Evaluation of compound options using perturbation approximation

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Abstract

This paper proposes a fast, efficient and robust way to compute the prices of compound options such as the popular call-on-call options within the context of multiscale stochastic volatility models. Recent empirical studies indicate the existence of at least two characteristic time scales for volatility factors including one highly persistent factor and one quickly mean-reverting factor. Here we introduce one relatively slow time scale and another relatively fast time scale, with respect to typical time to maturities, into our multiscale stochastic volatility models. Using a combination of singular and regular perturbations techniques we approximate the price of a compound option by the price under constant volatility of the corresponding option corrected in order to take in account the effects of stochastic volatility. We provide formulas for these corrections which involve universal parameters calibrated to the term structure of implied volatility. Our method is not model sensitive, and the calibration and computational efforts are drastically reduced compared with solving fully specified models.

Key Words: compound options, multiscale stochastic volatility model, regular perturbation, singular perturbation, price approximation, effective volatility.

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1 Introduction

A compound option, or an option on option, gives the holder the right, but not the obligation to buy (long) or sell (short) the underlying option. For market participants, compound options are known to be very sensitive to the volatility changes. For instance Brenner et al [2] introduced an instrument, option on straddle, to hedge volatility risk since at-the-money forward straddle is a traded asset in the market and its value is mainly affected by the change in volatility. Compound options are also used to evaluate American put options (see for instance [9]). When the volatility is constant, Geske [8] developed a close-form solution for the price of a vanilla European call on a European call.

In this paper, we explore the evaluation problem for compound options within the context of stochastic volatility. In this environment, payoff functions of compound options do depend on the driving volatility level. This feature, volatility dependence in the terminal condition, is very different from the case of European-style options, in which only underlying risky assets are variables in payoff functions. It also explains the reason for the sensitivity of compound options to volatility. Despite the complex behavior of volatility under the pricing measure and the fact that volatility is not directly observed from the market, evaluating compound options by numerical partial differential equation (PDE) methods can be problematic, at least very time consuming. For instance in the case of two-factor stochastic volatility models, one has to specify the full stochastic volatility model, then solve two iterative three-dimensional PDEs in order to obtain the price of an option-on-option. The first PDE gives the underlying option price and the second PDE gives the compound option price.

Recent empirical studies document that two-factor stochastic volatility models with well-separate characteristic time scales can produce stylized facts like the observed kurtosis, fat-tailed return distribution and long memory effect. See [1], [4] and [12] for detailed discussions. Hence we consider two-factor stochastic volatility models characterized by one relatively slow time scale and another relatively fast time scale with respect to typical time to maturities. This setting of models allow us to use a combination of singular and regular perturbation methods, first proposed by Fouque et al [6, 7], to derive price approximations to compound options. This technique provides a simple parametrization of the observed implied volatilities in terms of Greeks. In Section 3, we derive an approximated payoff function for a call-on-call based on the expansion of a price approximation to the underlying call option price. Using the close-form solution of a call-on-call [8], we show that only one one-dimensional PDE needs to be solved. All the parameters we need to compute the approximated price can be easily calibrated the implied volatility surface without the full specification of a particular stochastic volatility model. Thus, this article describes a fast, efficient and robust procedure to approximate compound option prices by taking the observed implied volatility skew into account. It is compared with solving two three-dimensional PDEs in the full specified models for a call-on-call price. The accuracy of the approximation is provided. This methodology can be easily generalized to European call on put, put on call, put on put, and the other structured product like options on straddle, etc [10]. As mentioned above, payoffs of compound options do depend on volatility level in a stochastic volatility environment. Up to the first order price approximation obtained from our perturbation techniques, only a constant volatility defined in (14) is actually used. Hence, compound options are weakly dependent on driving volatility level which can be removed at the first order of approximation.

This paper is organized as follows. Section 2 contains the introduction of multiscale stochastic volatility models. Compound options and their price approximations are presented in Section 3. Calibration of relevant parameters from the term structure of implied volatility is discussed in Section 4. Numerical illustrations are presented in Section 5. A discussion on parameters reduction is presented in Section 6.

2 Multiscale Stochastic Volatility Models

We consider a family of stochastic volatility models (S_t, Y_t, Z_t), where S_t is the underlying price, Y_t evolves as an Ornstein-Uhlenbeck (OU) process, as a prototype of an ergodic diffusion, and (Z_t) follows another diffusion process. To be specific, under the physical probability measure P, our model can be written as

$$\begin{aligned} dS_t &= \mu S_t dt + \sigma_t S_t dW_t^{(0)}, \qquad (1) \\ \sigma_t &= f(Y_t, Z_t), \\ dY_t &= \alpha(m - Y_t)dt + \nu\sqrt{2\alpha}\, dW_t^{(1)}, \\ dZ_t &= \delta c(Z_t)dt + \sqrt{\delta}\, g(Z_t) dW_t^{(2)}, \end{aligned}$$

where $\left(W_t^{(0)}, W_t^{(1)}, W_t^{(2)}\right)$ are standard Brownian motions correlated according to the following cross-variations:

$$d\langle W^{(0)}, W^{(1)}\rangle_t = \rho_1 dt, \qquad (2)$$

$$d\langle W^{(0)}, W^{(2)}\rangle_t = \rho_2 dt,$$
$$d\langle W^{(1)}, W^{(2)}\rangle_t = \rho_{12} dt,$$

with constants $|\rho_1| < 1, |\rho_2| < 1$ and $|\rho_{12}| < 1$. The stock price S_t is governed by a diffusion with a constant rate of return μ and the random volatility σ_t. One driving volatility process Y_t is mean-reverting around its long run mean m, with a rate of mean reversion $\alpha > 0$, and a "vol-vol" $\nu\sqrt{2\alpha}$ corresponding to a long run standard deviation ν. Here we choose to write an OU process with long run distribution $\mathcal{N}(m, \nu^2)$ as a prototype of more general ergodic diffusions. Another driving volatility process Z_t is a general diffusion which evolves on the time scale $1/\delta$. We assume that the coefficients c and g are smooth and bounded. The volatility function f in (1) is assumed to be bounded and bounded away from 0.

