Some Exotic Options Pricing under Subordinated Brownian Motion Models: A Variance Reduction Approach

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- Introduction

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  - Simulation & Variance Reduction Methods

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- Therefore, choose Y such that Y is highly (either + or -) correlated to X

Biased Control Variate Estimation

- Estimator

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$\widetilde{Y}$: approximation of mean value of random variable Y with error ε

$$\widetilde{Y} = E\left(Y\right) + \varepsilon$$

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- **Why use SBM Models?**
- More accurate & realistic than the Black-Scholes-Merton model

Subordinated Brownian Motion Models-Single Asset

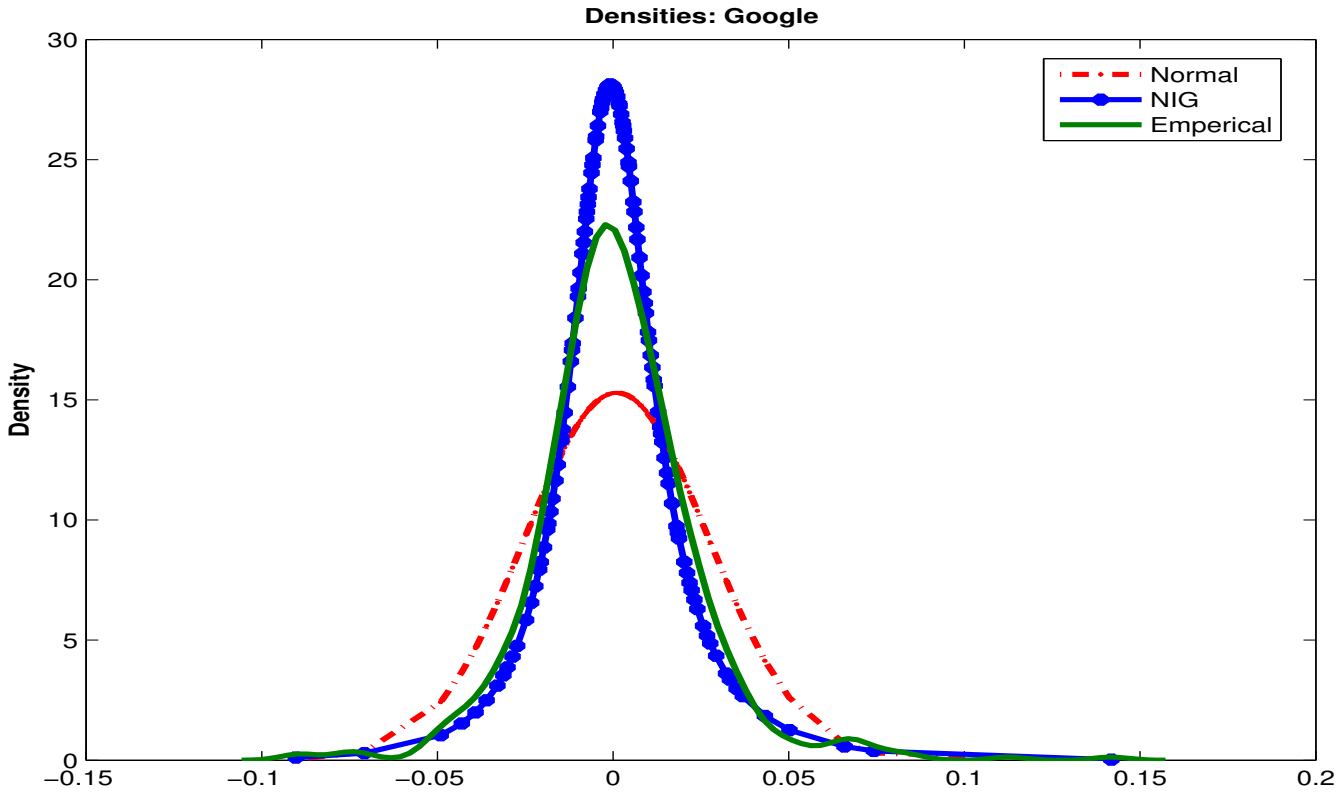

Figure : Comparisons of densities for Google data set

Subordinated Brownian Motion Models-Single Asset

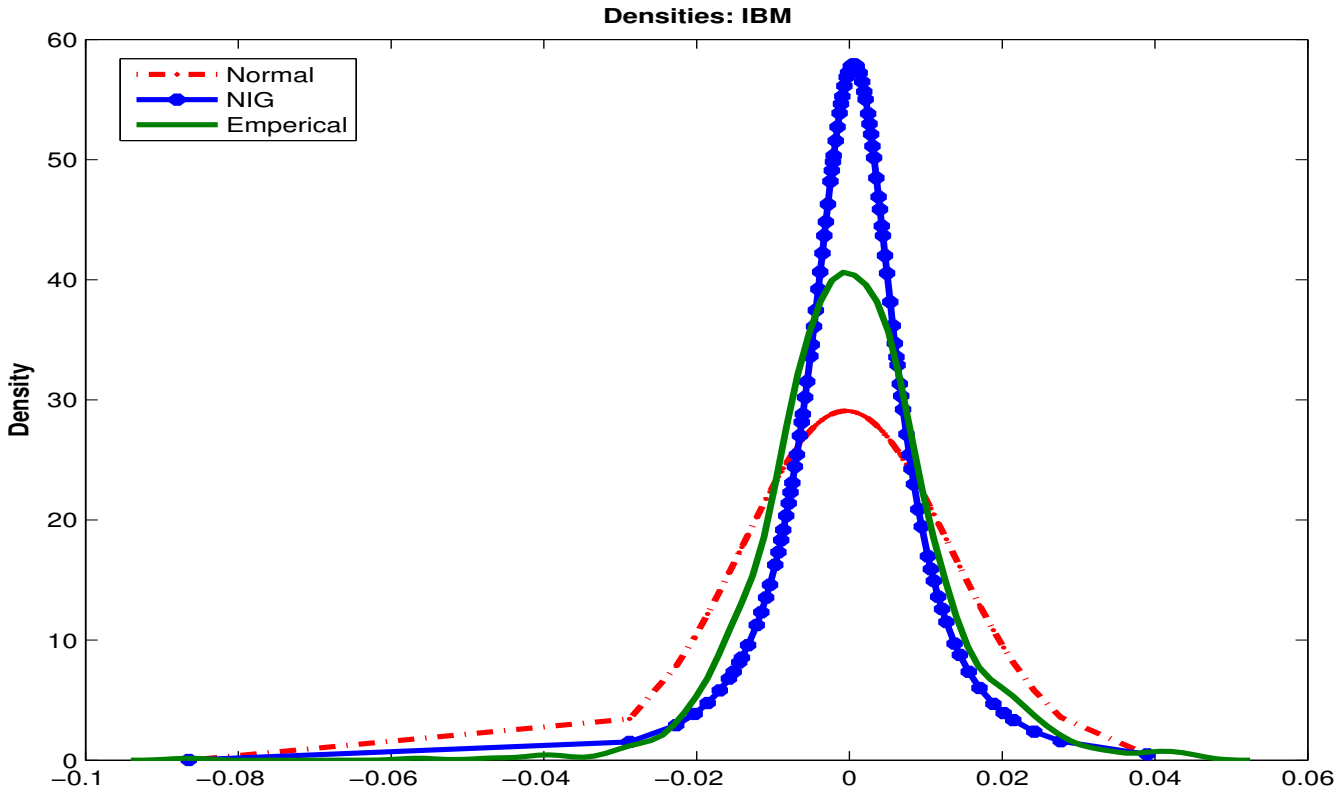

Figure : Comparisons of densities for IBM data set

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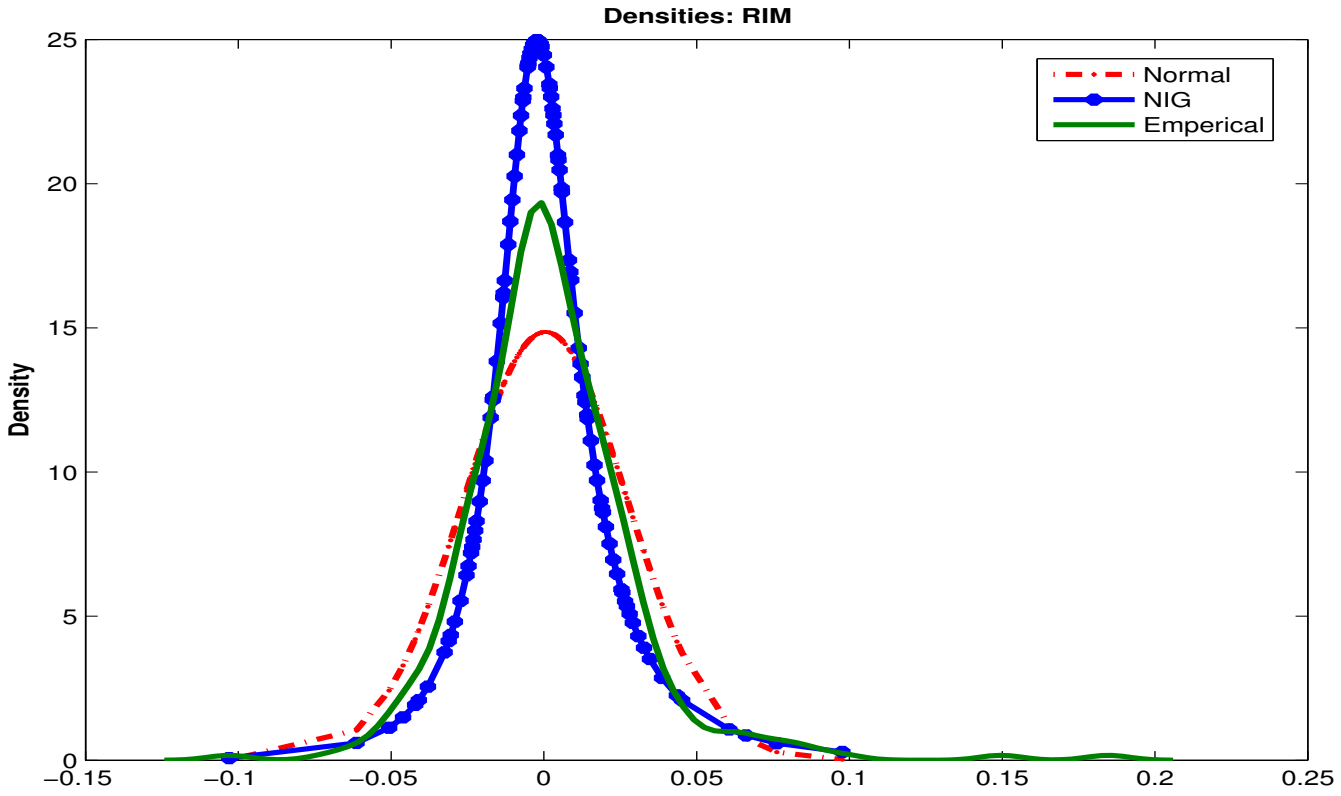

