Asian Options under Multiscale Stochastic Volatility

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Abstract. We study the problem of pricing arithmetic Asian options when the underlying is driven by stochastic volatility models with two well-separated characteristic time scales. The inherently path-dependent feature of Asian options can be efficiently treated by applying a change of numeraire, introduced by Vercer. In our previous work on pricing Asian options, the volatility is modeled by a fast mean-reverting process. A singular perturbation expansion is used to derive an approximation for option prices. In this paper, we consider an additional slowly varying volatility factor so that the pricing partial differential equation becomes four-dimensional. Using the singular-regular perturbation technique introduced by Fouque-Papanicolaou-Sircar-Solna, we show that the four-dimensional pricing partial differential equation can be approximated by solving a pair of one-dimensional partial differential equations, which takes into account the full term structure of implied volatility.

1. Introduction

Asian options are known as path dependent options whose payoffs depend on the average stock price and a fixed or floating strike price during a specific period of time before maturity. The problem of pricing arithmetic Asian options under a stochastic volatility environment has been studied by Fouque and Han [**3**], where only one fast mean-reverting volatility factor was considered in their model. However, it is well documented in empirical studies that two-factor stochastic volatility models can produce the observed kurtosis, fat-tailed return distributions and long memory effect. For example, Alizadeh *et al.* [**1**] used ranged-based estimation to indicate the existence of two volatility factors including one highly persistent factor and one quickly mean-reverting factor. Chernov *et al.* [**2**] used the efficient method of moments (EMM) to calibrate multiple stochastic volatility factors and jump components. One of their main results is that two factors are necessary for log-linear models. A recent paper by Molina *et al.* [**7**] used Markov Chain Monte Carlo (MCMC) methods to show the appearance of two well separated time scales in foreign exchange data.

From the view point of derivatives evaluation, Fouque *et al.* [**5**] used a combination of singular and regular perturbations to approximate option prices. Within this

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methodology, they observed that the introduction of a short and a long time scales indeed provides a much better fit on the term structure of implied volatility. This is done by comparing to their previous work [**4**]. In [**5**], the fixed-strike arithmetic Asian call option was considered and the price approximation was carried out to solve a pair of two-dimensional PDEs. Our goal in this paper is to show that the capability of dimension reduction techniques presented in [**3**] can be extended to multiscale stochastic volatility models.

Based on the results in Section 3, the approximated price, or so-called corrected price, does not depend on estimates of the current level of the unobservable stock price volatility. All the parameters we need to compute the approximated price can be easily calibrated from the observed historical stock prices and the implied volatility surface. Thus, this article describes a robust procedure to correct Asian option prices by taking the observed implied volatility skew into account. Since there is no close-form solution for Asian option prices, numerical computation of the corrected Asian option price is certainly needed.

This paper is organized as follows. Section 2 contains the introduction of multiscale stochastic volatility models, and a review of the Asian option pricing problem and its asymptotics. The dimension reduction technique is applied to derive the Asian option pricing PDEs, and their asymptotics are presented in Section 3. Calibration of the relevant parameters from the implied volatility surface is discussed in Section 4. Seasoned Asian options prices and Asian Put-Call parity are presented in Section 5. Numerical illustration are presented in Section 6 and the conclusion is in Section 7.

2. Multiscale Stochastic Volatility Models

We consider a family of stochastic volatility models (S_t, Y_t, Z_t), where S_t is the underlying price, Y_t evolves as an Ornstein-Uhlenbeck (OU) process, as a prototype of an ergodic diffusion, and (Z_t) follows another diffusion process. To be specific, under the physical probability measure P, our model can be written as

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma_t S_t dW_t, \\ \sigma_t &= f(Y_t, Z_t), \\ dY_t &= \alpha(m - Y_t)dt + \beta(\rho_1 dW_t + \rho'_S dZ_{S,t}), \qquad (2.1) \\ dZ_t &= \delta c(Z_t)dt + \sqrt{\delta} g(Z_t)(\rho_2 dW_t + \tilde{\rho}_{12} dZ_{S,t} + \rho'_L dZ_{L,t}). \qquad (2.2) \end{aligned}$$

where $(W_t, Z_{S,t}, Z_{L,t})$ are independent standard Brownian motions, and the constant coefficients ρ'_S and ρ'_L are defined by $\rho'_S = \sqrt{1-\rho_1^2}$ and $\rho'_L = \sqrt{1-\rho_2^2-\tilde{\rho}_{12}^2}$, where the instant correlations ρ_1, ρ_2, and $\tilde{\rho}_{12}$ satisfy $|\rho_1| < 1$ and $|\rho_2^2 + \tilde{\rho}_{12}^2| < 1$ respectively. The stock price S_t evolves as a diffusion with a constant μ in the drift and the random process σ_t in the volatility. The volatility factor Y_t evolves with a long-run mean m, a rate of mean reversion $\alpha > 0$, and a "volatility of the volatility" β. The other volatility factor Z_t evolves as a general diffusion process with a time scale δ, where we assume that the functions $c(z)$ and $g(z)$ in equation (2.2) are smooth and at most linearly growing at infinitely.

To incorporate two characteristic time scales, namely one short (fast) and the other long (slow), into the stochastic volatility models, we assume that α is large and δ small. That is, the two characteristic time scales $1/\alpha$ and $1/\delta$ correspond to a fast varying mean-reverting process Y_t and a slowly varying diffusion process Z_t, as

shown in (2.1) and (2.2), respectively. These time scales are meant to be relatively fast or slow by comparing to time to maturities of contracts. In order to perform asymptotic analysis, we introduce a small parameter $0 < \varepsilon << 1$ such that the rate of mean reversion defined by $\alpha = 1/\varepsilon$ becomes large. To capture the volatility clustering behavior, we define $\nu^2 = \beta^2/2\alpha$, the long-run variance of Y_t, and we assume that it is a fixed $\mathcal{O}(1)$ constant.
Under the pricing risk-neutral probability measure P^*, our model becomes

$$\begin{aligned}
(2.3)\ dS_t &= rS_t dt + f(Y_t, Z_t)S_t dW_t^*, \\
(2.4)\ dY_t &= \left(\frac{1}{\epsilon}(m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\Lambda(Y_t, Z_t)\right) dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\left(\rho_1 dW_t^* + \rho_S' dZ_{S,t}^*\right), \\
(2.5)\ dZ_t &= \left(\delta c(Z_t) - \sqrt{\delta} g(Z_t)\Gamma(Y_t, Z_t)\right) dt \\
&\quad + \sqrt{\delta} g(Z_t)\left(\rho_2 dW_t^* + \tilde{\rho}_{12} dZ_{S,t}^* + \rho_L' dZ_{L,t}^*\right),
\end{aligned}$$