In order to incorporate two characteristic volatility time scales, namely one short (fast) and another long (slow), into the stochastic volatility models, we assume that the rate of mean-reversion α is large and that δ is small. These two characteristic time scales $1/\alpha$ and $1/\delta$ are meant to be relatively short or long by comparing with time to typical maturities of contracts. To perform asymptotic analysis, we introduce a small parameter $0 < \varepsilon << 1$ such that the rate of mean reversion defined by $\alpha = 1/\varepsilon$ becomes large. To capture the volatility clustering behaviors, we assume ν to be a fixed $\mathcal{O}(1)$ constant.

Under the risk-neutral probability measure P^*, it follows from Girsanov theorem that our model can be written as

$$\begin{aligned} dS_t &= rS_t dt + f(Y_t, Z_t) S_t dW_t^{(0)*}, \\ dY_t &= \left(\frac{1}{\epsilon}(m - Y_t) - \frac{\nu\sqrt{2}}{\sqrt{\epsilon}}\Lambda_1(Y_t, Z_t)\right) dt + \frac{\nu\sqrt{2}}{\sqrt{\epsilon}} dW_t^{(1)*}, \\ dZ_t &= \left(\delta c(Z_t) - \sqrt{\delta} g(Z_t)\Lambda_2(Y_t, Z_t)\right) dt + \sqrt{\delta} g(Z_t) dW_t^{(2)*}, \end{aligned} \tag{3}$$

where $W_t^{(0)*}$, $W_t^{(1)*}$, and $W_t^{(2)*}$ are standard Brownian motions with the same covariations as in (2). The small parameter ϵ corresponds to the the fast scale and large parameter δ corresponds to the slow scale. The combined market prices of volatility risk $\Lambda_1(y, z)$ and $\Lambda_2(y, z)$ are assumed bounded such that the joint process (S_t, Y_t, Z_t) is Markovian. The risk-free interest rate r is assumed to be constant. Due to the presence of the combined market price of risk, there exists a (Λ_1, Λ_2)-dependent family of equivalent risk-neutral measures. However, we assume that the market chooses one measure through (Λ_1, Λ_2).

3 Price Approximation to Compound Options

In the context of multiscale stochastic volatility environment (3), we consider a compound option to be a European option defined on another European option. In Section 3.1 we first recall results by Fouque et al. [6] on price approximations to the underlying European options and the accuracy result. Then in Section 3.2 we study the pricing of compound options when payoffs are smooth with bounded derivatives. For the case of non-smooth payoffs of compound options such as call-on-call options, their price approximations are derived in Section 3.3.

3.1 Underlying Option Price Approximation

The no arbitrage European option price $P^{\varepsilon,\delta}$ under assumptions of the stochastic volatility model (3), is defined by

$$P^{\varepsilon,\delta}(t,x,y;T,h) = E^*_{t,x,y,z}\{e^{-r(T-t)}h(S_T)\}, \tag{4}$$

where the payoff $h(x)$ is assumed to be smooth with bounded derivatives and we denote by $E^*_{t,x,y,z}\{\cdot\}$ the conditional expectation $E^*\{\cdot \mid S_t = x, Y_t = y, Z_t = z\}$ on the current time t, the stock price $S_t = x$, the driving volatility level $(Y_t = y, Z_t = z)$. Based on a combination of singular and regular perturbation methods [6], the accuracy result of the approximation

$$\begin{aligned} &P^{\varepsilon,\delta}(t,x,y,z;T,h) - \left(P_0(t,x;T,h;\bar\sigma(z)) + \tilde P_1(t,x;T,h;\bar\sigma(z))\right) \\ &= \mathcal{O}(\max\{\varepsilon,\delta\}) \end{aligned} \tag{5}$$

can be obtained in the point-wise sense while ε and δ are small enough. The zero order term P_0 is the homogenization of 4 in y under a small expansion in z; namely, P_0 is regarded as a homogenized Black-Scholes price with the effectively-froze constant volatility $\bar\sigma(z)$, which will be defined in (14). The first order correction $\tilde P_1$ is of order $\sqrt{\varepsilon}$ or $\sqrt{\delta}$. Notice that in the price approximation, $P_0 + \tilde P_1$ there is no dependence on the fast varying volatility level y and the slowly varying volatility level z appears in the volatility $\bar\sigma(z)$ as a parameter. Using short notations, $P_0(t,x;\bar\sigma(z))$ and $\tilde P_1(t,x;\bar\sigma(z))$ solve the following two PDEs with the constant volatility $\bar\sigma(z)$, respectively,

$$\begin{cases} \mathcal{L}_{BS}(\bar\sigma(z))P_0(t,x;\bar\sigma(z)) = 0 \\ P_0(T,x) = h(x), \end{cases} \tag{6}$$

$$\begin{cases} \mathcal{L}_{BS}(\bar\sigma(z))\tilde P_1(t,x;\bar\sigma(z)) = -\left(\mathcal{A}^\varepsilon + 2\mathcal{B}^\delta\right) P_0(t,x;\bar\sigma(z)) \\ \tilde P_1(T,x) = 0. \end{cases} \tag{7}$$

The Black-Scholes differential operator $\mathcal{L}_{BS}(\bar{\sigma})(z)$ is defined in (21), and the other operators are defined as follows:

$$\mathcal{A}^{\varepsilon} = V_2^{\varepsilon} x^2 \frac{\partial^2}{\partial x^2} + V_3^{\varepsilon} x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2}{\partial x^2}\right), \tag{8}$$

$$\mathcal{B}^{\delta} = V_0^{\delta} \frac{\partial}{\partial \sigma} + V_1^{\delta} x \frac{\partial}{\partial x}\left(\frac{\partial}{\partial \sigma}\right). \tag{9}$$

