Some Exotic Options Pricing under Subordinated Brownian Motion Models: A Variance Reduction Approach

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  - Simulation & Variance Reduction Methods
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  - Fixed Strike Lookback Option

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Consider the problem: To estimate $\mu_X = E(X)$.
Control Variates

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Consider the problem: To estimate $\mu_X = E(X)$

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Define $Z = X - \theta(Y - \mu_Y)$, $\mu_Y = E(Y)$, $\theta \in \mathbb{R}$

$\implies E(Z) = E(X) = \mu_X$, i.e., $Z$ is an unbiased estimator of $\mu_X$
Control Variates

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- If $\mu_Y = E(Y)$ has a closed formula $\implies$ classical CV
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- If $\mu_Y = E(Y)$ has a closed formula $\Rightarrow$ classical CV
- Otherwise $\Rightarrow$ Biased control variate (BCV),
Control Variates

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- If $\mu_Y = E(Y)$ has a closed formula $\implies$ classical CV
- Otherwise $\implies$ Biased control variate (BCV),

- Hope $\sigma_Z^2 < \sigma_X^2$
Control Variates

\[ \sigma_Z^2 = \sigma_X^2 - 2\theta \text{cov}(X, Y) + \theta^2 \sigma_Y^2 \]
Control Variates

- \[ \sigma_Z^2 = \sigma_X^2 - 2\theta \text{cov}(X, Y) + \theta^2 \sigma_Y^2 \]
- The optimal parameter to minimize \( \sigma_Z^2 \) is given by

\[
\theta^* = \frac{\text{cov}(X, Y)}{\text{var}(Y)} = \frac{\rho_{X,Y} \sigma_X}{\sigma_Y}
\]
Control Variates

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- With this optimal \( \theta^* \), \( \sigma_{Z^*}^2 = \sigma_X^2 (1 - \rho^2) < \sigma_X^2 \) for \( \rho \neq 0 \),
Control Variates

\[ \sigma^2_Z = \sigma^2_X - 2\theta \text{cov}(X, Y) + \theta^2 \sigma^2_Y \]

The optimal parameter to minimize \( \sigma^2_Z \) is given by

\[ \theta^* = \frac{\text{cov}(X, Y)}{\text{var}(Y)} = \frac{\rho_X, Y \sigma_X}{\sigma_Y} \]

With this optimal \( \theta^* \), \( \sigma^2_Z^* = \sigma^2_X (1 - \rho^2) < \sigma^2_X \) for \( \rho \neq 0 \),

Thus, the corresponding variance reduction ratio

\[ \eta_{X, Z^*} = \frac{\sigma^2_X}{\sigma^2_Z^*} = \frac{1}{1 - \rho^2} \rightarrow \infty, \text{ as } \rho \rightarrow \pm 1 \]
\[
\sigma_Z^2 = \sigma_X^2 - 2\theta \text{cov}(X, Y) + \theta^2 \sigma_Y^2
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\]

Therefore, choose \(Y\) such that \(Y\) is highly (either + or -) correlated to \(X\)
Biased Control Variate Estimation

- Estimator

\[ \tilde{Z} = X - b \left[ Y - \tilde{Y} \right] \]

\( \tilde{Y} \): approximation of mean value of random variable \( Y \) with error \( \varepsilon \)

\[ \tilde{Y} = E(Y) + \varepsilon \]
Biased Control Variate Estimation

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- **Mean-Squared-Error**

\[ MSE(\tilde{Z}, \mu) \equiv E \left[ (\tilde{Z} - \mu)^2 \right] = Var(\tilde{Z}) + (b\varepsilon)^2. \]
Biased Control Variate Estimation

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  \[ MSE(\tilde{Z}, \mu) \equiv E \left[ (\tilde{Z} - \mu)^2 \right] = Var(\tilde{Z}) + (b\varepsilon)^2. \]

- **The optimal \( b^* \) is**
  \[ b^* = \frac{Cov(X, Y)}{MSE[Y, E(Y)]]} \]
Biased Control Variate Estimation

- Estimator

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- The optimal \( b^* \) is

\[ b^* = \frac{Cov( X, Y )}{MSE[ Y, E( Y )]} \]

- The corresponding MSE

\[ MSE( \tilde{Z}^*, \mu ) = \left( 1 - \rho^2 \frac{Var( Y )}{MSE[ Y, E( Y )]} \right) Var( X ) . \]
Control Variates Using Estimated Means

- Estimator

\[ \hat{Z} = X - b \left[ Y - \hat{Y} \right] \]

\( \hat{Y} \): a random variable with mean \( E(Y) \) and variance \( Var(Y) / N' \)
(with prior simulations sample size \( N' \))

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\[
MSE \left( \hat{Z}, \mu \right) = \frac{Var(\hat{Z})}{N} + b^2 \frac{Var(\hat{Y})}{N} - 2b \frac{Cov(X, Y)}{N} + b^2 \frac{Var(\hat{Y})}{N'}
\]
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\]

- The optimal \( b \)

\[
b^* = \frac{\text{Cov}(X, Y)}{\text{Var}(Y)} \left( \frac{1}{1 + N/N'} \right)
\]
Control Variates Using Estimated Means