Figure : Comparisons of densities for RIM data set

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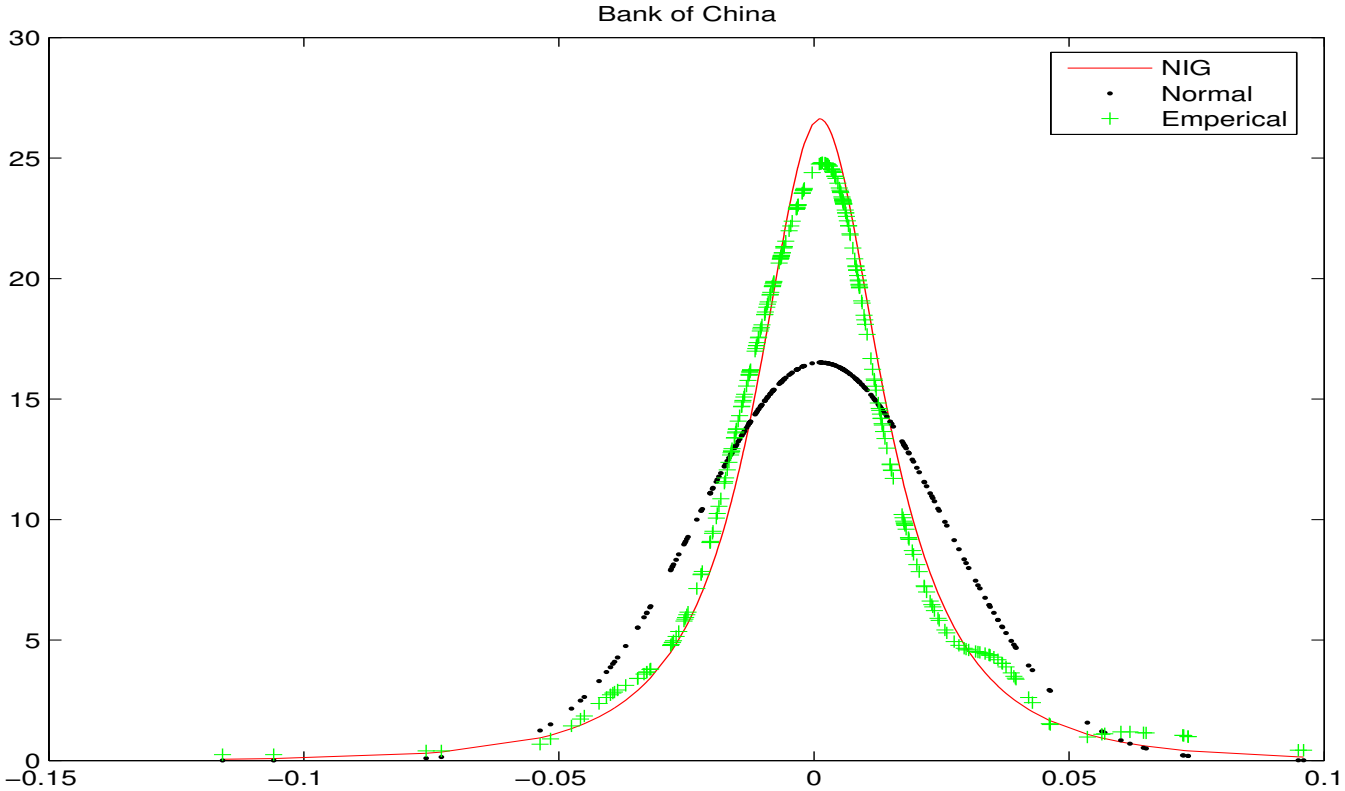