The universal parameters are defined by

$$V_0^{\delta} = -\frac{\sqrt{\delta}}{2} g(z) \langle \Lambda_2(y,z) \rangle \bar{\sigma}'(z), \tag{10}$$

$$V_1^{\delta} = \frac{\sqrt{\delta}}{2} \rho_2 g(z) \langle f(y,z) \rangle \bar{\sigma}'(z), \tag{11}$$

$$V_2^{\varepsilon} = \frac{\nu\sqrt{\varepsilon}}{\sqrt{2}} \langle \Lambda_1(y,z) \frac{\partial \phi(y,z)}{\partial y} \rangle, \tag{12}$$

$$V_3^{\varepsilon} = -\frac{\nu\sqrt{\varepsilon}}{\sqrt{2}} \rho_1 \langle f(y,z) \frac{\partial \phi(y,z)}{\partial y} \rangle. \tag{13}$$

Parameters V_0^{δ} and V_1^{δ} (resp. to V_2^{ε} and V_3^{ε}) are small of order $\sqrt{\delta}$ (resp. to $\sqrt{\varepsilon}$). The parameters V_0^{δ} and V_2^{ε} reflect the effect of the market prices of volatility risk (Λ_1, Λ_2). The parameters V_1^{δ} and V_3^{ε} are proportional to the correlation coefficients ρ_2 and ρ_1 respectively. The bracket $\langle \cdot \rangle$ denotes an expectation with respect to the invariant distribution, $\mathcal{N}(m, \nu^2)$, of the OU-process with infinitesimal generator $\mathcal{L}_0$ defined in (19). That is for a bounded function l, we define

$$\langle l(\cdot) \rangle := \frac{1}{\sqrt{2\pi\nu^2}} \int_{-\infty}^{\infty} l(y) e^{-\frac{(y-m)^2}{2\nu^2}} dy.$$

The z-dependent effective volatility $\bar{\sigma}(z)$ is defined by

$$\bar{\sigma}(z)^2 = \langle f(\cdot, z)^2 \rangle. \tag{14}$$

In practice $\bar{\sigma}(z)^2$ is estimated from historical stock returns over a period of time of order one (shorter than δ^{-1}) so that the z-dependence is automatically incorporated in the estimate. The function $\phi(y,z)$ is a solution of the Poisson equation

$$\mathcal{L}_0 \phi(y,z) = f^2(y,z) - \bar{\sigma}^2(z)$$

up to an additional function depending on the variable z only, which will not affect the operator $\mathcal{A}^{\varepsilon}$. Note that there exists an explicit expression for $\tilde{P}_1$ [6]:

$$\tilde{P}_1(t,x;\bar{\sigma}(z)) = (T-t)\left(\mathcal{A}^{\varepsilon} + \mathcal{B}^{\delta}\right) P_0(t,x;\bar{\sigma}(z)). \tag{15}$$

3.2 Compound Options with Smooth Payoffs

Considering a two-factor stochastic volatility environment (3), the payoff for a compound option is given as:

$$h_1\left(P^{\varepsilon,\delta}(T_1, S_{T_1}, Y_{T_1}, Z_{T_1}; T_2, h_2)\right), \tag{16}$$

where S_{T_1} is the value of the stock underlying the underlying option $P^{\varepsilon,\delta}$, (Y_{T_1}, Z_{T_1}) is the driving volatility level at the expiry date of the compound T_1, the expiry date of the underlying option is T_2, and the function h_2 is the payoff for the underlying option. Here payoffs $h_1(x)$ and $h_2(x)$ are assumed to be smooth with bounded derivatives.
The no arbitrage price of a compound option is defined by a conditional expectation under the pricing measure P^*

$$\begin{aligned} &U^{\varepsilon,\delta}(t, x, y, z; T_1, T_2, h_1, h_2) \\ &= E^*_{t,x,y,z}\left\{e^{-r(T_1-t)} h_1\left(P^{\varepsilon,\delta}(T_1, S_{T_1}, Y_{T_1}, Z_{T_1}; T_2, h_2)\right)\right\}. \end{aligned} \tag{17}$$

Regarding the composite function $h_1 \circ P^{\varepsilon,\delta}$ as a new payoff, the compound option defined in (17) becomes a nontraditional European option whose payoff depend on forward volatility level (Y_{T_1}, Z_{T_1}) explicitly. Hence compound options are inherently volatility-dependent in payoffs such that they are sensitive to volatility changes. This feature is very different from traditional European options whose payoffs do not depend on volatility level at all.

From an application of the Feynman-Kac formula, the no-arbitrage compound option price $U^{\varepsilon,\delta}$ solves a three-dimensional parabolic type PDE with a terminal condition:

$$\begin{aligned} &\mathcal{L}^{\varepsilon,\delta} U^{\varepsilon,\delta} = 0, \\ &U^{\varepsilon,\delta}(T_1, x, y, z) = h_1(P^{\varepsilon,\delta}(T_1, x, y, z)). \end{aligned} \tag{18}$$