- Estimator
  \[ \hat{Z} = X - b \left[ Y - \hat{Y} \right] \]
  \( \hat{Y} \): a random variable with mean \( E(Y) \) and variance \( \text{Var}(Y)/N' \) (with prior simulations sample size \( N' \))

- Mean-Squared-Error
  \[
  \text{MSE} \left( \hat{Z}, \mu \right) = \frac{\text{Var} \left( \hat{Z} \right)}{N} + b^2 \frac{\text{Var} \left( \hat{Y} \right)}{N} - 2b \frac{\text{Cov} \left( X, Y \right)}{N} + b^2 \frac{\text{Var} \left( \hat{Y} \right)}{N'}
  \]

- The optimal \( b \)
  \[
  b^* = \frac{\text{Cov} \left( X, Y \right)}{\text{Var} \left( Y \right)} \left( \frac{1}{1 + N/N'} \right)
  \]

- The corresponding MSE
  \[
  \text{MSE} \left( \hat{Z}^*, \mu \right) = \left( 1 - \rho^2 \frac{1}{1 + N/N'} \right) \text{Var} \left( X \right)
  \]
Randomized Quasi-Monte Carlo Methods

- Low-Discrepancy Sequence (LDS): \( \left( \xi^{(i)} \right)_{i \in \mathbb{N}} \subset [0, 1]^d \)
Randomized Quasi-Monte Carlo Methods

- Low-Discrepancy Sequence (LDS): \( \left( \xi^{(i)} \right)_{i \in \mathbb{N}} \subset [0, 1]^d \)

- Random shifted LDS \( \left\{ U + \xi^{(i)} \right\} \) \( i \in \mathbb{N} \) \( \subset [0, 1]^d \) has the same asymptotic convergent rate as \( \left( \xi^{(i)} \right) \) \( i \in \mathbb{N} \)
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  - \( U \sim \text{Unif} \left( [0, 1]^d \right) \)
Randomized Quasi-Monte Carlo Methods

- Low-Discrepancy Sequence (LDS): \( \left( \tilde{\xi}^{(i)} \right)_{i \in \mathbb{N}} \subset [0, 1]^d \)
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  - \( U \sim Unif \left( [0, 1]^d \right) \)
  - \( \{x\} \): the vector of fractional parts of \( x \)
Randomized Quasi-Monte Carlo Methods

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  - \( U \sim \text{Unif} \left( [0, 1]^d \right) \)
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With \( N_2 \) iid \( U_1, \cdots, U_{N_2} \sim \text{Unif} \left( [0, 1]^d \right) \), the randomized QMC (RQMC) estimate of \( \mu \) over \( N = N_1 N_2 \) samples is

\[
\mu \approx \mu_N = \frac{1}{N_2} \sum_{j=1}^{N_2} \mu_j
\]
Randomized Quasi-Monte Carlo Methods

- Low-Discrepancy Sequence (LDS): \( \left( \tilde{\xi}(i) \right)_{i \in \mathbb{N}} \subset [0, 1]^d \)

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  \[
  \mu \approx \mu_N = \frac{1}{N_2} \sum_{j=1}^{N_2} \mu_j
  \]
  \[
  \mu_j = \frac{1}{N_1} \sum_{i=1}^{N_1} h \left( \left\{ U_j + \tilde{\xi}(i) \right\} \right)
  \]
Subordinated Brownian Motion Models-Single Asset

- $S_t = S_0 \exp(\alpha Y_t + \sigma W_Y_t)$

Why use SBM Models?
More accurate & realistic than the Black-Scholes-Merton model

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Subordinated Brownian Motion Models-Single Asset

- $S_t = S_0 \exp\left(rt + \eta Y_t + \sigma W_{Y_t}\right)$
- $Y_t$ - subordinator, a positive & nondecreasing stochastic process
Subordinated Brownian Motion Models-Single Asset

- \( S_t = S_0 \exp (rt + \eta Y_t + \sigma W_{Y_t}) \)
- \( Y_t \) - subordinator, a positive & nondecreasing stochastic process
- \( Y_t \equiv t \implies \{ S_t \} \) - geometric Brownian motion
Subordinated Brownian Motion Models-Single Asset

- $S_t = S_0 \exp (rt + \eta Y_t + \sigma W_{Y_t})$
- $Y_t$ - subordinator, a positive & nondecreasing stochastic process
- $Y_t \equiv t \implies \{S_t\}$ - geometric Brownian motion
- $Y_t = \text{an inverse Gaussian (IG)} \implies \{S_t\}$ - normal inverse Gaussian (NIG) process

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Subordinated Brownian Motion Models-Single Asset

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**Why use SBM Models?**

- More accurate & realistic than the Black-Scholes-Merton model
Figure: Comparisons of densities for Google data set
Subordinated Brownian Motion Models-Single Asset