where W_t^*, $Z_{S,t}^*$, and $Z_{L,t}^*$ are independent standard Brownian motions. The small parameter ϵ corresponds to the fast scale and the large parameter δ^{-1} corresponds to the slow scale. The combined market prices of volatility risk associated with Y_t and Z_t are

$$\begin{aligned}
\Lambda(y, z) &= \frac{\rho_1(\mu - r)}{f(y, z)} + \gamma(y, z)\sqrt{1 - \rho_1^2} \\
\Gamma(y, z) &= \frac{\rho_2(\mu - r)}{f(y, z)} + \gamma(y, z)\tilde{\rho}_{12} + \xi(y, z)\sqrt{1 - \rho_2^2 - \tilde{\rho}_{12}^2}.
\end{aligned}$$

We assume that the risk-free interest rate r is constant and that the market prices of volatility risks $\gamma(y, z)$ and $\xi(y, z)$ are bounded and depend only on the volatility levels y and z. At the leading order $1/\varepsilon$ in (2.4), that is omitting the Λ-term in the drift, Y_t is an OU process which is fast mean-reverting with a normal invariant distribution $N(m, \nu^2)$. The volatility factor Y_t fluctuates randomly around its mean level m and the long run magnitude ν of volatility remains fixed for values of ε. The other volatility factor Z fluctuates slowly on a long time-scale of order δ^{-1}. Furthermore, due to the presence of other sources of noise modeled by the Brownian motions Z_S and Z_L , there exists a (γ, ξ)-dependent family of equivalent risk-neutral measures. However, we assume that the market chooses one measure P^* through the market price of volatility risk (γ, ξ).

2.1. Asian Option Pricing PDE and its Asymptotics. The usual way to deal with the continously sampled arithmetic average Asian option problem is to introduce a new process

$$I_t = \int_0^t S_s ds, \tag{2.6}$$

which represents the running sum stock process. Here we assume the stochastic volatility model obeys (2.3, 2.4, 2.5) in addition to the differential form of (2.6), i.e.

$$dI_t = S_t dt,$$

with the initial condition $I_0 = 0$. Under the risk-neutral probability measure P^* the joint process (S_t, Y_t, Z_t, I_t) is a Markov process. The equation (2.6) remains unchanged under the change of measure. Thus, we define the price of the Asian floating-strike call option at time $0 \le t \le T$ by

$$P^{\varepsilon,\delta}(t,s,y,z,I) = E^* \left\{ e^{-r(T-t)} \left(S_T - \frac{I_T}{T} \right)^+ \mid S_t = s, Y_t = y, Z_t = z, I_t = I \right\}.$$

From the Feynman-Kac formula, $P^{\epsilon,\delta}(t,s,y,z,I)$ solves

$$\begin{aligned} \mathcal{L}^{\varepsilon,\delta} P^{\varepsilon,\delta} &= 0, \\ P^{\varepsilon,\delta}(T,s,y,z,I) &= \left(s - \frac{I}{T} \right)^+, \end{aligned} \tag{2.7}$$

where we define the partial differential operator $\mathcal{L}^{\varepsilon,\delta}$ by

$$\mathcal{L}^{\varepsilon,\delta} = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3,$$

and each component operator is given by

$$\mathcal{L}_0 = \nu^2 \frac{\partial^2}{\partial y^2} + (m-y)\frac{\partial}{\partial y} \tag{2.8}$$

$$\mathcal{L}_1 = \rho_1 \sqrt{2}\nu s f(y,z)\frac{\partial^2}{\partial s \partial y} - \sqrt{2}\nu\Lambda(y,z)\frac{\partial}{\partial y} \tag{2.9}$$

$$\mathcal{L}_2(f(y,z)) = \frac{\partial}{\partial t} + \frac{f^2(y,z)}{2}s^2\frac{\partial^2}{\partial s^2} + r(s\frac{\partial}{\partial s} - \cdot) + s\frac{\partial}{\partial I} \tag{2.10}$$

$$\mathcal{M}_1 = -g(z)\Gamma(y,z)\frac{\partial}{\partial z} + \rho_2 g(z) f(y,z) s \frac{\partial^2}{\partial s \partial z} \tag{2.11}$$

$$\mathcal{M}_2 = c(z)\frac{\partial}{\partial z} + \frac{g(z)^2}{2}\frac{\partial^2}{\partial z^2}, \tag{2.12}$$

$$\mathcal{M}_3 = \nu\sqrt{2}\rho_{12} g(z)\frac{\partial^2}{\partial y \partial z}, \tag{2.13}$$

where the correlation parameter ρ_{12} is defined by $\rho_{12} = \rho_1\rho_2 + \tilde{\rho}_{12}\sqrt{1-\rho_1^2}$. The price of the Asian option is obtained at any given current time t, stock price S_t, volatility levels Y_t and Z_t, and cumulated stock price I_t, by solving the linear PDE (2.7) . However, this pricing PDE is four-dimensional in space and any numerical PDE scheme to solve it requires significant computation efforts.

Fouque *et al.* derived in [**5**] a y-independent approximated Asian floating-strike call option price $Q(t,s,z,I)$ as the sum of two terms $Q_0(t,s,z,I)$ and $\tilde{Q}_1(t,s,z,I)$. The zeroth order price $Q_0(t,s,z,I)$ solves the two-dimensional PDE

$$\mathcal{L}_2(\overline{\sigma}(z))Q_0 = 0, \tag{2.14}$$

with the terminal condition

$$Q_0(T,s,z,I) = \left(s - \frac{I}{T} \right)^+.$$

It corresponds to the price of the option under the Black-Scholes model with a volatility $\overline{\sigma}(z)$ independent of s. The correction $\tilde{Q}_1(t,s,z,I)$ solves the same two-dimensional PDE with a zero terminal condition but with a source term:

$$\mathcal{L}_2(\overline{\sigma}(z))\tilde{Q}_1 = \mathcal{L}_S Q_0, \tag{2.15}$$

where the operator $\mathcal{L}_S$ is given by

$$\mathcal{L}_S = \frac{2}{\overline{\sigma}(z)}\left[V_0^\delta \frac{\partial}{\partial \sigma} + V_1^\delta s \frac{\partial^2}{\partial s \partial \sigma}\right] + \left[V_2^\varepsilon s^2 \frac{\partial^2}{\partial s^2} + V_3^\varepsilon s^3 \frac{\partial^3}{\partial s^3}\right]. \tag{2.16}$$