The partial differential operator $\mathcal{L}^{\varepsilon,\delta}$ is defined by

$$\mathcal{L}^{\varepsilon,\delta} = \frac{1}{\varepsilon}\mathcal{L}_0 + \frac{1}{\sqrt{\varepsilon}}\mathcal{L}_1 + \mathcal{L}_2 + \sqrt{\delta}\mathcal{M}_1 + \delta\mathcal{M}_2 + \sqrt{\frac{\delta}{\varepsilon}}\mathcal{M}_3$$

and each component operator is given by

$$\mathcal{L}_0 \;=\; (m-y)\frac{\partial}{\partial y} + \nu^2\frac{\partial^2}{\partial y^2}, \tag{19}$$

$$
\begin{aligned}
\mathcal{L}_1 &= \sqrt{2}\nu\left(\rho_1 x f(y,z)\frac{\partial^2}{\partial x\partial y} - \Lambda_1(y,z)\frac{\partial}{\partial y}\right), &(20)\\
\mathcal{L}_2(f(y,z)) &= \frac{\partial}{\partial t} + \frac{f^2(y,z)}{2}x^2\frac{\partial^2}{\partial x^2} + r\left(x\frac{\partial}{\partial x} - \cdot\right), &(21)\\
\mathcal{M}_1 &= -g(z)\Lambda_2(y,z)\frac{\partial}{\partial z} + \rho_2 g(z) f(y,z) x\frac{\partial^2}{\partial x\partial z}, &(22)\\
\mathcal{M}_2 &= c(z)\frac{\partial}{\partial z} + \frac{g(z)^2}{2}\frac{\partial^2}{\partial z^2}, &(23)\\
\mathcal{M}_3 &= \nu\sqrt{2}\rho_{12} g(z)\frac{\partial^2}{\partial y\partial z}. &(24)
\end{aligned}
$$

Based on the price approximation in (5) and assumptions on h_1, a Taylor expansion of the payoff defined in (17)

$$
\begin{aligned}
&h_1\left(P^{\varepsilon,\delta}(T_1, S_{T_1}, Y_{T_1}, Z_{T_1}; T_2, h_2)\right) = h_1\left(P_0(T_1, S_{T_1}; T_2, h_2; \bar{\sigma}(Z_{T_1}))\right) \qquad (25)\\
&+h_1^{'}\left(P_0(T_1, S_{T_1}; T_2, h_2; \bar{\sigma}(Z_{T_1}))\right)\tilde{P}_1(T_1, S_{T_1}; T_2, h_2; \bar{\sigma}(Z_{T_1})) + \mathcal{O}(\max\{\varepsilon,\delta\})
\end{aligned}
$$

is obtained. From this payoff expansion, it is readily observed that the first two terms are not dependent on the fast varying volatility level Y_{T_1} but only on slowly varying level Z_{T_1}. This fact characterizes the long memory property of compound options because that Z_{T_1} varies little from its starting point $Z_t = z$ while assuming the large time scale $1/\delta >> T_1 - t$. Therefore, up to the accuracy of orders ε and δ, the compound option price can be approximated by two conditional expectations

$$
\begin{aligned}
&U^{\varepsilon,\delta}(t,x,y,z;T_1,T_2,h_1,h_2) \qquad (26)\\
&\approx E^*_{t,x,y,z}\left\{e^{-r(T_1-t)}h_1\left(P_0(T_1, S_{T_1}; T_2, h_2; \bar{\sigma}(Z_{T_1}))\right)\right\}\\
&+E^*_{t,x,y,z}\left\{e^{-r(T_1-t)}h_1^{'}\left(P_0(T_1, S_{T_1}; T_2, h_2; \bar{\sigma}(Z_{T_1}))\right)\tilde{P}_1(T_1, S_{T_1}; T_2, h_2; \bar{\sigma}(Z_{T_1}))\right\}.
\end{aligned}
$$

Our goal here is to further lay out asymptotics for this compound option price approximation. As a matter of fact, the presence of slowly varying component Z_{T_1} in payoffs $h_1 \circ P_0$ and $h_1'(P_0)\tilde{P}_1$, does not affect the singular and regular perturbation analysis shown in [6]. Indeed it is obtained straightforwardly in [5] that the leading order term in the expansion (26)

$$
E^*_{t,x,y,z}\left\{e^{-r(T_1-t)}h_1\left(P_0(T_1, S_{T_1}; T_2, h_2; \bar{\sigma}(Z_{T_1}))\right)\right\} \qquad (27)
$$

has the following expansion

$$
U_0(t,x;T_1,T_2,h_1,h_2;\bar{\sigma}(z)) + \tilde{U}_1(t,x;T_1,T_2,h_1,h_2;\bar{\sigma}(z)) + \mathcal{O}(\max\{\varepsilon,\delta\}),
$$

where the first term U_0 solves the Black-Scholes PDE with the effective volatility $\bar{\sigma}(z)$ and a terminal condition

$$\begin{aligned} &\mathcal{L}_{BS}(\bar{\sigma}(z))U_0(t,x;T_1,T_2,h_1,h_2;\bar{\sigma}(z)) = 0 \\ &U_0(T_1,x;T_1,T_2,h_1,h_2;\bar{\sigma}(z)) = h_1\left(P_0(T_1,x;T_2,h_2;\bar{\sigma}(z))\right). \end{aligned} \tag{28}$$

It is worthy to observe that U_0 depends on z through the volatility $\bar{\sigma}(z)$; namely, $\bar{\sigma}(z)$ is only a parameter in the problem that defines U_0. The second term $\tilde{U}_1$ solves the same PDE with a source

$$\mathcal{L}_{BS}(\bar{\sigma}(z))\tilde{U}_1(t,x;\bar{\sigma}(z)) = -\left(\mathcal{A}^{\varepsilon} + 2\mathcal{B}^{\delta}\right)U_0(t,x;\bar{\sigma}(z)),$$

but its terminal condition is zero. Based on the formula (15),

$$\tilde{U}_1(t,x;T_1,T_2,h_1,h_2;\bar{\sigma}(z)) = (T_1 - t)\left(\mathcal{A}^{\varepsilon} + \mathcal{B}^{\delta}\right)U_0$$

is obtained.

