:R\rightarrow R$ is a deterministic function such that $h_T=2\lambda$. Note that $\alpha_t=-\frac{h_tX_t}{2\theta}$ and

$$dX_t=-\frac{h_tX_t}{2\theta}dt+\sigma dW_t+\int_R z\tilde{N}\left(dt,dz\right); \quad X_0=x.$$

Moreover, differentiate the ansatz

$$\begin{aligned} dp_t &= h_tdX_t+h_t^{'}X_tdt \\ &= \left[-\frac{h_t^2}{2\theta}+h_t^{'}\right]X_tdt+h_t\sigma dW_t+h_t\int_R z\tilde{N}\left(dt,dz\right). \end{aligned}$$

Hence, h_t is the solution of

$$h_t^{'} = \frac{h_t^2}{2\theta} - 2$$

and $h_T = 2\lambda, t < T$.The solution is then given by

$$h_t = 2\sqrt{\theta}\frac{1 + \beta e^{\frac{2t}{\sqrt{\theta}}}}{1 - \beta e^{\frac{2T}{\sqrt{\theta}}}},$$

where $\beta = \frac{\lambda - \sqrt{\theta}}{\lambda + \sqrt{\theta}} e^{-\frac{2T}{\sqrt{\theta}}}$. By using the stochastic maximum principle, we can conclude that

$$\alpha_t^* = -\frac{h_t X_t}{2\theta}$$

is the optimal control, $p_t = h_t X_t$ and $q_t = \sigma h_t$, $r\left(t^-, z\right) = h_t z$.

Outline

- Systemic risk with a game feature

Systemic risk with a game feature

- The dynamics with the central bank

$$dX_t^i = \alpha_t^i dt + dL_t^i, \tag{1}$$

where $dL_t^i = \sigma^i dW_t^i + \int_R \gamma^i(t^-, z) N^i(dt, dz)$, R is the set such that the integral converges, W_t^i, $i = 1, ..., N$ are independent Brownian motions and $\int_R \gamma^i(t^-, z) N^i(dt, dz)$ are independent jump processes with Poisson random measure $N^i(dt, dz)$ and of jump size $\gamma^i(t^-, z)$.

- Rewrite (1) as

$$dX_t^i = \left[\alpha_t^i dt + v_t^i\right] dt + d\tilde{L}_t^i,$$

where $v_t^i = \int_R \gamma^i(t^-, z) \nu^i(dz)$ and now $d\tilde{L}_t^i = \sigma^i dW_t^i + \int_R \gamma^i(t^-, z) \tilde{N}^i(dt, dz)$, $i = 1, ..., N$ are independent martingales. $\tilde{N}^i(dt, dz) = N^i(dt, dz) - \nu^i(dz) dt$, for $i = 1, ..., N$ are compensated Poisson random measures.

Control problem

Bank $i \in \{1, ..., N\}$ controls its rate of lending and borrowing (to a central bank) at time t by choosing the control α_t^i in order to minimize

$$J^i\left(\alpha^i, ..., \alpha^N\right) = E\left\{\int_0^T f_i\left(X_t, \alpha_t^i\right) dt + g_i\left(X_T^i\right)\right\}$$

with

$$\begin{aligned} f_i\left(x, \alpha^i\right) &= \frac{1}{2}\left(\alpha^i\right)^2 - q\alpha^i\left(\bar{x} - x^i\right) + \frac{\epsilon}{2}\left(\bar{x} - x^i\right)^2, \\ g_i(x) &= \frac{c}{2}\left(\bar{x} - x^i\right)^2, \end{aligned}$$

where $f_i(x, \alpha)$ is convex in (x, α) under the assumption $q^2 \leq \epsilon$.

Closed-Loop Equilibria

$\alpha = \alpha(t, x)$, feedback strategies

- Using the Pontryagin approach, the Hamiltonian for bank i is given by

$$
\begin{aligned}
& H^i\left(x, y^{i,1}, ..., y^{i,N}, \alpha^1(t,x), ..., \alpha^i_t, ..., \alpha^N(t,x)\right) \\
= \ & \sum_{k \neq i} \left[a\left(\bar{x} - x^k\right) + \alpha^k(t,x)\right] y^{i,k} \\
& + \left[a\left(\bar{x} - x^i\right) + \alpha^i\right] y^{i,i} \\
& + \frac{1}{2}\left(\alpha^i\right)^2 - q\alpha^i\left(\bar{x} - x^i\right) + \frac{\epsilon}{2}\left(\bar{x} - x^i\right)^2
\end{aligned}
$$