The z-dependent effective volatility $\overline{\sigma}(z)$ is defined by

$$\overline{\sigma}(z)^2 = \int f(y,z)^2 d\Pi(y),$$

where Π denotes the invariant measure of the OU-process with infinitesimal generator $\mathcal{L}_0$, that is a $\mathcal{N}(m,\nu^2)$ distribution. In practice $\overline{\sigma}(z)^2$ is estimated from historical stock returns over a period of time of order one (shorter than δ^{-1}) so that the z-dependence is automatically incorporated in the estimate. The small parameters V_0^δ, V_1^δ, V_2^ε and V_3^ε are calibrated from the term structure of implied volatility as explained in Section 4.

3. Dimension Reduction: Asian Option Problems with Multiscale Stochastic Volatility and their Asymptotics

Our goal in this section is two-fold. First, we show that the dimension reduction technique presented in our previous work [**3**] can be extended to the Asian option pricing problem when the volatility is driven by multiscale volatilty. The case of discretely sampled average stock prices is also considered. As a result, the four-dimensional pricing PDE (2.7) can be reduced to three-dimensional. Second, we carry out the singular-regular perturbation technique to derive the approximation expansion for Asian option prices. These asymptotics reduces the problem to a pair of one-dimensional PDEs, and therefore simplifies the pair of two-dimensional PDEs (2.14, 2.15).

3.1. Three-Dimensional Pricing PDE. Because of the path dependent nature of the payoff, it is important to distinguish whether the Asian option contract already starts. When the current time t is exactly at the contract starting date 0, the Asian option is called "fresh". When the current time t is between the contract starting date 0 and maturity date T, it is "seasoned." For ease of exposition, we limit our discussion to the case Asian option is fresh. The seasoned case and the Asian put-call parity will be considered in Section 5.
The general payoff function of arithmetic average Asian options is

$$h\left(\int_0^T S_t d\lambda(t) - K_1 S_T - K_2\right),$$

where the sampling function λ has finite variation. For the case of continuously sampled Asian option, $\lambda(t)$ is chosen as $\lambda(t) = \frac{t}{T}$, and for the case of discretely sampled Asian option we have $\lambda(t) = \frac{1}{n}\lfloor \frac{nt}{T} \rfloor$ where $\lfloor\cdot\rfloor$ denotes the integer part.
For the pricing of Asian options, the idea of dimension reduction technique introduced by Vecer [**8**] and later by Vecer and Xu [**9**] is to construct a wealth or portfolio process, which can replicate the stock price average by self-financing trading in the stock and bond when the stock is modeled by some semimartingale processes. Fouque and Han [**3**] generalize their result to one factor stochastic volatility when the characteristic time scale is fast and the sampling function $\lambda(t)$ is continuous.
To include the discrete-sampled scenario within the multiscale stochastic volatility

model, we use a time-dependent trading strategy function introduced in [**9**] and given by

$$q_t = e^{-rt} \int_t^T e^{rs} d\lambda(s). \tag{3.1}$$

In this strategy the finite-variation process q_t is the number of units held at time t of the underlying stock. Since the price of the bond at time t is e^{rt}, the quantity $(X_t - q_t\, S_t)e^{-rt}$ is the number of units held in bonds. We assume this portfolio is to be self-financing so that the variation of the wealth process can be expressed in differential form as

$$\begin{aligned} dX_t &= q_{t-} dS_t + (X_{t-} - q_{t-}\, S_t)e^{-rt} d(e^{rt}) \\ &= q_{t-} dS_t + +r(X_{t-} - q_{t-} S_t)dt. \end{aligned} \tag{3.2}$$

From Proposition 2.2 in [**9**] or a straight forwardly extension of [**3**], the payoff of Asian contract can be replicated by X_T, namely

$$X_T = \int_0^T S_t d\lambda(t) - K_2,$$

if the initial wealth is chosen equal to

$$X_0 = q_0 S_0 - e^{-rT} K_2. \tag{3.3}$$

Hence the general payoff function for arithmetic average Asian options can be described as

$$h\left(\int_0^T S_t d\lambda(t) - K_1 S_T - K_2\right) = h(X_T - K_1 S_T). \tag{3.4}$$

When $K_1 = 0$, we have a fixed strike Asian option; when $K_2 = 0$, we have the floating strike Asian option. The price $P^{\varepsilon,\delta}(0, s, y, z; T, K_1, K_2)$ of an arithmetic average Asian option with multiscale stochastic volatility, is given by

$$P^{\varepsilon,\delta}(0, s, y, z; T, K_1, K_2) = e^{-rT} E^*\{h(X_T - K_1 S_T) \mid S_0 = s, Y_0 = y, Z_0 = z\}, \tag{3.5}$$

where (S_t, Y_t, Z_t, X_t) follow (2.3), (2.4), (2.5) and (3.2), respectively, under the pricing risk-neutral measure P^*.

By change of numeraire

$$\psi_t = \frac{X_t}{S_t}, \tag{3.6}$$

and from Ito's formula, the dynamics of this numeraire process is given by

$$d\psi_t \;=\; (q_{t-} - \psi_t) f(Y_t) d\tilde{W}_t^{\,*}, \tag{3.7}$$

where the shifted Brownian motion $\tilde{W}_t^*$ is defined by

$$\tilde{W}_t^{\,*} = W_t^* - \int_0^t f(Y_s, Z_s) ds. \tag{3.8}$$

By Girsanov Theorem, under the probability measure $\tilde{P}^*$ defined by

$$\begin{aligned} \frac{d\tilde{P}^*}{dP^*} &= e^{-rT} \frac{S_T}{S_0} \\ &= \exp\left(\int_0^T f(Y_t, Z_t) dW_t^* - \frac{1}{2}\int_0^T f(Y_t, Z_t)^2 dt\right), \end{aligned}$$

the process $\tilde{W}_t^*$ given by (3.8) becomes a standard Brownian motion. Hence, the driving volatility processes can be expressed as

$$
\begin{aligned}
(3.9)\qquad dY_t &= \left[\frac{1}{\varepsilon}(m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}\left(\Lambda(Y_t, Z_t) - \rho_1 f(Y_t, Z_t)\right)\right] dt \\
&\quad + \frac{\nu\sqrt{2}}{\sqrt{\varepsilon}}(\rho_1 d\tilde{W}_t^* + \rho_S' \, dZ_{S,t}^*), \\
(3.10)\qquad dZ_t &= \left[\delta c(Z_t) - \sqrt{\delta}\left(g(Z_t)\Gamma(Y_t, Z_t) - \rho_2 g(Z_t) f(Y_t, Z_t)\right)\right] dt \\
&\quad + \sqrt{\delta} g(Z_t)\left(\rho_2 d\tilde{W}_t^* + \tilde{\rho}_{12} dZ_{S,t}^* + \rho_L' dZ_{L,t}^*\right).
\end{aligned}
$$