Next we derive an approximation to the second conditional expectation in the expansion (26)

$$E^*_{t,x,y,z}\left\{e^{-r(T_1-t)}h_1'\left(P_0(T_1,S_{T_1};T_2,h_2;\bar{\sigma}(Z_{T_1}))\right)\tilde{P}_1(T_1,S_{T_1};T_2,h_2;\bar{\sigma}(Z_{T_1}))\right\}. \tag{29}$$

Since $h_1'(P_0)\tilde{P}_1$ is of order $\sqrt{\varepsilon}$ or $\sqrt{\delta}$ and we are only interested in the approximation up to $\mathcal{O}(\sqrt{\varepsilon},\sqrt{\delta})$, it is enough to take the homogenization of (29) in y under a small expansion in z. As a consequence, the conditional expectation (29) is approximated by only its homogenization $\tilde{U}_2$, which solves

$$\begin{aligned} &\mathcal{L}_{BS}(\bar{\sigma}(z))\tilde{U}_2(t,x;T_1,T_2,h_1,h_2;\bar{\sigma}(z)) = 0 \\ &\tilde{U}_2(T_1,x;T_1,T_2,h_1,h_2;\bar{\sigma}(z)) = h_1'\left(P_0(T_1,x;\bar{\sigma}(z))\right)\tilde{P}_1(T_1,x;\bar{\sigma}(z)) \end{aligned}$$

Because h_1 and h_2 are smooth with bounded derivatives and $\tilde{P}_1$ is of order $\sqrt{\varepsilon}$ or $\sqrt{\delta}$, it is easy to generalize the accuracy result in (5) to claim that the approximation $\tilde{U}_2$ is valid within an error $\mathcal{O}(\max\{\varepsilon,\delta\})$. We therefore conclude the following theorem.

Theorem 3.1 *When the payoffs h_1 and h_2 are smooth and have bounded derivatives, for fixed (t,x,y,z) and $t < T_1 < T_2$, there exists a constant $C > 0$ such that*

$$\begin{aligned} &\left|U^{\varepsilon,\delta}(t,x,y,z;T_1,T_2,h_1,h_2) - \left(U_0 + \tilde{U}_1 + \tilde{U}_2\right)(t,x;T_1,T_2,h_1,h_2;\bar{\sigma}(z))\right| \\ &\leq C\max\{\varepsilon,\delta,\sqrt{\varepsilon\delta}\} \end{aligned}$$

To summarize, we derive the price approximation for a compound option

$$U^{\varepsilon,\delta}(t,x,y,z;T_1,T_2,h_1,h_2) \approx \hat{U}(t,x;T_1,T_2,h_1,h_2;\bar{\sigma}(z)),$$

where we define

$$\hat{U} = U_0 + \left(\tilde{U}_1 + \tilde{U}_2\right).$$

The homogenized compound option price U_0 solves the one-dimensional Black-Scholes PDE with a constant volatility $\bar{\sigma}(z)$

$$\boxed{\begin{array}{l} \mathcal{L}_{BS}(\bar{\sigma}(z))U_0(t,x;T_1,T_2,h_1,h_2;\bar{\sigma}(z)) = 0 \\ U_0(T_1,x;T_1,T_2,h_1,h_2;\bar{\sigma}(z)) = h_1\left(P_0(T_1,x;T_2,h_2;\bar{\sigma}(z))\right) \end{array}} \tag{30}$$

the first correction $\tilde{U}_1$ has an explicit solution

$$\boxed{\tilde{U}_1(t,x;T_1,T_2,h_1,h_2;\bar{\sigma}(z)) = (T_1 - t)\left(\mathcal{A}^\varepsilon + \mathcal{B}^\delta\right)U_0(t,x;\bar{\sigma}(z))} \tag{31}$$

and the second correction solves another one-dimensional Black-Scoles PDE with nontrivial terminal condition

$$\boxed{\begin{array}{l} \mathcal{L}_{BS}(\bar{\sigma}(z))\tilde{U}_2(t,x;T_1,T_2,h_1,h_2;\bar{\sigma}(z)) = 0 \\ \tilde{U}_2(T_1,x;T_1,T_2,h_1,h_2;\bar{\sigma}(z)) = h_1'\left(P_0(T_1,x;\bar{\sigma}(z))\right)\tilde{P}_1(T_1,x;T_2,h_2;\bar{\sigma}(z)) \\ = (T_2 - T_1)h_1'\left(P_0(T_1,x;\bar{\sigma}(z))\right)\left(\mathcal{A}^\varepsilon + \mathcal{B}^\delta\right)P_0(T_1,x;\bar{\sigma}(z)) \end{array}} \tag{32}$$

From this summary of computational procedure, we conclude that under the stochastic volatility model (3), compound options are weakly dependent on the driving volatility level. In contrast to European options, payoffs for compound options shown in (25) indeed depend on the driving volatility level (Y_{T_1}, Z_{T_1}). However, the perturbation analysis through a Taylor expansion on the payoff reveals that the first order approximations shown in (30,31,32) to a compound option price merely depend on the constant $\bar{\sigma}(z)$ at the level of volatility. The reason is that volatility changes are embedded into the underlying European option price changes. Hence up to the first order price approximation the reduction from a random volatility to a constant volatility explains the weak dependence of compound options on volatility changes. In fact, there are strongly volatility-dependent derivatives traded in the market whose payoffs are directly defined by a running sum of realized volatility or variance [3]. We refer to [10, 11] for deriving price approximations based on perturbation analysis, in which additional effects such as local times and boundary layers needed to be taken in to account.

