金融隨機計算:第一章

Black-Scholes-Merton Theory of Derivative Pricing and Hedging

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Derivative Contracts

- Derivatives, also called *contingent claims*, are contracts based on some underlying assets.
- A typical option is a contract that gives its holder the right, but not the obligation, to but one unit of a financial (underlying) asset for a predetermined *strike* (or *exercise*) *price* K by the *maturity* date T.

Option

- The type of option: the option to buy is called a *call* option; the option to sell is called a *put* option.
- The underlying asset: it can be a stock, bond, currency, etc.
- The expiration date: if the option can be exercised at any time before maturity, it is called an *American* option; if it can only be exercised at maturity, it is called a *European* option.
- The amount of an underlying asset to be purchased or sold.
- The price of an option is also called the *premium*.

European Options

- Let (S_t) denote the underlying risky asset such as a stock price.
- A European option is a contract to pay $h(S_T)$ at a maturity time *T*, which is *path independent*.

Call and Put

- Call payoff:
$h(S_T) = (S_T - K)^+ = max\{S_T - K, 0\}$
- That means if $S_T > K$, the holder will exercise the option to make a profit $S_T - K$ by buying the stock for the price *K* and selling it immediately at the market price S_T; otherwise the option is not worthy to be exercised.
- Put payoff:
$h(S_T) = (K - S_T)^+ = max\{K - S_T, 0\}$

American Options

- A holder of an American option can exercise the option at any time before maturity *T*.
- Let τ denote the time to exercise the option, which is a stopping time because up to time $t \leq T$, the holder has the right to hold or exercise the option based on the information flow.
- Call payoff:
$h(S_\tau) = (S_\tau - K)^+ = max\{S_\tau - K, 0\}$
- Put payoff:
$h(S_T) = (K - S_\tau)^+ = max\{K - S_\tau, 0\}$

Exotic Option

- Exotic options are those options for which their contracts are not of European or American style defined previously.
- Barrier options. Ex, for a down-and-out call option, $h(S_T) = (S_T - K)^+ I_{\{inf_{t \leq T} S_t > B\}}$
- Lookback options. Ex, $h = \left(S_T - \inf_{t \leq T} S_t\right)^+ = S_T - \inf_{t \leq T} S_t$
- Asian options. Ex, $h = \left(S_T - \frac{1}{T}\int_0^T S_t\, dt\right)^+$

Underlying Risky Asset

- The stock price (S_t) is governed by the geometric Brownian motion under the *historical* filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathcal{P})$:

$$dS_t = \mu S_t dt + \sigma S_t dW_t \tag{1}$$

$$S_0 = x$$

Replication of Trading Strategy

- A dynamic *trading strategy* is a pair (α_t, β_t) of adapted processes specifying the number of units held at time *t* of the stock (underlying risky asset) S_t and bond (riskless asset) e^{rt}, where *r* is the risk-free interest rate.
- At the maturity *T*, the value this portfolio will *replicate* the terminal payoff $h(S_T)$; namely $\alpha_T S_T + \beta_T e^{rT} = h(S_T)$.

Self-Financing Portfolio

- That is the variations of portfolio value are due only to the variations of the stock price and bond price.
- $d(\alpha_t S_t + \beta_t e^{rt}) = \alpha_t dS_t + r\beta_t e^{rt} dt$
- In integral form $\alpha_t S_t + \beta_t e^{rt} = \alpha_0 S_0 + \beta_0 + \int_0^t \alpha_s \, dS_s + \int_0^t \beta_s \, de^{rs}, 0 \leq t \leq T$

Perfect Replication

Based on the arbitrage pricing theory, the fair price of a derivatives is equal to the value of its portfolio:

$$\alpha_t S_t + \beta_t e^{rt} = P(t, S_t)$$

where $P(t, x)$ denotes the option price at time t and stock price $S_t = x$.

Ito's Formula

- $(\alpha_t \mu S_t + \beta_t e^{rt})dt + \alpha_t \sigma S_t dW_t = \left(\frac{\partial P}{\partial t} + \mu S_t \frac{\partial P}{\partial x} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 P}{\partial x^2}\right) dt + \sigma S_t \frac{\partial P}{\partial x} dW_t$
- $\alpha_t = \frac{\partial P}{\partial x}(t, S_t)$
- $\beta_t = (P(t, S_t) - \alpha_t S_t)e^{-rt}$

Black-Scholes PDE

$$\mathcal{L}_{BS}P(t,x) = 0$$

With the terminal condition $P(T,x) = h(x)$, where the operator

$$\mathcal{L}_{BS} = \frac{\partial}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2}{\partial x^2} + rx\frac{\partial}{\partial x} - r$$

is define on the domain of $[0,T] \times (0,\infty)$.