Figure: Comparisons of densities for IBM data set
Figure: Comparisons of densities for RIM data set
Figure: Comparisons of densities for Bank of China data set
Figure: Comparisons of densities for Sinopec data set
Figure: Comparisons of densities for Maotai data set
\[ S^{(j)}_t = S^{(j)}_0 \exp \left( r_j t + \eta_j Y_t + \sum_{l=1}^{d} c_{jl} W^{(j)}_{Y_t} \right) \]

\[ S^{(j)}_t \big|_{t=0} = S^{(j)}_0, \quad j = 1, \ldots, d. \]
Subordinated Brownian Motion Models-MultiAsset

\[ S_t^{(j)} = S_0^{(j)} \exp \left( r_j t + \eta_j Y_t + \sum_{l=1}^{d} c_{jl} W_\gamma^{(j)} \right) \]

\[ S_t^{(j)} \bigg|_{t=0} = S_0^{(j)}, \quad j = 1, \ldots, d. \]

If \( Y_t \sim IG(at, b) \) (a > 0, b > 0)

\[ r_j = r + a \left[ \sqrt{b^2 - 2(\eta_j + \sigma_j^2 / 2)} \right] - b, \quad b \geq \sqrt{\max_{1 \leq j \leq d} \left| 2\eta_j + \sigma_j^2 \right|} \]
Subordinated Brownian Motion Models-MultiAsset

\[ S_t^{(j)} = S_0^{(j)} \exp \left( r_j t + \eta_j Y_t + \sum_{l=1}^{d} c_{jl} W_t^{(j)} \right) \]

\[ S_t^{(j)} \mid t=0 = S_0^{(j)}, \quad j = 1, \cdots, d. \]

- If \( Y_t \sim IG(at, b) \) (\( a > 0, b > 0 \))
  \[ r_j = r + a \left[ \sqrt{b^2 - 2(\eta_j + \sigma_j^2/2)} \right] - b, \quad b \geq \sqrt{\max_{1 \leq j \leq d} |2\eta_j + \sigma_j^2|} \]

- If \( Y_t \sim \Gamma(at, b) \) (\( a > 0, b > 0 \))
  \[ r_j = r + a \log [1 - (\eta_j + \sigma_j^2/2)/b], \quad b > \max_{1 \leq j \leq d} (\eta_j + \sigma_j^2/2) \]
Single Asset: Control Variates under NIG & VG Models

Fixed Strike Lookback Option

- Fixed Strike Lookback Option
Fixed Strike Lookback Option

Payoff (at maturity $T$)

$$X_{fixed} = \left( \max_{0 \leq i \leq d} S_{ti} - K \right)^+ = \left( \max_{0 \leq i \leq d} S_{ti} - K \right) \cdot 1_{\{\max_{0 \leq i \leq d} S_{ti} > K\}}$$
Fixed Strike Lookback Option

Payoff (at maturity $T$)

$$X_{\text{fixed}} = \left( \max_{0 \leq i \leq d} S_{t_i} - K \right)^+ = \left( \max_{0 \leq i \leq d} S_{t_i} - K \right) \cdot 1_{\left\{ \max_{0 \leq i \leq d} S_{t_i} > K \right\}}$$

Price (at $t = 0$)

$$V = E \left[ e^{-rT} X_{\text{fixed}} \right]$$
Single Asset: Control Variates under NIG & VG Models

Fixed Strike Lookback Option

- Construct CVs for this option
Construct CVs for this option

Under NIG Models

\[
Y_{\text{fixed}} = \begin{cases} 
    (\max_{0 \leq i \leq d} S_{t_i} - K) \cdot 1 \left\{ S_0 \exp\{\max_{0 \leq i \leq d} B_{t_i}\} > K \right\}, & K < S_0, \quad K > S_0 (CV_1) \\
    (\max_{0 \leq i \leq d} S_{t_i} - K) \cdot 1 \left\{ S_0 \exp\{\min_{0 \leq i \leq d} B_{t_i}\} < K \right\}, & K = S_0 \\
    (\max_{0 \leq i \leq d} S_{t_i} - K)^+ \cdot 1 \left\{ S_0 \exp\{\min_{0 \leq i \leq d} B_{t_i}\} < K \right\}, & K > S_0 (CV_2)
\end{cases}
\]

\(B_t\): the corresponding Brownian motion process
Construct CVs for this option

Under NIG Models

\[ Y_{\text{fixed}} = \begin{cases} 
  (\max_{0 \leq i \leq d} S_{t_i} - K) \cdot 1 \{ S_0 \exp \{ \max_{0 \leq i \leq d} B_{t_i} \} > K \}, & K < S_0, \ K > S_0(CV_1) \\
  (\max_{0 \leq i \leq d} S_{t_i} - K) \cdot 1 \{ S_0 \exp \{ \min_{0 \leq i \leq d} B_{t_i} \} < K \}, & K = S_0 \\
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\end{cases} \]