3.3 Compound Options with Non-Smooth Payoffs: Call-on-Call

We investigate a typical European style compound option: call-on-call, which specifies payoff functions in (17) as

$$h_1(x) = (x - K_1)^+$$

and

$$h_2(x) = (x - K_2)^+.$$

Both call payoffs h_1 and h_2 are not smooth at the strikes $x = K_1$ and $x = K_2$ respectively. To emphasize these call payoffs we substitute the notations h_1 and h_2 by K_1 and K_2 below. For example, the call-on-call option prices with strike prices K_1 and K_2 at expiry dates T_1 and T_2 respectively, is denoted by

$$\begin{aligned} &U^{\varepsilon,\delta}(t,x,y,z;T_1,T_2,K_1,K_2) \\ &= E^*_{t,x,y,z}\left\{e^{-r(T_1-t)}\left(P^{\varepsilon,\delta}(T_1,S_{T_1},Y_{T_1},Z_{T_1};T_2,K_2) - K_1\right)^+\right\}. \quad (33)\end{aligned}$$

where the underlying call option prices is denoted by

$$P^{\varepsilon,\delta}(T_1,x,y,z;T_2,K_2) = E^*_{T_1,x,y,z}\{e^{-r(T_2-T_1)}(S_{T_2} - K_2)^+\}, \qquad (34)$$

It is shown in [6] that the expansion accuracy for the underlying European call option is given as

$$\begin{aligned} &P^{\varepsilon,\delta}(T_1,x,y,z;T_2,K_2) - \left(P_0(T_1,x;T_2,K_2;\bar\sigma(z)) + \tilde P_1(T_1,x;T_2,K_2;\bar\sigma(z))\right) \\ &= \mathcal{O}(\max\{\varepsilon|\log\varepsilon|, \sqrt{\varepsilon\delta}, \delta\}) \qquad (35)\end{aligned}$$

in the point-wise sense.

For the call-on-call option, it is observed that a Taylor expansion shown in (25) does not hold because the kink of h_1 at K_1. To resolve this, we regularize the piecewise smooth payoff h_1 by the price of a European call option with time to maturity Δ, i.e.

$$h_1(x) = (x - K)^+ \approx h_1^\Delta(x,z) := P(T_1 - \Delta, x; T_1, K_1; \bar\sigma(z)),$$

where $h_1^\Delta(x,z)$ is a smooth function with bounded derivatives in x given any z. Consequently, the regularized compound option price is defined as

$$\begin{aligned} &U^{\epsilon,\delta}_\Delta(t,x,y,z;T_1,T_2,K_1,K_2) = U^{\epsilon,\delta}(t,x,y,z;T_1,T_2,h_1^\Delta,K_2) \\ &= E^*_{t,x,y,z}\left\{e^{-r(T_1-t)}h_1^\Delta\left(P_0(T_1,S_{T_1};T_2,K_2;\bar\sigma(Z_{T_1}))\right.\right. \\ &\left.\left.+\tilde P_1(T_1,S_{T_1};T_2,K_2;\bar\sigma(Z_{T_1})) + \mathcal{O}(\epsilon|\log\varepsilon| + \delta + \sqrt{\varepsilon\delta})\right)\right\},\end{aligned}$$

where the second equality is from (35). Using a Taylor expansion in h_1^Δ at P_0 and $\tilde{P}_1$ being of order $\sqrt{\varepsilon}$, the regularized price becomes

$$
\begin{aligned}
&U_\Delta^{\varepsilon,\delta}(t,x,y,z;T_1,T_2;K_1,K_2) = E^*_{t,x,y,z}\{e^{-r(T_1-t)}h_1^\Delta(P_0(T_1,S_{T_1};\bar\sigma(Z_{T_1})))\}\\
&+E^*_{t,x,y,z}\left\{e^{-r(T_1-t)}\left((h_1^\Delta)'(P_0(T_1,S_{T_1};\bar\sigma(Z_{T_1}))\right)\tilde{P}_1(T_1,S_{T_1};\bar\sigma(Z_{T_1}))\right\}\\
&+\mathcal{O}(\epsilon|\log\varepsilon| + \delta + \sqrt{\varepsilon\delta})
\end{aligned}
$$

Since payoffs $h_1^\Delta \circ P_0$ and $\left(h_1^\Delta\right)'(P_0)\,\tilde{P}_1$ in this expansion are smooth with bounded derivatives, as presented in Section 3.2, the first two conditional expectations in the price expansion can be further approximated by Δ-dependent functions, say $U_0^\Delta + \tilde{U}_1^\Delta$, and $\tilde{U}_2^\Delta$ respectively within an error of $\mathcal{O}(\max\{\varepsilon,\delta\})$.

Intuitively, as Δ goes to zero, we expect that U_0^Δ, $\tilde{U}_1^\Delta$, and $\tilde{U}_2^\Delta$ will converge pointwisely to U_0, $\tilde{U}_1$, and $\tilde{U}_2$ respectively, where these $U's$ are defined in (36), (37), and (38). The proof of the accuracy result for the price approximation, $U_0+\tilde{U}_1+\tilde{U}_2$, to a call-on-call option price, $U^{\varepsilon,\delta}$ defined in (33), requires to control bounds for the following terms

$$
\begin{aligned}
&|U^{\varepsilon,\delta} - \left(U_0+\tilde{U}_1+\tilde{U}_2\right)| \le |U^{\varepsilon,\delta} - U_\Delta^{\varepsilon,\delta}| + |U_\Delta^{\varepsilon,\delta} - \left(U_0^\Delta+\tilde{U}_1^\Delta+\tilde{U}_2^\Delta\right)|\\
&+|U_0-U_0^\Delta| + |\tilde{U}_1+\tilde{U}_2 - \left(\tilde{U}_1^\Delta+\tilde{U}_2^\Delta\right)|.
\end{aligned}
$$