Figure : Comparisons of densities for Bank of China data set

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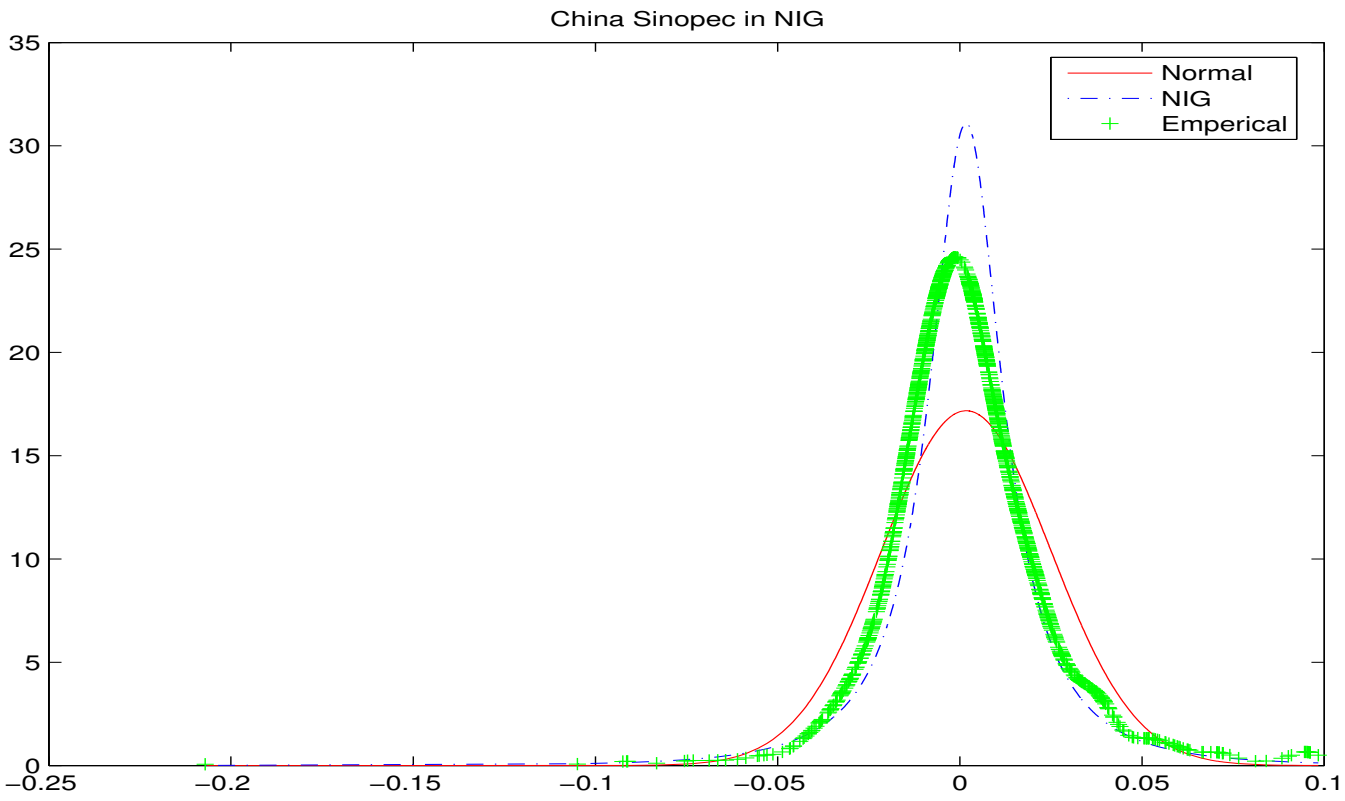