We assume that the payoff function h satisfies the homogeneous property, i.e.

$$h(xy) = xh(y),$$

for each nonnegative x. Payoffs of calls and puts are typical examples. When $t = 0$, the Asian option price (3.5) becomes

$$
\begin{aligned}
& sE^*\left\{e^{-rT}\frac{S_T}{S_0} h\left(\frac{X_T}{S_T} - K_1\right) \mid S_0 = s, Y_0 = y, Z_0 = z\right\} \\
&= s\tilde{E}^*\{h(\psi_T - K_1) \mid \psi_0 = \psi, Y_0 = y, Z_0 = z\},
\end{aligned}
$$

where, by using (3.3), we have

$$\psi = \frac{x}{s} = q(0) - e^{-rT}\frac{K_2}{s}.$$

We define the quantity of interest $u^{\varepsilon,\delta}$ by

$$(3.11)\qquad u^{\varepsilon,\delta}(0, \psi, y, z; T, K_1, K_2) \equiv \tilde{E}^*\{h(\psi_T - K_1) \mid \psi_0 = \psi, Y_0 = y, Z_0 = z\},$$

such that the Asian option price (3.5) can be expressed as

$$(3.12)\qquad P^{\varepsilon,\delta}(0, s, y, z; T, K_1, K_2) = su^{\varepsilon,\delta}(0, \psi, y, z; T, K_1, K_2).$$

Note that from (3.7) and (3.9) the joint process (ψ_t, Y_t, Z_t) is Markovian. If, for $t \le T$, we introduce

$$u^{\varepsilon,\delta}(t, \psi, y, z; T, K_1, K_2) \equiv \tilde{E}^*\{h(\psi_T - K_1) \mid \psi_t = \psi, Y_t = y, Z_t = z\},$$

then by an application of the Feynman-Kac formula, $u^{\varepsilon,\delta}$ solves

$$(3.13)\qquad \left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \hat{\mathcal{L}}_2 + \sqrt{\delta}\hat{\mathcal{M}}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3\right) u^{\varepsilon,\delta} = 0,$$

with the terminal condition $u^{\varepsilon,\delta}(T, \psi, y, z) = h(\psi - K_1)$. The partial differential operators $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{M}_2$, and $\mathcal{M}_3$ are given the same as (2.8, 2.9, 2.12, 2.13), and $\hat{\mathcal{L}}_2$ and $\hat{\mathcal{M}}_1$ are given by

$$
\begin{aligned}
\hat{\mathcal{L}}_2(f(y,z)) &= \frac{\partial}{\partial t} + \frac{1}{2}(\psi - q_{t-})^2 f(y,z)^2 \frac{\partial^2}{\partial \psi^2}, \\
\hat{\mathcal{M}}_1 &= -\left(g(z)\Gamma(y,z) - \rho_2 g(z) f(y,z)\right)\frac{\partial}{\partial z} + \rho_2 g(z) f(y,z)(q_{t-} - \psi)\frac{\partial^2}{\partial x \partial z}
\end{aligned}
$$

It is remarkable that the PDE (3.13) has one less spatial dimension than (2.7). However, the solution of this backward PDE can only be the price of the fresh Asian option when the current time $t = 0$. The same PDE may not be a pricing equation for the seasoned Asian option, which is discussed in Section 5.

3.2. Asymptotics. We expand the solution $u^{\varepsilon,\delta}$ of (3.13) in powers of $\sqrt{\delta}$

$$u^{\varepsilon,\delta}(t,\psi,y,z) = u_0^\varepsilon(t,\psi,y,z) + \sqrt{\delta}u_1^\varepsilon(t,\psi,y,z) + \delta u_2(t,\psi,y,z) + \cdots \tag{3.14}$$

and substitute into (3.13) to obtain

$$\begin{aligned} 0 &= \left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \hat{\mathcal{L}}_2\right) u_0^\varepsilon \\ &+\sqrt{\delta}\left(\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \hat{\mathcal{L}}_2\right) u_1^\varepsilon + \hat{\mathcal{M}}_1 u_0^\varepsilon + \frac{1}{\sqrt{\varepsilon}}\mathcal{M}_3 u_0^\varepsilon\right) + \cdots . \end{aligned}$$

The leading order term u_0^ε solves the problem

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \hat{\mathcal{L}}_2\right) u_0^\varepsilon = 0,$$

with the terminal condition $u_0^\varepsilon = h(\psi - K_1)$. Performing the singular perturbation detailed in [**4**] we obtain the following approximation

$$u_0^\varepsilon \approx u_0(t,\psi,z) + \tilde{u}_{1,0}(t,\psi,z) \tag{3.15}$$

where the leading order term $u_0(t,\psi,z)$ solves

$$\begin{aligned} < \hat{\mathcal{L}}_2 > u_0 &= \frac{\partial u_0}{\partial t} + \frac{1}{2}(\psi - q_{t-})^2 < f(y,z)^2 > \frac{\partial^2 u_0}{\partial \psi^2} = 0, \\ u_0(T,\psi,z) &= h(\psi - K_1), \end{aligned} \tag{3.16}$$

and the correction $\tilde{u}_{1,0}(t,\psi,z) \equiv \sqrt{\varepsilon}u_{1,0}(t,\psi,z)$ solves

$$\begin{aligned} < \hat{\mathcal{L}}_2 > \tilde{u}_{1,0} &= \mathcal{A}u_0, \\ \tilde{u}_{1,0}(T,\psi,z) &= 0. \end{aligned} \tag{3.17}$$