The details of deriving these bounds reply on expressions of Greeks of a call-on-call, successively derivatives-in-x of a call on call, and suitable estimates on the growth at infinity. To limit discussions here we simply outline some results in [5] where a general piecewise smooth payoffs will be discussed. The first term $|U^{\varepsilon,\delta} - U_\Delta^{\varepsilon,\delta}|$ is of $\mathcal{O}(\Delta)$ and we will choose $\Delta = \varepsilon$. The second term is of $\mathcal{O}(\max\{\varepsilon,\delta\})$ by following Theorem 3.1. The third term $|U_0-U_0^\Delta|$ is of $\mathcal{O}(\Delta)$. The last term is of order $\mathcal{O}(\varepsilon|\log\varepsilon| + \delta + \sqrt{\varepsilon\delta})$. Consequently the following accuracy result for a call-on-call option price can be obtained:

Theorem 3.2 *When the payoffs h_1 and h_2 are typical calls of a compound option with the strikes K_1 and K_2 respectively, for fixed (t,x,y,z) and $t < T_1 < T_2$, there exists a constant $C > 0$ such that*

$$
\begin{aligned}
&\left|U^{\varepsilon,\delta}(t,x,y,z;T_1,T_2,K_1,K_2) - \left(U_0+\tilde{U}_1+\tilde{U}_2\right)(t,x;T_1,T_2,K_1,K_2;\bar\sigma(z))\right|\\
&\le C\max\{\varepsilon \mid \log\varepsilon \mid, \delta, \sqrt{\varepsilon\delta}\}
\end{aligned}
$$

We summarize computational procedures for the price approximation of a call-on-call option. Based on the result of Geske [8], $U_0(t,x;\bar\sigma(z))$ has a close-form solution

$$\boxed{\begin{array}{l} U_0(t,x;T_1,T_2,K_1,K_2;\bar{\sigma}(z)) = x\mathcal{N}_2(g+\bar{\sigma}(z)\sqrt{\tau_1}, k+\bar{\sigma}\sqrt{\tau_2};\rho) \\ -K_1 e^{-r\tau_2}\mathcal{N}_2(g,k;\rho) - K_2 e^{-r\tau_1}\mathcal{N}_1(g) \end{array}} \tag{36}$$

Parameters are defined as

$$\begin{aligned}
\tau_1 &= T_1 - t, \\
\tau_2 &= T_2 - t, \\
g(\bar{x}, z) &= \frac{\log(x/\bar{x}) + (r - \bar{\sigma}(z)^2/2)\,\tau_1}{\bar{\sigma}(z)\sqrt{\tau_1}}, \\
k(z) &= \frac{\log(x/K_1) + (r - \bar{\sigma}(z)^2/2)\,\tau_2}{\bar{\sigma}(z)\sqrt{\tau_2}} \\
\rho &= \sqrt{\frac{\tau_1}{\tau_2}}.
\end{aligned}$$

The critical point $\bar{x}$ is the value of x solved uniquely from the following nonlinear algebraic equation

$$x\mathcal{N}_1(k+\bar{\sigma}(z)\sqrt{\tau_2}) - K_2 e^{-r\tau_2}\mathcal{N}_1(k) - K_1 = 0.$$

We denote the univariate cumulative normal integral by $\mathcal{N}_1(\cdot)$ and the bivariate cumulative integral by

$$\mathcal{N}_2(a,b;\rho) := \frac{1}{2\pi}\int_{-\infty}^{a}\int_{-\infty}^{b} e^{-\frac{w^2-2\rho wz+z^2}{2(1-\rho^2)}}\,dwdz,$$

where a and b are arbitrary real numbers and ρ is the correlation.
The first correction $\tilde{U}_1$ has an explicit form

$$\boxed{\begin{array}{l} \tilde{U}_1(t,x;T_1,T_2,K_1,K_2;\bar{\sigma}(z)) \\ = (T_1-t)(\mathcal{A}^{\varepsilon}+\mathcal{B}^{\delta})U_0(t,x;T_1,T_2,K_1,K_2;\bar{\sigma}(z)) \end{array}} \tag{37}$$

The second correction $\tilde{U}_2$ solves the Black-Scholes PDE with a terminal condition

$$\boxed{\begin{array}{l} \mathcal{L}_{BS}(\bar{\sigma}(z))\tilde{U}_2(t,x;T_1,T_2,K_1,K_2;\bar{\sigma}(z)) = 0 \\ \tilde{U}_2(T_1,x;\bar{\sigma}(z)) = \\ (T_2-T_1)I_{\{P_0(T_1,x;T_2,K_2;\bar{\sigma}(z))\geq K_1\}}(\mathcal{A}^{\varepsilon}+\mathcal{B}^{\delta})P_0(T_1,x;T_2,K_2;\bar{\sigma}(z)) \end{array}} \tag{38}$$

For ease of discussion on calibration in Section 4, the second correction $\tilde{U}_2$ is decomposed into four different pieces by linearity of the Black-Scholes PDE in (38):

$$\tilde{U}_2(T_1, x, z) = (T_2 - T_1)\left(V_0^\delta R_0 + V_1^\delta R_1 + V_2^\varepsilon R_2 + V_3^\varepsilon R_3\right), \tag{39}$$

where $R_i(t, x, z)$ solves the same Black-Scholes PDE but with different terminal conditions defined by

$$\begin{aligned} R_i(T_1, x; \bar{\sigma}(z)) &= I_{\{P_0 \geq K_1\}} V_i^\delta \frac{\partial^{1+i}}{\partial x^i \partial \sigma} P_0, \\ R_j(T_1, x; \bar{\sigma}(z)) &= I_{\{P_0 \geq K_1\}} V_j^\varepsilon x^{4-j} \frac{\partial}{\partial x}\left(x^{2j-4} \frac{\partial^{j-1}}{\partial x^{j-1}}\right) P_0, \end{aligned}$$

where $i \in \{0, 1\}$ and $j \in \{2, 3\}$.