B_t: the corresponding Brownian motion process

Under VG Models

\[ Y_{\text{fixed}} = \begin{cases} 
  (\max_{0 \leq i \leq d} S_{t_i} - K) \cdot 1 \{ \max_{0 \leq i \leq d} S_{t_i}^{BS} > K \}, & K < S_0 \\
  (\max_{0 \leq i \leq d} S_{t_i} - K) \cdot 1 \{ \min_{0 \leq i \leq d} S_{t_i}^{BS} < K \}, & K = S_0, \ K > S_0(CV_1) \\
  (\max_{0 \leq i \leq d} S_{t_i} - K)^+ \cdot 1 \{ \min_{0 \leq i \leq d} S_{t_i}^{BS} < K \}, & K > S_0(CV_2) 
\end{cases} \]

S_{t}^{BS}: the corresponding geometric Brownian motion process
Floating Strike Lookback Option, with payoff

\[
\left( S_T - \min_{0 \leq i \leq d} S_{t_i} \right)^+ = \left( S_T - \min_{0 \leq i \leq d} S_{t_i} \right) \cdot 1_{\{ S_T > \min_{0 \leq i \leq d} S_{t_i} \}}
\]
Floating Strike Lookback Option, with payoff

\[
\left( S_T - \min_{0 \leq i \leq d} S_{t_i} \right)^+ = \left( S_T - \min_{0 \leq i \leq d} S_{t_i} \right) \cdot 1_{\{S_T > \min_{0 \leq i \leq d} S_{t_i}\}}
\]

Construct CVs for this option under NIG & VG Models

\[
\left( S_T - \min_{0 \leq i \leq d} S_{t_i} \right) \cdot 1_{\{S_T^BS > \min_{0 \leq i \leq d} S_{t_i}^{BS}\}}
\]

\(S_t^{BS}\): the corresponding geometric Brownian motion process
Up-and-Out Barrier Option, with payoff

\[(S_T - K)^+ \cdot 1\{\max_{0 \leq i \leq d} S_{t_i} < B\} = (S_T - K) \cdot 1\{S_T > K\} \cdot 1\{\max_{0 \leq i \leq d} S_{t_i} < B\}\]
Up-and-Out Barrier Option, with payoff

\[(S_T - K)^+ \cdot 1\{\max_{0 \leq i \leq d} S_{t_i} < B\} = (S_T - K) \cdot 1\{S_T > K\} \cdot 1\{\max_{0 \leq i \leq d} S_{t_i} < B\}\]

Construct CVs for this option
Barrier Option

- **Up-and-Out Barrier Option**, with payoff
  \[
  (S_T - K)^+ \cdot 1_{\max_{0 \leq i \leq d} S_{t_i} < B} = (S_T - K) \cdot 1_{S_T > K} 1_{\max_{0 \leq i \leq d} S_{t_i} < B}
  \]

- Construct CVs for this option
- **Under NIG Models**
  \[
  (S_T - K) \cdot 1_{S_T^{BS} > K} 1_{\max_{0 \leq i \leq d} S_{t_i} < B}
  \]

  $S_T^{BS}$: the corresponding geometric Brownian motion process
Barrier Option

- **Up-and-Out Barrier Option**, with payoff

\[
(S_T - K)^+ \cdot 1_{\{\max_{0 \leq i \leq d} S_{t_i} < B\}} = (S_T - K) \cdot 1_{\{S_T > K\}} 1_{\{\max_{0 \leq i \leq d} S_{t_i} < B\}}
\]

- Construct CVs for this option
- **Under NIG Models**

\[
(S_T - K) \cdot 1_{\{S_T^{BS} > K\}} 1_{\{\max_{0 \leq i \leq d} S_{t_i} < B\}}
\]

\(S_T^{BS}\): the corresponding geometric Brownian motion process

- **Under VG Models**

\[
(S_T - K)^+ = (S_T - K) \cdot 1_{\{S_T > K\}}
\]

the corresponding European vanilla call option under the same model
Numerical Test Results

Single Asset Problems: Parameters

- $S_0 = $100, $K = $90, $100, $110, r = 0.1, T = 1 \text{ year}$

Barrier level $B = $160
Numerical Test Results
Single Asset Problems: Parameters

- \( S_0 = $100, \ K = $90, \ $100, \ $110, \ r = 0.1, \ T = 1 \text{ year} \)

  Barrier level \( B = $160 \)

- NIG model parameter set
  \[ \{\alpha, \beta, \delta, \mu\} = \{75.49, -4.089, 3, 0.0149508\}, \]
  \[ a = 1, \ b = \delta \sqrt{\alpha^2 - \beta^2} \text{ with } |1 + \beta| < \alpha, \]
  \[ \eta = \beta \delta^2, \ \sigma = \delta \]
Numerical Test Results

Single Asset Problems: Parameters

- \( S_0 = $100, \ K = $90, \ $100, \ $110, \ r = 0.1, \ T = 1 \text{ year} \)

- Barrier level \( B = $160 \)

- NIG model parameter set
  \[ \{\alpha, \beta, \delta, \mu\} = \{75.49, -4.089, 3, 0.0149508\}, \]
  \[ a = 1, \ b = \delta \sqrt{\alpha^2 - \beta^2} \text{ with } |1 + \beta| < \alpha, \ \eta = \beta \delta^2, \ \sigma = \delta \]