The operator $\mathcal{A}$ is defined by

$$\mathcal{A} = \overline{V}_2(z)(q_{t-} - \psi)^2 \frac{\partial^2}{\partial \psi^2} + \overline{V}_3(z)(q_{t-} - \psi)^3 \frac{\partial^3}{\partial \psi^3},$$

where the z-dependent functions $\overline{V}_2$ and $\overline{V}_3$ are given by

$$\overline{V}_2(z) = \frac{\nu\sqrt{\varepsilon}}{\sqrt{2}}\left(-\rho_1 < f(y,z)\frac{\partial \phi(y,z)}{\partial z} > - < \Lambda(y,z)\frac{\partial \phi(y,z)}{\partial z} >\right), \tag{3.18}$$

$$\overline{V}_3(z) = \frac{\rho_1\nu\sqrt{\varepsilon}}{\sqrt{2}} < f(y,z)\frac{\partial \phi(y,z)}{\partial z} > . \tag{3.19}$$

We consider next the second $u_1^\varepsilon(t,\psi,y,z)$ in the expansion (3.14): it solves

$$\left(\frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \hat{\mathcal{L}}_2\right) u_1^\varepsilon = -\left(\hat{\mathcal{M}}_1 + \frac{1}{\sqrt{\varepsilon}}\mathcal{M}_3\right) u_0^\varepsilon \tag{3.20}$$

with the zero terminal condition. Similarly, we look for the solution whose expansion is given by

$$u_1^\varepsilon(t,\psi,y,z) = u_{0,1}(t,\psi,y,z) + \sqrt{\varepsilon}u_{1,1}(t,\psi,y,z) + \varepsilon u_{2,1}(t,\psi,y,z) + \cdots .$$

Substituting this expansion into the PDE (3.20) and using the expansion (3.15) for u_0^ε, it follows that $u_{0,1}$, $u_{1,1}$, and $u_{2,1}$ solve the following PDEs

$$\begin{aligned} &\mathcal{L}_0 u_{0,1} = 0, \\ &\mathcal{L}_1 u_{0,1} + \mathcal{L}_0 u_{1,1} = -\mathcal{M}_3 u_0 = 0, \\ &\hat{\mathcal{L}}_2 u_{0,1} + \mathcal{L}_1 u_{1,1} + \mathcal{L}_0 u_{2,1} = -\hat{\mathcal{M}}_1 u_0. \end{aligned}$$

We conclude that $u_{0,1}$ and $u_{1,1}$ are independent of the variable y, and $u_{0,1}$ solves

$$< \hat{\mathcal{L}}_2 > u_{0,1} = - < \hat{\mathcal{M}}_1 > u_0,$$

where the homogenized partial differential operator $< \hat{\mathcal{M}}_1 >$ is written as

$$\begin{aligned} < \hat{\mathcal{M}}_1 > &= \Big(-\left(g(z) < \Gamma(y,z) > -\rho_2 g(z) < f(y,z) >\right) \\ &\quad +\rho_2 g(z) < f(y,z) > (q_{t-} - \psi)\frac{\partial}{\partial \psi}\Big)\,\overline{\sigma}'\frac{\partial}{\partial \sigma}, \end{aligned}$$

and $\overline{\sigma}'$ denotes the derivative with respect to z. We define $\tilde{u}_{0,1} = \frac{\sqrt{\delta}}{2}u_{0,1}$, so that $\tilde{u}_{0,1}(t,\psi,z)$ solves

$$\begin{aligned} < \hat{\mathcal{L}}_2 > \tilde{u}_{0,1} &= \mathcal{B}u_0, \\ \tilde{u}_{0,1}(T,\psi,z) &= 0. \end{aligned} \tag{3.21}$$

The differential operator $\mathcal{B}$ is defined as

$$\mathcal{B} = \frac{1}{\overline{\sigma}}\left(\overline{V}_0^{\delta}\frac{\partial}{\partial \sigma} + \overline{V}_1^{\delta}(q_{t-} - \psi)\frac{\partial^2}{\partial\psi\partial\sigma}\right),$$

where

$$\begin{aligned} \overline{V}_0^{\delta} &= \frac{\sqrt{\delta}}{2}\left(g(z) < \Gamma(y,z) > -\rho_2 g(z) < f(y,z) >\right)\overline{\sigma}\,\overline{\sigma}', \\ \overline{V}_1^{\delta} &= -\frac{\sqrt{\delta}}{2}\rho_2 g(z) < f(y,z) > \overline{\sigma}\,\overline{\sigma}'. \end{aligned}$$

To summarize we have obtained that

$$u^{\varepsilon,\delta}(t,\psi,y,z) = u_0(t,\psi,z) + \tilde{u}_{1,0}(t,\psi,z) + \tilde{u}_{0,1}(t,\psi,z) + \mathcal{O}(\varepsilon + \delta + \sqrt{\varepsilon\,\delta}),$$

where u_0 solves (3.16), $\tilde{u}_{1,0}$ is of order $\mathcal{O}(\sqrt{\varepsilon})$ and solves (3.17), and $\tilde{u}_{0,1}$ is of order $\mathcal{O}(\sqrt{\delta})$ and solves (3.21). Consequently, according to (3.12), the price of a fresh Asian option is approximated by

$$P^{\varepsilon,\delta}(0,s,y,z) = su_0(0,\psi,z) + s\tilde{u}_{1,0}(0,\psi,z) + s\tilde{u}_{0,1}(0,\psi,z) + \mathcal{O}(\varepsilon + \delta + \sqrt{\varepsilon\,\delta}).$$

The accuracy of this approximation is obtained in the case where the payoff function h being smooth and bounded by a straight forward generalization of Theorem 3.6 in [**5**]. For the case of h corresponding to a call or a put, the order of accuracy is $\mathcal{O}(\varepsilon \log|\varepsilon|)$; we refer to the discussion in [**3**] and skip it here.

4. Implied Volatilities and Calibration

When volatility comprises fast mean-reverting and a slowly varying diffusion processes on time-scales respectively smaller and larger than typical maturities, one can apply the asymptotic analysis on the pricing PDEs (2.7) and (3.13), in order to obtain an approximated price. We find out that the quantities of interest derived through the analysis depend on the same parameters than in the approximated European option prices. Thus, in this section we describe a robust procedure to correct Black-Scholes Asian option prices to account for the observed implied volatility skew. The methodology is to observe both the underlying stock prices and the European option prices, which is encapsulated in the skew surface, such that the Asian option price under the stochastic volatility environment can be calculated.