Figure : Comparisons of densities for Sinopec data set

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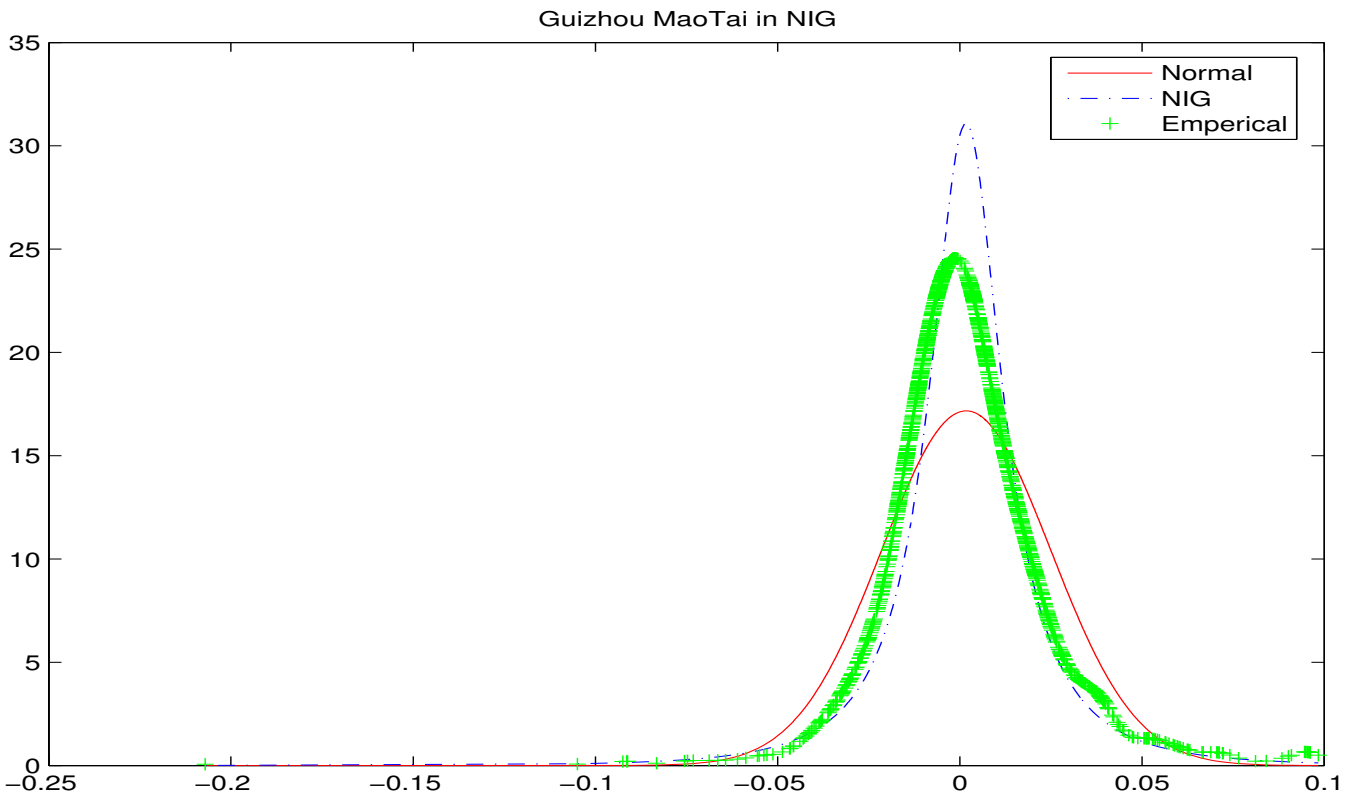

Figure : Comparisons of densities for Maotai data set

Subordinated Brownian Motion Models-MultiAsset

- $$S_t^{(j)} = S_0^{(j)} \exp\left(r_j t + \eta_j \Upsilon_t + \sum_{l=1}^{d} c_{jl} W_{\Upsilon_t}^{(j)} \right)$$

  $$S_t^{(j)}|_{t=0} = S_0^{(j)}, \qquad j = 1, \cdots, d.$$

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- If $\Upsilon_t \sim IG(at, b)$ $(a > 0,\ b > 0)$
  $r_j = r + a\left[\sqrt{b^2 - 2(\eta_j + \sigma_j^2/2)}\right] - b,\ b \geq \sqrt{\max_{1\leq j\leq d} |2\eta_j + \sigma_j^2|}$

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- If $\Upsilon_t \sim \Gamma(at, b)$ $(a > 0,\ b > 0)$
  $r_j = r + a\log[1 - (\eta_j + \sigma_j^2/2)/b],\ b > \max_{1\leq j\leq d}(\eta_j + \sigma_j^2/2)$

Single Asset: Control Variates under NIG & VG Models

Fixed Strike Lookback Option

- Fixed Strike Lookback Option

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Fixed Strike Lookback Option

- Fixed Strike Lookback Option
- Payoff (at maturity T)

$$X_{fixed} = \left(\max_{0 \le i \le d} S_{t_i} - K \right)^+ = \left(\max_{0 \le i \le d} S_{t_i} - K \right) \cdot \mathbf{1}_{\left\{ \max_{0 \le i \le d} S_{t_i} > K \right\}}$$

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- Price (at $t = 0$)

$$V = E\left[e^{-rT} X_{fixed} \right]$$

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Fixed Strike Lookback Option

- Construct CVs for this option

Single Asset: Control Variates under NIG & VG Models

Fixed Strike Lookback Option

- Construct CVs for this option
- Under NIG Models

$$Y_{fixed} = \begin{cases} (\max_{0\le i\le d} S_{t_i} - K)\cdot \mathbf{1}_{\left\{S_0 \exp\left\{\max_{0\le i\le d} B_{t_i}\right\}>K\right\}}, & K < S_0,\ K > S_0 (CV_1 \\ (\max_{0\le i\le d} S_{t_i} - K)\cdot \mathbf{1}_{\left\{S_0 \exp\left\{\min_{0\le i\le d} B_{t_i}\right\}<K\right\}}, & K = S_0 \\ (\max_{0\le i\le d} S_{t_i} - K)^+\cdot \mathbf{1}_{\left\{S_0 \exp\left\{\min_{0\le i\le d} B_{t_i}\right\}<K\right\}}, & K > S_0 (CV_2) \end{cases}$$