- VG model parameter set
  \[ \{\mu, \nu, \sigma\} = \{-0.1436, 0.3, 0.12136\}, \]
  \[ a = b = \frac{1}{\nu}, \ \eta = \mu \]
Numerical Test Results

Single Asset Problems: Parameters

- $S_0 = $100, $K = $90, $100, $110, r = 0.1, T = 1$ year

  Barrier level $B = $160

- NIG model parameter set
  \[
  \{\alpha, \beta, \delta, \mu\} = \{75.49, -4.089, 3, 0.0149508\},
  \]
  \[a = 1, b = \delta \sqrt{\alpha^2 - \beta^2} \text{ with } |1 + \beta| < \alpha, \eta = \beta \delta^2, \sigma = \delta\]

- VG model parameter set
  \[
  \{\mu, \nu, \sigma\} = \{-0.1436, 0.3, 0.12136\},
  \]
  \[a = b = \frac{1}{\nu}, \eta = \mu\]

- For the RQMC method, the number of shifts = 10
Numerical Test Results
Single Asset Problems: Parameters

- \( S_0 = $100, \ K = $90, \ $100, \ $110, \ r = 0.1, \ T = 1 \text{ year} \)

  Barrier level \( B = $160 \)

- NIG model parameter set
  \( \{a, \beta, \delta, \mu\} = \{75.49, -4.089, 3, 0.0149508\} \),
  \( a = 1, \ b = \delta \sqrt{a^2 - \beta^2} \text{ with } |1 + \beta| < a, \eta = \beta \delta^2, \sigma = \delta \)

- VG model parameter set
  \( \{\mu, \nu, \sigma\} = \{-0.1436, 0.3, 0.12136\} \),
  \( a = b = \frac{1}{\nu}, \eta = \mu \)

- For the RQMC method, the number of shifts = 10

- Sample size \( N' \) for preliminary simulations to compute estimated means of CVs is 1,000,000.
**Numerical Test Results**

Single Asset Problems: Fixed Strike Lookback

**Table:** Correlations of the CVs method for fixed strike lookback options under NIG models

<table>
<thead>
<tr>
<th>N</th>
<th>$K &lt; S_0$ $d=256$</th>
<th>$K = S_0$ $d=256$</th>
<th>$K &gt; S_0$, $(CV_1)$ $d=16$</th>
<th>$K &gt; S_0$, $(CV_1)$ $d=64$</th>
<th>$K &gt; S_0$, $(CV_1)$ $d=256$</th>
<th>$K &gt; S_0$, $(CV_2)$ $d=256$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1024</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.999721</td>
<td>0.998104</td>
<td>0.993706</td>
<td>1.0000</td>
</tr>
<tr>
<td>2048</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.999798</td>
<td>0.998362</td>
<td>0.993616</td>
<td>1.0000</td>
</tr>
<tr>
<td>4096</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.999705</td>
<td>0.998423</td>
<td>0.993314</td>
<td>1.0000</td>
</tr>
<tr>
<td>8192</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.999716</td>
<td>0.998641</td>
<td>0.993837</td>
<td>1.0000</td>
</tr>
<tr>
<td>16384</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.999747</td>
<td>0.998552</td>
<td>0.994107</td>
<td>1.0000</td>
</tr>
<tr>
<td>32768</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.999719</td>
<td>0.998471</td>
<td>0.993421</td>
<td>1.0000</td>
</tr>
</tbody>
</table>
**Numerical Test Results**

**Single Asset Problems: Fixed Strike Lookback**

Table: Comparisons of fixed strike lookback call option simulations under NIG models with $d = 256$ and $K < S_0$

<table>
<thead>
<tr>
<th>$N$</th>
<th>Price</th>
<th>Var</th>
<th>Time</th>
<th>Price</th>
<th>Var</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MC</td>
<td>SB</td>
<td></td>
<td>MC</td>
<td>SB</td>
<td></td>
</tr>
<tr>
<td>2560</td>
<td>29.5613</td>
<td>2.63E+2</td>
<td>16.2</td>
<td>29.5943</td>
<td>2.89E-01</td>
<td>10.8</td>
</tr>
<tr>
<td>5120</td>
<td>29.7973</td>
<td>2.67E+2</td>
<td>32.3</td>
<td>29.4720</td>
<td>8.50E-02</td>
<td>21.8</td>
</tr>
<tr>
<td>10240</td>
<td>29.4712</td>
<td>2.56E+2</td>
<td>64.4</td>
<td>29.7151</td>
<td>1.33E-01</td>
<td>43.0</td>
</tr>
<tr>
<td>20480</td>
<td>29.7879</td>
<td>2.56E+2</td>
<td>129.2</td>
<td>29.6712</td>
<td>2.47E-02</td>
<td>85.4</td>
</tr>
<tr>
<td>40960</td>
<td>29.4437</td>
<td>2.58E+2</td>
<td>275.4</td>
<td>29.5700</td>
<td>3.14E-02</td>
<td>171.8</td>
</tr>
<tr>
<td>81920</td>
<td>29.6377</td>
<td>2.61E+2</td>
<td>552.9</td>
<td>29.6021</td>
<td>8.44E-03</td>
<td>347.6</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>MC+CV</th>
<th>SB+CV</th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>2560</td>
<td>29.6524</td>
<td>5.29E-11</td>
<td>15.1</td>
<td>29.6524</td>
<td>0.00E+00</td>
<td>12.0</td>
</tr>
<tr>
<td>5120</td>
<td>29.6524</td>
<td>2.07E-11</td>
<td>30.5</td>
<td>29.6524</td>
<td>0.00E+00</td>
<td>23.8</td>
</tr>
<tr>
<td>10240</td>
<td>29.6524</td>
<td>3.08E-10</td>
<td>60.0</td>
<td>29.6524</td>
<td>0.00E+00</td>
<td>48.2</td>
</tr>
<tr>
<td>20480</td>
<td>29.6524</td>
<td>7.29E-10</td>
<td>120.1</td>
<td>29.6524</td>
<td>0.00E+00</td>
<td>95.7</td>
</tr>
<tr>
<td>40960</td>
<td>29.6524</td>
<td>0.00E+00</td>
<td>239.8</td>
<td>29.6524</td>
<td>2.65E-13</td>
<td>190.8</td>
</tr>
<tr>
<td>81920</td>
<td>29.6524</td>
<td>0.00E+00</td>
<td>480.8</td>
<td>29.6524</td>
<td>5.30E-13</td>
<td>383.3</td>
</tr>
</tbody>
</table>