4.1. Review of Vanilla European Options Asymptotics and Calibration: Multi-Scales. We give here a brief review of the main results in [**5**] from the asymptotic analysis of the European options problem under multiscale stochastic volatility model as presented in (3.9, 3.10). Let $P^{\varepsilon,\delta}$ be the price of a European option which solves

$$\mathcal{L}^{\varepsilon,\delta}P^{\varepsilon,\delta} = 0,$$
$$P^{\varepsilon,\delta}(T,s,y,z) = h(s),$$

where we denote by $\mathcal{L}^{\varepsilon,\delta}$ the partial differential operator defined by

$$\mathcal{L}^{\varepsilon,\delta} = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_{BS} + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3,$$

and these differential operators share the same definition as listed in (2.8 - 2.13) except for $\mathcal{L}_{BS}$ which is defined by

$$\mathcal{L}_{BS}(f(y,z)) = \frac{\partial}{\partial t} + \frac{f^2(y,z)}{2}s^2\frac{\partial^2}{\partial s^2} + r(s\frac{\partial}{\partial s} - \cdot).$$

Recall the approximating results from Fouque *et al.* [**5**]. They use a singular-regular perturbation technique to derive an explicit formula for the price approximation:

$$P^{\varepsilon,\delta}(t,s,y,z) \approx P_0(t,s,z) - (T-t)\left(\mathcal{A}^\varepsilon + \mathcal{B}^\delta\right)P_0(t,s,z),$$

where the partial differential operator $\mathcal{A}^\varepsilon$ and $\mathcal{B}^\delta$ are given by

$$\begin{aligned}\mathcal{A}^\varepsilon &= V_2^\varepsilon s^2\frac{\partial^2}{\partial s^2} + V_3^\varepsilon s^3\frac{\partial^3}{\partial s^3},\\ \mathcal{B}^\delta &= \frac{1}{\overline{\sigma}}\left[V_0^\delta\frac{\partial}{\partial\sigma} + V_1^\delta s\frac{\partial^2}{\partial s\partial\sigma}\right],\end{aligned}$$

and the relevant parameters are defined by

$$V_2^\varepsilon = \frac{\nu\sqrt{\varepsilon}}{\sqrt{2}}\left(2\rho_1 < f(y,z)\frac{\partial\phi(y,z)}{\partial y} > - < \Lambda(y,z)\frac{\partial\phi(y,z)}{\partial y} >\right), \tag{4.1}$$

$$V_3^\varepsilon = \frac{\rho_1\nu\sqrt{\varepsilon}}{\sqrt{2}} < f(y,z)\frac{\phi(y,z)}{\partial y} >, \tag{4.2}$$

$$V_0^\delta = \frac{\sqrt{\delta}}{2}g < \Gamma > \overline{\sigma}\,\overline{\sigma}', \tag{4.3}$$

$$V_1^\delta = -\frac{\sqrt{\delta}}{2}\rho_2 g < f > \overline{\sigma}\,\overline{\sigma}'. \tag{4.4}$$

The effective volatility $\overline{\sigma}$ defined by $\overline{\sigma}^2(z) =< f^2(\cdot,z) >$, is a function of the slow factor z. The function $\phi(y,z)$ is a solution of the Poisson equation

$$\mathcal{L}_0\phi(y,z) = f^2(y,z) - \overline{\sigma}^2(z)$$

up to an additional function depending on the variable z only, which will not affect the operator $\mathcal{A}$. The leading order price $P_0(t,s,z)$ solves

$$\mathcal{L}_{BS}(\overline{\sigma}(z))P_0(t,s,z) = 0$$

with the terminal condition

$$P_0(T,s,z) = h(s).$$

Similar to the one-factor stochastic volatility models presented in [**4**], the parameters $V_2^\varepsilon, V_3^\varepsilon, V_0^\delta$, and V_1^δ can be calibrated from the implied volatility surface. It is shown in [**5**] that the implied volatility $I^{\varepsilon,\delta}$ of an European option price is approximated by

$$I^{\varepsilon,\delta} \approx \overline{\sigma} + \left[a^{\varepsilon} + a^{\delta}(T-t)\right] \frac{\log(K/s)}{T-t} + \left[b^{\varepsilon} + b^{\delta}(T-t)\right]$$

where the z-dependent parameters $a^{\varepsilon}, b^{\varepsilon}, a^{\delta}$, and b^{δ} are defined by

$$\begin{array}{ll} a^{\varepsilon} = -\frac{V_3^{\varepsilon}}{\overline{\sigma}^3}, & b^{\varepsilon} = \overline{\sigma} + \frac{V_3^{\varepsilon}}{\overline{\sigma}^3}\left(r + \frac{3}{2}\overline{\sigma}^2\right) - \frac{V_3^{\varepsilon}}{\overline{\sigma}}, \\ a^{\delta} = -\frac{V_1^{\delta}}{\overline{\sigma}^3}, & b^{\delta} = -\frac{V_0^{\delta}}{\overline{\sigma}} + \frac{V_1^{\delta}}{\overline{\sigma}^3}\left(r - \frac{\overline{\delta}^2}{2}\right). \end{array}$$

Therefore the calibration formulas

$$V_0^{\delta}/\overline{\sigma} = -\left(b^{\delta} + a^{\delta}\left(r - \frac{\overline{\sigma}^2}{2}\right)\right), \tag{4.5}$$

$$V_1^{\delta}/\overline{\sigma} = -a^{\delta}\overline{\sigma}^2, \tag{4.6}$$

$$V_2^{\varepsilon} = -\bar{\sigma}\left(b^{\varepsilon} + a^{\varepsilon}\left(r - \frac{\bar{\sigma}^2}{2}\right)\right), \tag{4.7}$$

$$V_3^{\varepsilon} = -a^{\varepsilon}\bar{\sigma}^3, \tag{4.8}$$

are deduced. It is shown in [**5**] that one factor stochastic volatility models with either a short time scale or a long time scale do not permit a good fit of the implied volatility surface over a range of maturities. However two factor models and the perturbation method summarized in this section give an excellent fit across strikes and over a wide range of maturities.