FBSDE

Minimizing H^i over α^i gives the choices :

$$\hat{\alpha}^i = -y^{i,i} + q\left(\bar{x} - x^i\right), \qquad i = 1, ..., N, \tag{2}$$

and we make the ansatz

$$Y_t^{i,j} = \left(\frac{1}{N} - \delta_{i,j}\right)\left[\eta_t\left(\bar{X}_t - X_t^i\right) + \varphi_t^i\right],$$

where η_t and φ_t^i are deterministic functions satisfying the terminal condition $\eta_T = c$ and $\varphi_T^i = 0$.With the choices (2) we get

$$\begin{aligned} \alpha^k &= \left[q + \eta_t\left(1 - \frac{1}{N}\right)\right]\left(\bar{x} - x^k\right) + \left(1 - \frac{1}{N}\right)\varphi_t^k, \\ \partial_{x^j}\alpha^k &= \left[q + \eta_t\left(1 - \frac{1}{N}\right)\right]\left(\frac{1}{N} - \delta_{k,j}\right). \end{aligned}$$

FBSDE

The forward equation is given by

$$\begin{aligned} dX_t^i &= \partial y^{i,i} H^i \left(X_t, Y_t^i, \alpha_t\right) dt + dL_t^i \\ &= \boldsymbol{\alpha}_t^i dt + dL_t^i \\ &= \left\{ \left[q + \eta_t \left(1 - \frac{1}{N}\right)\right] \left(\bar{x} - x^k\right) + \left(1 - \frac{1}{N}\right) \varphi_t^k \right\} dt + dL_t^i. \end{aligned}$$

The adjoint backward equation is given by

$$\begin{aligned} dY_t^{i,j} &= -\partial x^j H^i dt + \sum_{k=1}^N Z_t^{i,j,k} dW_t^k + \sum_{k=1}^N \int_R r^{i,j,k} \left(t^-, z\right) \tilde{N}^k \left(dt, dz\right) \\ &= \left(\frac{1}{N} - \delta_{i,j}\right) \left(\bar{X}_t - X_t^i\right) \left[q\eta_t + q^2 - \epsilon\right] dt \\ &\quad + \left(\frac{1}{N} - \delta_{i,j}\right) \left(a + q\right) \varphi_t^i dt \\ &\quad + \sum_{k=1}^N Z_t^{i,j,k} dW_t^k + \sum_{k=1}^N \int_R r^{i,j,k} \left(t^-, z\right) \tilde{N}^k \left(dt, dz\right). \end{aligned}$$

Using Ito's formula to differentiate the ansatz to obtain

$$
\begin{aligned}
dY_t^{i,j} &= d\left\{\left(\frac{1}{N}-\delta_{i,j}\right)\left[\eta_t\left(\bar{X}_t-X_t^i\right)+\varphi_t^i\right]\right\} \\
&= \left(\frac{1}{N}-\delta_{i,j}\right)\left[\dot{\eta}_t\left(\bar{X}_t-X_t^i\right)dt+\eta_t d\left(\bar{X}_t-X_t^i\right)+\dot{\varphi}_t^i dt\right] \\
&= \left(\frac{1}{N}-\delta_{i,j}\right)\left(\bar{X}_t-X_t^i\right)\left[\dot{\eta}_t-\eta_t\left(q+\left(1-\frac{1}{N}\right)\eta_t\right)\right]dt \\
&\quad +\left(\frac{1}{N}-\delta_{i,j}\right)\left[\dot{\varphi}_t^i+\eta_t\left(1-\frac{1}{N}\right)\left(\bar{\varphi}_t-\varphi_t^i\right)+\eta_t\left(\bar{v}_t-v_t^i\right)\right]dt \\
&\quad +\eta_t\left(\frac{1}{N}-\delta_{i,j}\right)\left(\frac{1}{N}\sum_{k=1}^{N}d\tilde{L}_t^k-d\tilde{L}_t^i\right)
\end{aligned}
$$

After some careful calculation, η_t must satisfy the scalar Riccati equation

$$\dot{\eta}_t = 2q\eta_t + \left(1 - \frac{1}{N}\right)\eta_t^2 - \left(\epsilon - q^2\right),$$

with the terminal condition $\eta_T = c$,and φ_t^i must satisfy the equation

$$\dot{\varphi}_t^i = \left[q + \eta_t\left(1 - \frac{1}{N}\right)\right]\varphi_t^i - \eta_t\left(\bar{v}_t - v_t^i\right), i = 1, ..., N,$$

with terminal condition $\varphi_T^i = 0$. Furthermore, we will also have $\bar{\varphi}_t = 0$.
The optimal strategies are then given by

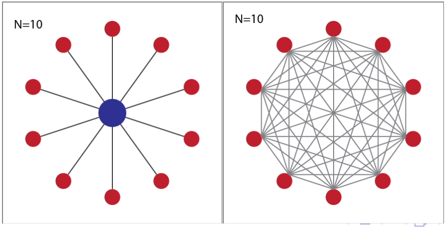
$$\alpha_t^i = \left[q + \left(1 - \frac{1}{N}\right)\eta_t\right]\left(\bar{X}_t - X_t^i\right) + \left(1 - \frac{1}{N}\right)\varphi_t^i, i = 1, ..., N.$$

Nash equailibrium

The dynamics with optimal strategies

$$
\begin{aligned}
dX_t^i &= \alpha_t^i dt + dL_t^i \\
&= \left[q + \left(1 - \frac{1}{N}\right)\eta_t \right] \left(\bar{X}_t - X_t^i\right) dt + \left[\left(1 - \frac{1}{N}\right) \varphi_t^i \right] dt + dL_t^i
\end{aligned}
$$

Under the Nash equailibrium, the system is operating as if banks were borrowing from and lending to each other at the rate $A_t \equiv q + \left(1 - \frac{1}{N}\right)\eta_t$ and the linear growth $B_t^i = \left(1 - \frac{1}{N}\right)\varphi_t^i$ contributed by the jumps. The net effect is liquidity and the central bank acts as a **clearing house**.