Yongzeng Lai & Qiuzi Tan (Wilfrid Laurier USome Exotic Options Pricing under Subordin)
Numerical Test Results
Single Asset Problems: Fixed Strike Lookback

Table: Correlations of the CVs method for fixed strike lookback options under VG models

<table>
<thead>
<tr>
<th>N</th>
<th>$K &lt; S_0$ d=256</th>
<th>$K = S_0$ d=256</th>
<th>$K &gt; S_0$, (CV$_1$) d=16</th>
<th>$K &gt; S_0$, (CV$_1$) d=64</th>
<th>$K &gt; S_0$, (CV$_2$) d=256</th>
</tr>
</thead>
<tbody>
<tr>
<td>1024</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.969664</td>
<td>0.974040</td>
<td>0.975840</td>
</tr>
<tr>
<td>2048</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.971094</td>
<td>0.972966</td>
<td>0.975173</td>
</tr>
<tr>
<td>4096</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.968546</td>
<td>0.973369</td>
<td>0.973334</td>
</tr>
<tr>
<td>8192</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.969242</td>
<td>0.973054</td>
<td>0.974897</td>
</tr>
<tr>
<td>16384</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.969785</td>
<td>0.972672</td>
<td>0.973943</td>
</tr>
<tr>
<td>32768</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.970092</td>
<td>0.972758</td>
<td>0.973870</td>
</tr>
</tbody>
</table>
Numerical Test Results
Single Asset Problems: Fixed Strike Lookback

**Table**: Comparisons of fixed strike lookback call option simulations under VG models with $d = 256$ and $K < S_0$

<table>
<thead>
<tr>
<th>$N$</th>
<th>Price</th>
<th>Var</th>
<th>Time</th>
<th>Price</th>
<th>Var</th>
<th>Time</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>MC</td>
<td>SB</td>
<td></td>
<td>MC</td>
<td>SB</td>
<td></td>
</tr>
<tr>
<td>2560</td>
<td>24.0301</td>
<td>8.40E+01</td>
<td>8.8</td>
<td>24.1418</td>
<td>6.30E-01</td>
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</tr>
<tr>
<td>5120</td>
<td>24.5274</td>
<td>8.81E+01</td>
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<td>24.2296</td>
<td>1.87E-01</td>
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</tr>
<tr>
<td>10240</td>
<td>24.4951</td>
<td>9.25E+01</td>
<td>37.3</td>
<td>24.2344</td>
<td>7.23E-02</td>
<td>41.5</td>
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<tr>
<td>20480</td>
<td>24.1886</td>
<td>8.70E+01</td>
<td>72.4</td>
<td>24.1338</td>
<td>3.36E-02</td>
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<tr>
<td>40960</td>
<td>24.3247</td>
<td>8.87E+01</td>
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<td>24.3203</td>
<td>2.31E-02</td>
<td>166.1</td>
</tr>
<tr>
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<td>24.2559</td>
<td>8.73E+01</td>
<td>284.9</td>
<td>24.2710</td>
<td>2.04E-02</td>
<td>332.0</td>
</tr>
<tr>
<td></td>
<td>MC+CV</td>
<td>SB+CV</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2560</td>
<td>24.2539</td>
<td>3.69E-11</td>
<td>12.5</td>
<td>24.2537</td>
<td>0.00E+00</td>
<td>14.0</td>
</tr>
<tr>
<td>5120</td>
<td>24.2539</td>
<td>1.15E-10</td>
<td>26.3</td>
<td>24.2537</td>
<td>0.00E+00</td>
<td>29.2</td>
</tr>
<tr>
<td>10240</td>
<td>24.2539</td>
<td>0.00E+00</td>
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<td>24.2537</td>
<td>0.00E+00</td>
<td>58.6</td>
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<td>0.00E+00</td>
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<td>0.00E+00</td>
<td>467.8</td>
</tr>
</tbody>
</table>