B_t: the corresponding Brownian motion process

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- Construct CVs for this option
- Under NIG Models

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- Under VG Models

$$Y_{fixed} = \begin{cases} (\max_{0\le i\le d} S_{t_i} - K)\cdot \mathbf{1}_{\{\max_{0\le i\le d} S^{BS}_{t_i}>K\}}, & K < S_0 \\ (\max_{0\le i\le d} S_{t_i} - K)\cdot \mathbf{1}_{\{\min_{0\le i\le d} S^{BS}_{t_i}<K\}}, & K = S_0,\ K > S_0 (CV_1) \\ (\max_{0\le i\le d} S_{t_i} - K)^+\cdot \mathbf{1}_{\{\min_{0\le i\le d} S^{BS}_{t_i}<K\}}, & K > S_0 (CV_2) \end{cases}$$

S^{BS}_t: the corresponding geometric Brownian motion process

Single Asset: Control Variates under NIG & VG Models

Floating Strike Lookback Option

- Floating Strike Lookback Option, with payoff

$$\left(S_T - \min_{0\le i\le d} S_{t_i}\right)^+ = \left(S_T - \min_{0\le i\le d} S_{t_i}\right)\cdot \mathbf{1}_{\left\{S_T > \min_{0\le i\le d} S_{t_i}\right\}}$$

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- Construct CVs for this option under NIG & VG Models

$$\left(S_T - \min_{0\le i\le d} S_{t_i}\right)\cdot \mathbf{1}_{\left\{S_T^{BS} > \min_{0\le i\le d} S_{t_i}^{BS}\right\}}$$

S_t^{BS}: the corresponding geometric Brownian motion process

Single Asset: Control Variates under NIG & VG Models

Barrier Option

- Up-and-Out Barrier Option, with payoff

$$(S_T - K)^+ \cdot \mathbf{1}_{\left\{\max_{0\le i\le d} S_{t_i} < B\right\}} = (S_T - K) \cdot \mathbf{1}_{\{S_T > K\}} \mathbf{1}_{\left\{\max_{0\le i\le d} S_{t_i} < B\right\}}$$

Single Asset: Control Variates under NIG & VG Models

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- Under NIG Models

$$(S_T - K)\cdot \mathbf{1}_{\left\{S_T^{BS} > K\right\}} \mathbf{1}_{\left\{\max_{0\le i\le d} S_{t_i} < B\right\}}$$

S_t^{BS}: the corresponding geometric Brownian motion process

Single Asset: Control Variates under NIG & VG Models

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- Under NIG Models

$$(S_T - K)\cdot \mathbf{1}_{\left\{S_T^{BS} > K\right\}} \mathbf{1}_{\left\{\max_{0\le i\le d} S_{t_i} < B\right\}}$$

S_t^{BS}: the corresponding geometric Brownian motion process

- Under VG Models

$$(S_T - K)^+ = (S_T - K)\cdot \mathbf{1}_{\{S_T > K\}}$$

the corresponding European vanilla call option under the same model

Numerical Test Results

Single Asset Problems: Parameters

- $S_0 = \$100$, $K = \$90$, $\$100$, $\$110$, $r = 0.1$, $T = 1$ year

  Barrier level $B = \$160$

Numerical Test Results

Single Asset Problems: Parameters

- $S_0 = \$100$, $K = \$90$, $\$100$, $\$110$, $r = 0.1$, $T = 1$ year

  Barrier level $B = \$160$
- NIG model parameter set
  $\{\alpha, \beta, \delta, \mu\} = \{75.49, -4.089, 3, 0.0149508\}$,
  $a = 1$, $b = \delta\sqrt{\alpha^2 - \beta^2}$ with $|1 + \beta| < \alpha$, $\eta = \beta\delta^2$, $\sigma = \delta$

Numerical Test Results

Single Asset Problems: Parameters

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- VG model parameter set
  $\{\mu, \nu, \sigma\} = \{-0.1436, 0.3, 0.12136\}$,
  $a = b = \frac{1}{\nu}$, $\eta = \mu$

Numerical Test Results

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- For the RQMC method, *the number of shifts* $= 10$

Numerical Test Results

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  $\{\mu, \nu, \sigma\} = \{-0.1436, 0.3, 0.12136\}$,
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- For the RQMC method, *the number of shifts* $= 10$
- Sample size N' for preliminary simulations to compute estimated means of CVs is 1,000,000.