4.2. Asian Option Prices Asymptotics. As shown in Section 3.2, the price approximation for a fresh Asian option is given by

$$P^{\varepsilon,\delta}(0,s,y,z) = s u^{\varepsilon,\delta}(0,\psi,y,z) \approx s\left(u_0(0,\psi,z) + \tilde{u}_{1,0}(0,\psi,z) + \tilde{u}_{0,1}(0,\psi,z)\right).$$

The leading term $u_0(t,\psi,z)$ solves

$$< \hat{\mathcal{L}}_2 > u_0 \equiv \frac{\partial u_0}{\partial t} + \frac{1}{2}(\psi - q_{t-})^2 \overline{\sigma}(z)^2 \frac{\partial^2 u_0}{\partial \psi^2} = 0, \tag{4.9}$$

with the terminal condition $u_0(T,\psi,z) = h(\psi - K_1)$. The sum of $\tilde{u}_{1,0}$ and $\tilde{u}_{0,1}$ solves the source problem

$$\begin{aligned} < \hat{\mathcal{L}}_2 > (\tilde{u}_{1,0} + \tilde{u}_{0,1}) &= \overline{V}_2^{\varepsilon}(z)(q_{t-} - \psi)^2 \frac{\partial^2 u_0}{\partial \psi^2} + \overline{V}_3^{\varepsilon}(z)(q_{t-} - \psi)^3 \frac{\partial^3 u_0}{\partial \psi^3} \\ &+ \frac{1}{\overline{\sigma}}\left(\overline{V}_0^{\delta} \frac{\partial u_0}{\partial \sigma} + \overline{V}_1^{\delta}(q_{t-} - \psi) \frac{\partial^2 u_0}{\partial \psi \partial \sigma}\right), \end{aligned} \tag{4.10}$$

where the parameters $\left(\overline{V}_2^{\varepsilon}, \overline{V}_3^{\varepsilon}, \overline{V}_0^{\delta}, \overline{V}_1^{\delta}\right)$ are given in Section 3.2. Comparing to the group parameters $\left(V_2^{\varepsilon}, V_3^{\varepsilon}, V_0^{\delta}, V_1^{\delta}\right)$ as defined in (4.1) - (4.4) in Section 4.1, we obtain the linear relation:

$$\begin{array}{ll} \overline{V}_2^{\varepsilon} = V_2^{\varepsilon} - V_3^{\varepsilon}, & \overline{V}_3^{\varepsilon} = V_3^{\varepsilon}, \\ \overline{V}_0^{\delta} = V_0^{\delta} + V_1^{\delta}, & \overline{V}_1^{\delta} = V_1^{\delta}. \end{array}$$

Therefore by calibrating the term structure of implied volatility built from European call prices, one can easily deduce the parameters needed to approximate the price of an Asian option.

The correction for the homogenized Asian option prices (4.9) need to be solved numerically. We observe in equation (4.10) that the influence of the fast factor brings in the Greeks Gamma and Delta-Gamma in the source term, and the slow factor brings in the Vega and Delta-Vega in the source term. In Section 2.1 we have recalled that the approximated price of an Asian option, without reducing dimensions, solves a pair of two-dimensional PDEs. After adopting the dimension reduction technique, we derive that it is enough to solve a pair of one-dimensional PDEs, and therefore reducing the computational efforts significantly.

5. Seasoned Asian Option Prices and Asian Put-Call Parity

The argument for continuously-sampled seasoned Asian option prices under multiscale stochastic volatility model is identical to the one factor case discussed in [**3**], Section 3.3 . We merely provide formulas here.
Suppose we are at the current time t, which is between the Asian option contract starting date 0 and the maturity date T. Denote by $\mathcal{F}_t$ the $\sigma - algebra$ generated by the three-dimensional process $(S_u, Y_u, Z_u, 0 \leq u \leq t)$. Conditioning on $\mathcal{F}_t$, the price of the Asian call option is given by at time t

$$
\begin{aligned}
& E^* \left\{ e^{-r(T-t)} \left(\frac{1}{T} \int_0^T S_u du - K_1 S_t - K_2 \right)^+ \mid \mathcal{F}_t \right\} \\
&= E^* \left\{ e^{-r(T-t)} \left(\frac{1}{T} \int_0^T S_u du - K_1 S_t - K_2 \right)^+ \mid S_t = s, Y_t = y, Z_t = z, I_t = I \right\} \\
&= \frac{\tau}{T} E^* \left\{ e^{-r\tau} \left(\frac{1}{\tau} \int_0^\tau S_t dt - \hat{K}_1 S_\tau - \hat{K}_2 \right)^+ \mid S_0 = s, Y_0 = y, Z_0 = z \right\} \\
&\equiv \frac{\tau}{T} P^{\varepsilon,\delta}_{call}(0, s, y, z; \tau, \hat{K}_1, \hat{K}_2),
\end{aligned}
$$

where we denote by $\tau = T - t$ the time to maturity, and the updated strikes $\hat{K}_1$ and $\hat{K}_2$ are defined by: $\hat{K}_1 = \frac{T}{\tau} K_1$ and $\hat{K}_2 = \frac{T}{\tau} K_2 + \frac{1}{\tau} I$.
Similarly, for Asian put options, we obtained

$$
\begin{aligned}
& \frac{\tau}{T} P^{\varepsilon}_{put}(0, s, y; \tau, \hat{K}_1, \hat{K}_2) = \\
& \frac{\tau}{T} E^* \left\{ e^{-r\tau} \left(\frac{1}{\tau} \int_0^\tau S_t dt - \hat{K}_1 S_\tau - \hat{K}_2 \right)^- \mid S_0 = s, Y_0 = y, Z_0 = z \right\}.
\end{aligned}
$$

A simple computation gives

$$
\begin{aligned}
& \frac{\tau}{T} P^{\varepsilon}_{call}(0, s, y; \tau, \hat{K}_1, \hat{K}_2) + \frac{\tau}{T} P^{\varepsilon}_{put}(0, s, y; \tau, \hat{K}_1, \hat{K}_2) \\
= \;& \frac{\tau}{T} E^* \left\{ e^{-r\tau} \left(\frac{1}{\tau} \int_0^\tau S_t dt - \hat{K}_1 S_\tau - \hat{K}_2 \right) \mid S_0 = s, Y_0 = y, Z_0 = z \right\} \\
= \;& \frac{s}{T} \frac{1 - e^{-r\tau}}{r} + \frac{\tau}{T} \hat{K}_1 s - \frac{\tau}{T} e^{-r\tau} \hat{K}_2, \qquad (5.1)
\end{aligned}
$$

which is the seasoned Asian put-call parity.

6. Numerical Computation

We have seen in Section 4 that the zero order price P_0 is of the form $P_0 = S_0 u_0$, where u_0 solves equation (4.9) with an "effective" volatility $\overline{\sigma}(z)$.