Yongzeng Lai & Qiuzi Tan (Wilfrid Laurier University, Waterloo, Ontario, Canada)
### Numerical Test Results

**Single Asset Problems: Floating Strike Lookback**

**Table**: Comparisons of average VRRs for floating strike lookback option simulations under NIG models

<table>
<thead>
<tr>
<th></th>
<th>SB</th>
<th>MC+CV</th>
<th>SB+CV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d = 64$</td>
<td>1.98E+04</td>
<td>1.02E+04</td>
<td>3.77E+07</td>
</tr>
<tr>
<td>$d = 128$</td>
<td>2.56E+04</td>
<td>3.88E+03</td>
<td>9.41E+06</td>
</tr>
<tr>
<td>$d = 256$</td>
<td>1.03E+04</td>
<td>2.32E+03</td>
<td>5.40E+06</td>
</tr>
</tbody>
</table>

**Table**: Comparisons of average VRRs for floating strike lookback option simulations under VG models

<table>
<thead>
<tr>
<th></th>
<th>SB</th>
<th>MC+CV</th>
<th>SB+CV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d = 64$</td>
<td>2.68E+03</td>
<td>2.17E+01</td>
<td>6.27E+06</td>
</tr>
<tr>
<td>$d = 128$</td>
<td>2.51E+03</td>
<td>2.30E+01</td>
<td>5.49E+06</td>
</tr>
<tr>
<td>$d = 256$</td>
<td>4.91E+03</td>
<td>2.98E+01</td>
<td>7.33E+06</td>
</tr>
</tbody>
</table>
Numerical Test Results
Single Asset Problems: Up-and-Out Barrier

Table: Correlations of the CVs method for barrier options under NIG models

<table>
<thead>
<tr>
<th>N</th>
<th>$K &lt; S_0$</th>
<th>$K = S_0$</th>
<th>$K &gt; S_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>d=16</td>
<td>d=256</td>
<td>d=16</td>
</tr>
<tr>
<td>1024</td>
<td>0.999857</td>
<td>0.992132</td>
<td>0.998984</td>
</tr>
<tr>
<td>2048</td>
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<td>0.994298</td>
<td>0.999532</td>
</tr>
<tr>
<td>4096</td>
<td>0.999646</td>
<td>0.993947</td>
<td>0.999572</td>
</tr>
<tr>
<td>8192</td>
<td>0.999785</td>
<td>0.994807</td>
<td>0.999559</td>
</tr>
<tr>
<td>16384</td>
<td>0.999763</td>
<td>0.994092</td>
<td>0.999518</td>
</tr>
<tr>
<td>32768</td>
<td>0.999772</td>
<td>0.994327</td>
<td>0.999493</td>
</tr>
</tbody>
</table>
### Numerical Test Results

**Single Asset Problems: Up-and-Out Barrier**

**Table:** Comparison of average VRRs for UNO barrier option simulation under the NIG model

<table>
<thead>
<tr>
<th>$d$</th>
<th>$K = $90</th>
<th>$K = $100</th>
<th>$K = $110</th>
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<tbody>
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<td>$6.39E+03$</td>
<td>$9.62E+03$</td>
</tr>
<tr>
<td></td>
<td>$3.71E+02$</td>
<td>$1.58E+02$</td>
<td>$6.64E+01$</td>
</tr>
<tr>
<td></td>
<td>$7.97E+05$</td>
<td>$7.19E+05$</td>
<td>$1.36E+05$</td>
</tr>
<tr>
<td>128</td>
<td>$4.57E+03$</td>
<td>$5.67E+03$</td>
<td>$6.32E+03$</td>
</tr>
<tr>
<td></td>
<td>$1.82E+02$</td>
<td>$7.10E+01$</td>
<td>$2.97E+01$</td>
</tr>
<tr>
<td></td>
<td>$4.02E+05$</td>
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<td>$1.57E+05$</td>
</tr>
<tr>
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<td>$4.80E+03$</td>
<td>$4.66E+03$</td>
<td>$7.77E+03$</td>
</tr>
<tr>
<td></td>
<td>$8.78E+01$</td>
<td>$3.38E+01$</td>
<td>$1.51E+01$</td>
</tr>
<tr>
<td></td>
<td>$2.39E+05$</td>
<td>$2.15E+05$</td>
<td>$3.98E+04$</td>
</tr>
</tbody>
</table>
Table: Correlations of the CVs method for barrier options under VG models