Remarks

- The Black-Scholes PDE can be solved either by numerical PDE methods like finite difference methods or by Monte Carlo simulations.
- The dynamic trading strategy (α_t, β_t) is also called *hedging* strategy by emphasizing the elimination of risks.
- This strategy requires a holding number of risky asset $\alpha_t = \frac{\partial P}{\partial x}(t, S_t)$, which is called the *Delta* in financial engineering.

Independence of μ

- By inspecting the Black-Scholes PDE, the rate of returns μ in (1) does not appear at all. This is a remarkable feature of the Black-Scholes theory that two investors may have completely different speculative view for the rate of returns of the risky asset, but they have to agree that the no-arbitrage price P of the derivative does not depend on μ.

Black-Scholes Formula

The price of a European call option admits a closed-form solution.

$$C_{BS}(t,x) = x\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2),$$

where

$$d_1 = \frac{\ln(x/K) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1(t,x) - \sigma\sqrt{T-t}$$

$$\mathcal{N}(d) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{d} e^{-u^2/2}\, du$$

Put-Call Parity

- $C_{BS}(t, S_t) - P_{BS}(t, S_t) = S_t - Ke^{-r(T-t)}$
- $\mathcal{L}_{BS}(C_{BS} - P_{BS})(t, x) = 0$

with the terminal condition $h(x) = x - K$, admits a simple solution $(C_{BS} - P_{BS})(t, x) = x - Ke^{-r(T-t)}$

Parameter Estimation

- Within the Black-Scholes formula, there is only one non-observable parameter, namely the volatility σ.

1. the direct method: given a set of historical stock prices, one can use either (1) the approximation of the quadratic variation of log returns or (2) the maximum likelihood of log returns of estimate σ.

2. the implied method: given the quoted European call or put option prices, one can invert an “implied” volatility from each quoted option.

Change of Probability Measure

- By Girsanov's Theorem, one can construct a new probability measure $\mathcal{P}^*$ which is equivalent to $\mathcal{P}$ so that the drifted Brownian motion $\left(W_t + \frac{\mu-r}{\sigma}t\right)$ becomes a standard Brownian motion, denoted by (W_t^*).

$$
\begin{aligned}
dS_t &= \mu S_t dt + \sigma S_t dW_t \\
&= (\mu + r - \mu) S_t dt + \sigma S_t d\left(W_t + \frac{\mu - r}{\sigma} t\right) \\
&= r S_t dt + \sigma S_t dW_t^*
\end{aligned}
$$

Remarks

1. the discounted stock price becomes a martingale under $\mathcal{P}^*$. This is because of
$$d(e^{-rt}S_t) = (e^{-rt}S_t)\sigma dW_t^*.$$

2. the discounted value of portfolio becomes a martingale under $\mathcal{P}^*$. This is because of
$$de^{-rt}(\alpha_t S_t + \beta_t e^{rt}) = \alpha_t(e^{-rt}S_t)\sigma dW_t^*.$$

We call such probability measure $\mathcal{P}^*$ as the risk-neutral measure and the shifted drift $\frac{\mu - r}{\sigma}$ as the market price of risk.

Risk-Neutral Evaluation (I)

- The trading strategy must replicate the terminal payoff $h(S_T)$,
$$\alpha_T S_T + \beta_T e^{rT} = h(S_T).$$
- Since $(e^{-rt}(\alpha_t S_t + \beta_t e^{rt}))$ is a martingale under $\mathcal{P}^*$,
$$e^{-rt}(\alpha_t S_t + \beta_t e^{rt}) = \mathbb{E}^*\{e^{-rT}(\alpha_T S_T + \beta_T e^{rT})|\mathcal{F}_t\}$$
- The value of the portfolio is equal to
$$\alpha_t S_t + \beta_t e^{rt} = \mathbb{E}^*\left\{e^{-r(T-t)}h(S_T)|\mathcal{F}_t\right\}$$
$$= \mathbb{E}^*\left\{e^{-r(T-t)}h(S_T)|S_t\right\}$$

  by the Markov property of S_t.

Risk-Neutral Evaluation (II)

- Based on the definition of the fair price, the option price can be defined as a conditional expectation of the discounted payoff under the risk-neutral probability measure $\mathcal{P}^*$:

$$P(t, S_t) = \mathbb{E}^*\{e^{-r(T-t)}h(S_T)|S_t\}.$$

Derivation of The Black-Scholes Formula

$$S_T = S_t exp\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\sqrt{T-t}Z\right)$$

$$P(t, S_t) = \widetilde{\mathbb{E}}\left\{e^{-r(T-t)}\left(S_t exp\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\sqrt{T-t}Z\right) - K\right)^+\right\}$$

$$= \int e^{-r(T-t)}\left(S_t exp\left(\left(r - \frac{\sigma^2}{2}\right)(T-t) + \sigma\sqrt{T-t}Z\right)\right.$$