Numerical Test Results

Single Asset Problems: Fixed Strike Lookback

Table : Correlations of the CVs method for fixed strike lookback options under NIG models

	$K < S_0$	$K = S_0$	$K > S_0$, (CV_1)			$K > S_0(CV_2)$
N	d=256	d=256	d=16	d=64	d=256	d=256
1024	1.0000	1.0000	0.999721	0.998104	0.993706	1.0000
2048	1.0000	1.0000	0.999798	0.998362	0.993616	1.0000
4096	1.0000	1.0000	0.999705	0.998423	0.993314	1.0000
8192	1.0000	1.0000	0.999716	0.998641	0.993837	1.0000
16384	1.0000	1.0000	0.999747	0.998552	0.994107	1.0000
32768	1.0000	1.0000	0.999719	0.998471	0.993421	1.0000

Numerical Test Results

Single Asset Problems: Fixed Strike Lookback

Table : Comparisons of fixed strike lookback call option simulations under NIG models with $d = 256$ and $K < S_0$

N	Price	Var	Time	Price	Var	Time
	MC			SB		
2560	29.5613	2.63E+2	16.2	29.5943	2.89E-01	10.8
5120	29.7973	2.67E+2	32.3	29.4720	8.50E-02	21.8
10240	29.4712	2.56E+2	64.4	29.7151	1.33E-01	43.0
20480	29.7879	2.56E+2	129.2	29.6712	2.47E-02	85.4
40960	29.4437	2.58E+2	275.4	29.5700	3.14E-02	171.8
81920	29.6377	2.61E+2	552.9	29.6021	8.44E-03	347.6
	MC+CV			SB+CV		
2560	29.6524	5.29E-11	15.1	29.6524	0.00E+00	12.0
5120	29.6524	2.07E-11	30.5	29.6524	0.00E+00	23.8
10240	29.6524	3.08E-10	60.0	29.6524	0.00E+00	48.2
20480	29.6524	7.29E-10	120.1	29.6524	0.00E+00	95.7
40960	29.6524	0.00E+00	239.8	29.6524	2.65E-13	190.8
81920	29.6524	0.00E+00	480.8	29.6524	5.30E-13	383.3

Numerical Test Results

Single Asset Problems: Fixed Strike Lookback

Table : Correlations of the CVs method for fixed strike lookback options under VG models

N	$K < S_0$ d=256	$K = S_0$ d=256	$K > S_0$, (CV_1) d=16	d=64	d=256	$K > S_0(CV_2)$ d=256
1024	1.0000	1.0000	0.969664	0.974040	0.975840	1.0000
2048	1.0000	1.0000	0.971094	0.972966	0.975173	1.0000
4096	1.0000	1.0000	0.968546	0.973369	0.973334	1.0000
8192	1.0000	1.0000	0.969242	0.973054	0.974897	1.0000
16384	1.0000	1.0000	0.969785	0.972672	0.973943	1.0000
32768	1.0000	1.0000	0.970092	0.972758	0.973870	1.0000

Numerical Test Results

Single Asset Problems: Fixed Strike Lookback

Table : Comparisons of fixed strike lookback call option simulations under VG models with $d = 256$ and $K < S_0$

N	Price	Var	Time	Price	Var	Time
	MC			SB		
2560	24.0301	8.40E+01	8.8	24.1418	6.30E-01	10.0
5120	24.5274	8.81E+01	18.6	24.2296	1.87E-01	20.7
10240	24.4951	9.25E+01	37.3	24.2344	7.23E-02	41.5
20480	24.1886	8.70E+01	72.4	24.1338	3.36E-02	83.0
40960	24.3247	8.87E+01	142.7	24.3203	2.31E-02	166.1
81920	24.2559	8.73E+01	284.9	24.2710	2.04E-02	332.0
	MC+CV			SB+CV		
2560	24.2539	3.69E-11	12.5	24.2537	0.00E+00	14.0
5120	24.2539	1.15E-10	26.3	24.2537	0.00E+00	29.2
10240	24.2539	0.00E+00	52.2	24.2537	0.00E+00	58.6
20480	24.2539	0.00E+00	104.7	24.2537	0.00E+00	117.2
40960	24.2539	0.00E+00	209.1	24.2537	0.00E+00	234.2
81920	24.2539	0.00E+00	426.9	24.2537	0.00E+00	467.8

Numerical Test Results

Single Asset Problems: Floating Strike Lookback

Table : Comparisons of average VRRs for floating strike lookback option simulations under NIG models

SB	MC+CV	SB+CV
$d = 64$		
1.98E+04	1.02E+04	3.77E+07
$d = 128$		
2.56E+04	3.88E+03	9.41E+06
$d = 256$		
1.03E+04	2.32E+03	5.40E+06

Table : Comparisons of average VRRs for floating strike lookback option simulations under VG models