<table>
<thead>
<tr>
<th>N</th>
<th>( K &lt; S_0 ) ( d=16 )</th>
<th>( K &lt; S_0 ) ( d=256 )</th>
<th>( K = S_0 ) ( d=16 )</th>
<th>( K = S_0 ) ( d=256 )</th>
<th>( K &gt; S_0 ) ( d=16 )</th>
<th>( K &gt; S_0 ) ( d=256 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1024</td>
<td>1.000000</td>
<td>0.984916</td>
<td>1.000000</td>
<td>0.974035</td>
<td>0.948773</td>
<td>0.983082</td>
</tr>
<tr>
<td>2048</td>
<td>0.986852</td>
<td>1.000000</td>
<td>1.000000</td>
<td>0.984988</td>
<td>0.978092</td>
<td>0.968028</td>
</tr>
<tr>
<td>4096</td>
<td>0.984453</td>
<td>0.988920</td>
<td>0.987632</td>
<td>0.990459</td>
<td>0.989998</td>
<td>0.975124</td>
</tr>
<tr>
<td>8192</td>
<td>0.987246</td>
<td>0.982182</td>
<td>0.986800</td>
<td>0.983695</td>
<td>0.986746</td>
<td>0.983926</td>
</tr>
<tr>
<td>16384</td>
<td>0.988499</td>
<td>0.989701</td>
<td>0.982374</td>
<td>0.983141</td>
<td>0.985473</td>
<td>0.980684</td>
</tr>
<tr>
<td>32768</td>
<td>0.992156</td>
<td>0.990017</td>
<td>0.992064</td>
<td>0.987797</td>
<td>0.983229</td>
<td>0.982404</td>
</tr>
</tbody>
</table>
**Numerical Test Results**

**Single Asset Problems: Up-and-Out Barrier**

**Table:** Comparisons of average VRRs for up-and-out barrier option simulation under VG models

<table>
<thead>
<tr>
<th></th>
<th>SB</th>
<th>MC+CV</th>
<th>SB+CV</th>
</tr>
</thead>
<tbody>
<tr>
<td>$d = 64$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K &lt; S_0$</td>
<td>3.81E+03</td>
<td>N/A</td>
<td>1.98E+07</td>
</tr>
<tr>
<td>$K = S_0$</td>
<td>4.86E+01</td>
<td>7.56E+03</td>
<td>N/A</td>
</tr>
<tr>
<td>$K &gt; S_0$</td>
<td>2.28E+03</td>
<td>4.54E+01</td>
<td>9.02E+06</td>
</tr>
<tr>
<td>$d = 128$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K &lt; S_0$</td>
<td>2.52E+03</td>
<td>1.20E+12</td>
<td>1.22E+07</td>
</tr>
<tr>
<td>$K = S_0$</td>
<td>3.68E+03</td>
<td>4.94E+01</td>
<td>1.21E+07</td>
</tr>
<tr>
<td>$K &gt; S_0$</td>
<td>1.95E+03</td>
<td>3.36E+01</td>
<td>8.96E+06</td>
</tr>
<tr>
<td>$d = 256$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$K &lt; S_0$</td>
<td>2.85E+03</td>
<td>4.30E+01</td>
<td>8.93E+06</td>
</tr>
<tr>
<td>$K = S_0$</td>
<td>2.18E+03</td>
<td>3.92E+01</td>
<td>5.75E+06</td>
</tr>
<tr>
<td>$K &gt; S_0$</td>
<td>2.67E+03</td>
<td>3.21E+01</td>
<td>5.27E+06</td>
</tr>
</tbody>
</table>
Summary

- Single asset: new control variates are constructed (lookback and barrier options)
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- Randomized quasi-Monte Carlo methods are applied
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  - Use of multivariate CVs
  - Combining with other variance reduction methods: importance sampling, bridge sampling etc.
  - Use of GPU computation
Thank you!
Table: Control variate mean estimation for fixed strike lookback option, $K = 90$

<table>
<thead>
<tr>
<th>$d$</th>
<th>$\mu$</th>
<th>var</th>
<th>time</th>
</tr>
</thead>
<tbody>
<tr>
<td>16</td>
<td>30.54083441</td>
<td>1.7594E-05</td>
<td>340.23</td>
</tr>
<tr>
<td>32</td>
<td>31.39290350</td>
<td>1.7678E-05</td>
<td>1273.33</td>
</tr>
<tr>
<td>64</td>
<td>31.99675901</td>
<td>1.7699E-05</td>
<td>1273.33</td>
</tr>
<tr>
<td>128</td>
<td>32.46988045</td>
<td>1.7790E-05</td>
<td>2545.14</td>
</tr>
<tr>
<td>256</td>
<td>32.77231957</td>
<td>1.7833E-05</td>
<td>4999.43</td>
</tr>
</tbody>
</table>
Also tested for other parameter sets \{\alpha, \beta, \delta, \mu\} calibrated from historical stock prices.
Also tested for other parameter sets \( \{\alpha, \beta, \delta, \mu\} \) calibrated from historical stock prices

- Google stocks: \( \{28.8279, 3.89676, 0.000514679, 0.0149508\} \)
  - IBM stocks: \( \{51.1663, -10.299, 0.00104061, 0.0049899\} \)
  - Yahoo stocks: \( \{18.068, -2.59218, 0.00280093, 0.0103147\} \)

The results are similar to the previous case, even reached higher VRRs on average.
Numerical results: Tests for other parameter sets

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  - Google stocks: \( \{28.8279, 3.89676, 0.000514679, 0.0149508\} \)
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