Systemic risk

The dynamics with optimal strategies are

$$dX_t^i = A_t \left(\bar{X}_t - X_t^i\right) dt + B_t^i dt + dL_t^i.$$

The ensemble average is

$$d\bar{X}_t = \frac{1}{N} \sum_{i=1}^{N} dL_t^i$$

The systemic risk probability stays the same even in the control problem with jumps!

$$\mathbb{P}\left(\min_{0 \le t \le T} \bar{X}_t \le D\right) = \mathbb{P}\left(\min_{0 \le t \le T} \frac{1}{N} \sum_{i=1}^{N} L_t^i \le D\right)$$

Example (Compound Poisson processes with distinct jump rates)

Let $L_t^i = \sigma W_t^i + \sum_{j=1}^{N_t^i} \xi_j$, ξ_j has distribution $f(y;\theta) = \frac{\theta}{2} e^{-|y-\mu|\theta}, \theta > 0$ and N_t^i is a Poisson process with rate λ^i. Then, for i=1,...,N, $v_t^i = \lambda^i E(\xi_j) = \lambda^i \mu$. Assuming that $\bar{v}_t = \frac{1}{N}\sum_{i=1}^{N} \lambda^i \mu = 0$. Then, under the Nash equailibrium, the dynamics now become

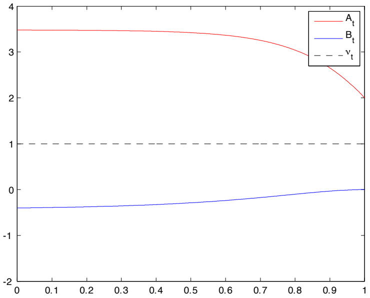
$$dX_t^i = \left[q + \left(1 - \frac{1}{N}\right)\eta_t \right] \left(\bar{X}_t - X_t^i\right) dt + \left(1 - \frac{1}{N}\right) \varphi_t^i dt + dL_t^i,$$

where η_t and φ_t^i must satisfy the following equations

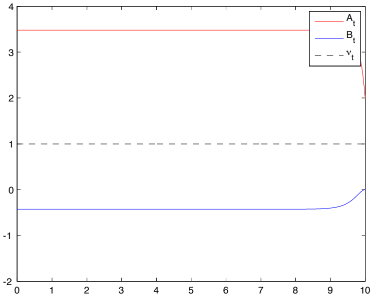
$$\begin{aligned} \dot{\eta}_t &= 2q\eta_t + \left(1 - \frac{1}{N^2}\right)\eta_t^2 - \left(\epsilon - q^2\right) \\ \dot{\varphi}_t^i &= \left[q + \eta_t \left(1 - \frac{1}{N^2}\right)\right] \varphi_t^i + \eta_t \lambda^i \mu, i = 1, ..., N, \end{aligned}$$

with the terminal condition $\eta_T = c$ and $\varphi_T^i = 0$.

Plots of additional liquidity and linear growth

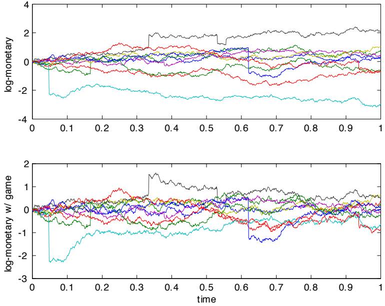


$q = 1, \epsilon = 10, v_t^i = 1, N = 10$ and $T = 1$.

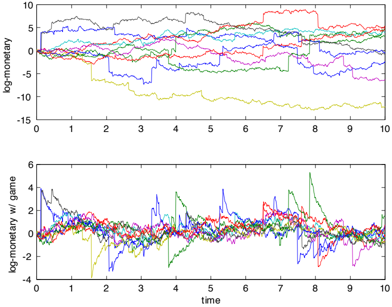


$q = 1, \epsilon = 10, v_t^i = 1, N = 10$ and $T = 10$.

Nash equailibrium illustrated



$v^i_t = -1, N = 10$ and $T = 1$.



$v^i_t = -1, N = 10$ and $T = 10$.

Conclusion and future works

Conclusion

1. The systemic risk is higher when we add jumps in our model.
2. When coming a game feature with jumps, the effect is the additional liquidity and linear growth contributed by jumps. The central bank acts as a clearing house.

Future works

1. Consider to generalize the rate of borrowing/lending as a funcion of $\bar{X}_t$. i.e. $a = a\left(\bar{X}_t\right)$.
2. Major player and minor players.
3. Consider the dynamics of monetary reserve instead of log-monetary reserve.

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Thank you!