SB	MC+CV	SB+CV
$d = 64$		
2.68E+03	2.17E+01	6.27E+06
$d = 128$		
2.51E+03	2.30E+01	5.49E+06
$d = 256$		
4.91E+03	2.98E+01	7.33E+06

Numerical Test Results

Single Asset Problems: Up-and-Out Barrier

Table : Correlations of the CVs method for barrier options under NIG models

	$K < S_0$		$K = S_0$		$K > S_0$	
N	d=16	d=256	d=16	d=256	d=16	d=256
1024	0.999857	0.992132	0.998984	0.988849	0.99889	0.969134
2048	0.999678	0.994298	0.999532	0.984853	0.999132	0.972845
4096	0.999646	0.993947	0.999572	0.985452	0.998852	0.970132
8192	0.999785	0.994807	0.999559	0.986432	0.999146	0.97294
16384	0.999763	0.994092	0.999518	0.985321	0.99892	0.970852
32768	0.999772	0.994327	0.999493	0.98613	0.999052	0.970225

Numerical Test Results

Single Asset Problems: Up-and-Out Barrier

Table : Comparison of average VRRs for UNO barrier option simulation under the NIG model

	SB	MC+CV	SB+CV
	$d = 64$		
$K = \$90$	6.33E+03	3.71E+02	7.97E+05
$K = \$100$	6.39E+03	1.58E+02	7.19E+05
$K = \$110$	9.62E+03	6.64E+01	1.36E+05
	$d = 128$		
$K = \$90$	4.57E+03	1.82E+02	4.02E+05
$K = \$100$	5.67E+03	7.10E+01	2.33E+05
$K = \$110$	6.32E+03	2.97E+01	1.57E+05
	$d = 256$		
$K = \$90$	4.80E+03	8.78E+01	2.39E+05
$K = \$100$	4.66E+03	3.38E+01	2.15E+05
$K = \$110$	7.77E+03	1.51E+01	3.98E+04

Numerical Test Results

Single Asset Problems: Up-and-Out Barrier

Table : Correlations of the CVs method for barrier options under VG models

	$K < S_0$		$K = S_0$		$K > S_0$	
N	d=16	d=256	d=16	d=256	d=16	d=256
1024	1.000000	0.984916	1.000000	0.974035	0.948773	0.983082
2048	0.986852	1.000000	1.000000	0.984988	0.978092	0.968028
4096	0.984453	0.988920	0.987632	0.990459	0.989998	0.975124
8192	0.987246	0.982182	0.986800	0.983695	0.986746	0.983926
16384	0.988499	0.989701	0.982374	0.983141	0.985473	0.980684
32768	0.992156	0.990017	0.992064	0.987797	0.983229	0.982404

Numerical Test Results

Single Asset Problems: Up-and-Out Barrier

Table : Comparisons of average VRRs for up-and-out barrier option simulation under VG models

	SB	MC+CV	SB+CV
		$d = 64$	
$K < S_0$	3.81E+03	N/A	1.98E+07
$K = S_0$	4.86E+01	7.56E+03	N/A
$K > S_0$	2.28E+03	4.54E+01	9.02E+06
		$d = 128$	
$K < S_0$	2.52E+03	1.20E+12	1.22E+07
$K = S_0$	3.68E+03	4.94E+01	1.21E+07
$K > S_0$	1.95E+03	3.36E+01	8.96E+06
		$d = 256$	
$K < S_0$	2.85E+03	4.30E+01	8.93E+06
$K = S_0$	2.18E+03	3.92E+01	5.75E+06
$K > S_0$	2.67E+03	3.21E+01	5.27E+06

Summary

- Single asset: new control variates are constructed (lookback and barrier options)

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Summary

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- Randomized quasi-Monte Carlo methods are applied
- Very significant variance reductions achieved
- Further improvement?
  - Use of multivariate CVs
  - Combining with other variance reduction methods: importance sampling, bridge sampling etc.
  - Use of GPU computation

Thank you!

Mean estimation for CV

Table : Control variate mean estimation for fix strike lookback option, $K = 90$

d	μ	var	time
16	30.54083441	1.7594E-05	340.23
32	31.39290350	1.7678E-05	1273.33
64	31.99675901	1.7699E-05	1273.33
128	32.46988045	1.7790E-05	2545.14
256	32.77231957	1.7833E-05	4999.43

Numerical results: Tests for other parameter sets

- Also tested for other parameter sets $\{\alpha, \beta, \delta, \mu\}$ calibrated from historical stock prices

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  - Google stocks: $\{28.8279, 3.89676, 0.000514679, 0.0149508\}$
    IBM stocks: $\{51.1663, -10.299, 0.00104061, 0.0049899\}$
    Yahoo stocks: $\{18.068, -2.59218, 0.00280093, 0.0103147\}$

Numerical results: Tests for other parameter sets

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    Yahoo stocks: $\{18.068, -2.59218, 0.00280093, 0.0103147\}$
- The results are similar to the previous case, even reached higher VRRs on average.