To illustrate with examples, we consider a continuously-sampled arithmetic average Asian option with a fixed strike price, i.e. $K_1 = 0$. Parameters are chosen so that the effective volatility $\overline{\sigma}(z) = 0.5$, the risk-free interest rate $r = 0.06$, the strike price $K_2 = 2$, time to maturity $\tau = 1$, stock price $s \in [1, 2.5]$, and the small parameters are chosen as $\overline{V}_0^\delta = -0.01, \overline{V}_1^\delta = -0.005, \overline{V}_2^\varepsilon = -0.01$ and $\overline{V}_3^\varepsilon = 0.004$. Numerical results for the homogenized price $P_0(0, s, z)$ are shown in Figure 1.

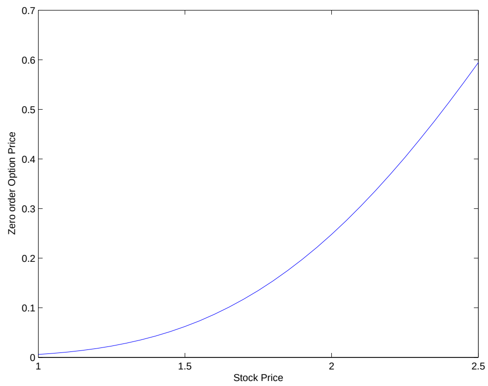

FIGURE 1. Finite difference numerical solution for the effective volatility price $P_0(0, s, z)$ of an arithmetic average Asian call option with parameters $\overline{\sigma}(z) = 0.5, r = 0.06, K_1 = 0, K_2 = 2$, and time to maturity $T = 1$.

Next we compare the effect of the correction $s\tilde{u}_{1,0}(0, \psi, z)$ in (3.17), due to the fast scale only, with the effect of the combined correction $s(\tilde{u}_{1,0}(0, \psi, z) + \tilde{u}_{01}(0, \psi, z))$ in (4.10), due to both fast and slow scales. In Figure 2, we plot $s\tilde{u}_{1,0}$ on the left and $s(\tilde{u}_{1,0} + \tilde{u}_{01})$ on the right. It is observed that the magnitude of the correction is larger when both fast and slow scales are present with this choice of parameters. Most importantly, as commented at the end of Section 4.1, it is necessary to incorporate the combination of the fast and slow volatility factors in order to obtained Asian options prices which are consistent with the observed term structure of implied volatility.

7. Conclusion

We have shown that the dimension reduction technique introduced in [**8**] can be applied to multiscale stochastic volatility models for a class of arithmetic average. When the volatility contains two factors with well-separated time scales, the singular-regular perturbation analysis can be applied such that the full term structure of implied volatility can be taken into account. The approximated price of an Asian option is characterized by two one-dimensional PDEs (4.9, 4.10). Compared to the usual two two-dimensional PDEs (2.14, 2.15) derived in [**4**], our results reduce significantly the computational efforts. Furthermore, the main parameters $\bar{\sigma}$, $\overline{V}_0^\delta$, $\overline{V}_1^\delta$, $\overline{V}_2^\varepsilon$, and $\overline{V}_3^\varepsilon$ needed in the PDEs are estimated from the historical stock returns and the implied volatility surface. The procedure is robust and no specific model of stochastic volatility is actually needed.

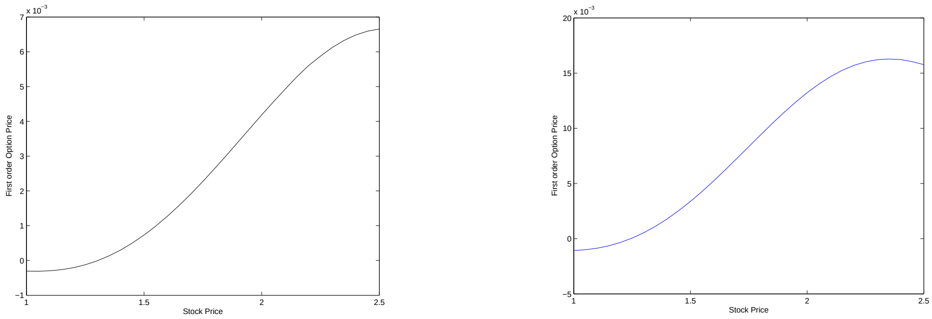

FIGURE 2. Finite difference numerical solution for the corrections to for an arithmetic average Asian call option price with parameters $\overline{\sigma} = 0.5, r = 0.06, T = 1, \overline{V}_0 = -0.01, \overline{V}_1 = -0.005, \overline{V}_2 = -0.01, \overline{V}_3 = 0.004$. In practice the last four parameters would have been calibrated from the observed implied volatility surface. The correction on the left corresponds to the effect of the volatility fast scale and the correction on the right is due to the combination of fast and slow volatility time scales.

References

[1] S. Alizadeh, M. Brandt, and F. Diebold, "Range-based estimation of stochastic volatility models," Journal of Finance, 57, 1047-1091, 2002.

[2] M. Chernov, R. Gallant, E. Ghysels, and G. Tauchen, "Alternative models for stock price dynamics," Preprint, Columbia University, January 2001.

[3] J.-P. Fouque and Chuan-Hsiang Han, "Pricing Asian Options with Stochastic Volatility," Quantitative Finance, volume 3 (2003) 353 - 362.

[4] J.-P. Fouque, G. Papanicolaou, and R. Sircar, "Derivatives in Financial Markets with Stochastic Volatility,"Cambridge University Press, 2000.

[5] J.-P. Fouque, G. Papanicolaou, R. Sircar, and K. Solna, "Multiscales Stochastic Volatility Asymptotics," to appear in the SIAM Journal on Multiscale Modeling and Simulation, 2003.

[6] B. LeBaron, "Stochastic volatility as a simple generator of apparent financial power laws and long memory," Quantitative Finance, 1(6):621-31, November 2001.

[7] G. Molina, C.H. Han and J.P. Fouque, " MCMC Estimation of Multiscale Stochastic Volatility Models," Preprint, 2003.

[8] Jan Vecer, "A New PDE Approach for Pricing Arithmetic Average Asian Options," The Journal of Computational Finance, Summer, 2001, 105-113.

[9] J. Vecer and M. Xu, "Pricing Asian Options in a Semimartingale Model," preprint, 2001.

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