$$\left. - K\right)^+ \frac{1}{\sqrt{2\pi}} e^{-z^2/2}\, dz$$

$$= \cdots$$

$$= S_t \mathcal{N}(d_1) - K e^{-r(T-t)} \mathcal{N}(d_2)$$

Hedging with One Stock

- $d(e^{-rt}P(t,S_t)) =$
  $e^{-rt}\left(\frac{\partial P}{\partial t} + \frac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 P}{\partial x^2} + rS_t \frac{\partial P}{\partial x} - rP\right)(t,S_t)dt +$
  $e^{-rt}\frac{\partial P}{\partial x}(t,S_t)\sigma S_t dW_t^*$
- $e^{-rT}P(T,S_T) =$
  $P(0,S_0) + \int_0^T e^{-rt}\frac{\partial P}{\partial x}(t,S_t)\sigma S_t dW_t^*$
- $\mathrm{P}(0,S_0) = \mathbb{E}^*[e^{-rT}h(S_T)|S_0]$

Martingale Representation Theorem

- **Theorem** *Let W_t, $0 \le t \le T$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathcal{P})$ and the filtration is generated by this Brownian motion. Let M_t, $0 \le t \le T$ be a martingale with respect to this filtration. Then there is an adapted process Γ_u, $0 \le u \le T$, such that*
$$M_t = M_0 + \int_0^t \Gamma_u \, dW_u, 0 \le t \le T$$
- Γ_u is typically related to the hedging strategy associated with a trading martingale M_t. This theorem only guarantees the existence condition of the hedging strategy. One can use Ito formula to compute Γ explicitly.

Arbitrage

- An arbitrage is a way of trading so that one starts with zero capital and at some later time T is sure not to have lost money and furthermore has a positive probability of having made money
- **Definition** *An arbitrage is a portfolio value process* V_t *satisfying* $V_0 = 0$ *and also satisfying for some time* $T > 0$

$$\mathcal{P}\{V_T \geq 0\} = 1, \mathcal{P}\{V_T > 0\} > 0$$

Fundamental Theorems of Asset Pricing

- **First Theorem** *If a market model has a risk-neutral probability measure, then it does not admit arbitrage.*
- **Definition** *A market model is complete if every derivative security can be hedged.*
- **Second Theorem** *Consider a market model that has a risk-neutral probability measure. The model is complete if and only if the risk-neutral probability measure is unique.*

Forward Contract

- **Definition** *A forward contract is an agreement to pay a specified delivery price L at a delivery date T for the asset price* S_t *at time t. The T-forward price* $For_S(t,T)$ *of this asset, is the value of K that makes the forward contract have no-arbitrage price* ***zero*** *at time t.*
- **Theorem** *The T-forward price is*

$$For_S(t,T) = \frac{S_t}{B(t,T)},$$

where the price of a zero-coupon is

$$B(r,T) = \frac{1}{D_t} \mathbb{E}^* \left\{ e^{-\int_t^T r_u du} 1 | r_t \right\}$$

Futures

- **Definition** *The futures price of an asset whose value at time T is S_T is given by the formula*
$$Fut_S(t,T) = \mathbb{E}^*\{S_T|\mathcal{F}_t\}, 0 \leq t \leq T$$

A long position in the futures contract is an agreement to receive as a cash flow the changes in the futures price during the time the position is held. A short position in the futures contract receives the opposite cash flow.

Forward-Futures Spread

- When the (instantaneous) interest-rate r_t is random, the *forward-futures spread* is defined by

$$For_S(0,T) - Fut_S(0,T) = \frac{S_0}{\mathbb{E}^* D_T} - \mathbb{E}^* S_T$$
$$= \frac{1}{\mathbb{E}^* D_T} (\mathbb{E}^*\{D_T S_T\} - \mathbb{E}^* D_T \mathbb{E}^* S_T)$$
$$= \frac{1}{\mathbb{E}^* D_T} Cov^*(D_T, S_T\ )$$

where $Cov^*(D_T, S_T\ )$ denotes the covariance of D_T and S_T under the risk-neutral probability measure.

Appendix: Girsanov Theorem

Let (W_t) *be a standard Brownian motion under a probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, \mathcal{P})$. *Let* θ_t *be an adapted process. Define*

$$Z_t = exp\left(-\int_0^t \theta_s \, dW_s - \frac{1}{2}\int_0^t \theta_s^2 \, ds \right)$$

$$W_t^* = W_t + \int_0^t \theta_s \, ds$$

and assume that

$$\mathbb{E}\left\{ \int_0^T \theta_s^2 Z_s^2 \, ds < \infty \right\}$$

Set $Z = Z_T$. *Then* $\mathbb{E}Z = 1$ *and under the probability measure* $\tilde{\mathcal{P}}$ *defined by* $\frac{d\tilde{\mathcal{P}}}{d\mathcal{P}} = Z_T$, *the process* $W_t^*, 0 \le t \le T$, *is a Brownian motion.*