4 Implied Volatility and Calibration

When volatility consists of a fast mean-reverting and a slowly varying diffusion processes, we have performed a combination of singular and regular perturbation techniques to approximate the compound option price (17) (resp. to (33)) under the stochastic volatility model by the price of a compound option (30) (resp. to (36)) under constant volatility $\bar{\sigma}(z)$ and its corrections (31, 32) (resp. to (37,38)). The main parameters of interests $\bar{\sigma}(z)$, V_0^δ, V_1^δ, V_2^ε, and V_3^ε used to calculate the compound option price approximation can be estimated from historical data and the underlying European option prices, which is encapsulated in the implied volatility surface. That is, $\bar{\sigma}(z)$ is estimated from the historical underlying risky asset prices and $\left(V_0^\delta, V_1^\delta, V_2^\varepsilon, V_3^\varepsilon\right)$ are calibrated to the term structure of implied volatility. This procedure is robust and described in Section 4.1. Moreover in Section 4.2 we define the implied compound volatility and discuss the calibration issue from compound option prices.

4.1 Calibration of Vanilla European Options Asymptotics

Recall [6] that the implied volatility $I^{\varepsilon,\delta}$ of a European option price can be approximated by

$$I^{\varepsilon,\delta} \approx \bar{\sigma}(z) + b^\varepsilon + a^\varepsilon \frac{\log(K/x)}{T-t} + a^\delta \log(K/x) + b^\delta (T-t), \tag{40}$$

where z-dependent parameters $a^\varepsilon, b^\varepsilon, a^\delta$, and b^δ are defined by

$$a^\varepsilon = \tfrac{V_3^\varepsilon}{\bar\sigma(z)^3}, \quad b^\varepsilon = \tfrac{V_2^\varepsilon}{\bar\sigma(z)} - \tfrac{V_3^\varepsilon}{\bar\sigma(z)^3}\left(r - \tfrac{\bar\sigma(z)^2}{2}\right),$$
$$a^\delta = \tfrac{V_1^\delta}{\bar\sigma(z)^2}, \quad b^\delta = V_0^\delta - \tfrac{V_1^\delta}{\bar\sigma(z)^2}\left(r - \tfrac{\bar\sigma(z)^2}{2}\right).$$

Therefore,

$$V_0^\delta = b^\delta + \frac{a^\delta}{\bar\sigma(z)}\left(r - \frac{\bar\sigma(z)^2}{2}\right), \tag{41}$$
$$V_1^\delta = a^\delta\bar\sigma(z), \tag{42}$$
$$V_2^\varepsilon = \bar\sigma(z)\left(b^\varepsilon + a^\varepsilon\left(r - \frac{\bar\sigma^2(z)}{2}\right)\right), \tag{43}$$
$$V_3^\varepsilon = a^\varepsilon\bar\sigma^3, \tag{44}$$

are deduced. The relation (40) provides a regression procedure to estimate the group of parameters. In practice, $\bar\sigma(z)$ is first estimated from high frequency historical data over a period of time less than $1/\delta$. Then $\left(a^\varepsilon, b^\varepsilon, a^\delta, b^\delta\right)$ are calibrated to the observed term structure of implied volatility by using (40) so that $V_0^\delta, V_1^\delta, V_2^\varepsilon$, and V_3^ε are obtained.

4.2 Implied Compound Volatility and Calibration

The implied compound volatility $I_{cc}^{\varepsilon,\delta}$ of a call-on-call option is defined to satisfy

$$U_0(t,x;T_1,T_2,K_1,K_2;I_c^{\varepsilon,\delta}) = U^{obs}, \tag{45}$$

where U_0 is defined in (36) and U^{obs} denotes an observed call-on-call option price given $(t,x;T_1,T_2,K_1,K_2)$. The compound implied volatility is assumed to have the following expansion in ε and δ:

$$I_{cc}^{\varepsilon,\delta} = I_0 + \sqrt{\epsilon}I_1 + \sqrt{\delta}I_2 + \cdots. \tag{46}$$

From Theorem 3.2 the observed true price of a call-on-call can be approximated by

$$U^{obs} \approx U_0 + \tilde U_1 + \tilde U_2.$$

Calibration procedure are done by successive comparison with expansions on both sides in (45). Up to the first order expansion, we get

$$U_0(t,x;I_0) + \left(\sqrt{\epsilon}I_1 + \sqrt{\delta}I_2\right)\frac{\partial U_0}{\partial\sigma}(t,x;I_0) = U_0(t,x;\bar\sigma(z)) + \left(\tilde U_1 + \tilde U_2\right)(t,x;\bar\sigma(z)).$$

Equating the leading order term $U_0(t,x;I_0) = U_0(t,x;\bar\sigma(z))$,

$$I_0 = \bar\sigma(z) \tag{47}$$

are deduced because that Vega,

$$\frac{\partial U_0}{\partial \sigma} = \frac{N_2(h+\bar\sigma\sqrt{\tau_1}, k+\bar\sigma\sqrt{\tau_2};\rho)}{N_1(k+\bar\sigma\sqrt{\tau_2})} K_1 e^{-r\tau_2} N_1'(k)\sqrt{\tau_2}$$

is positive. Next the first order term gives

$$\sqrt{\epsilon} I_1 + \sqrt{\delta} I_2 = \left(\tilde{U}_1 + \tilde{U}_2\right)(t,x;\bar\sigma(z)) \left[\frac{\partial U_0}{\partial \sigma}(t,x;\bar\sigma(z))\right]^{-1},$$

where (47) is used. Substituting this result with the expression (37, 39) and (47) into (46), a regression procedure to calibrate V_0^δ, V_1^δ, V_2^ε and V_3^ε is obtained by fitting

$$\begin{aligned} &I_c^{\varepsilon,\delta} - \bar\sigma(z) = \\ &\left\{(T_1 - t)\left[V_0^\delta \frac{\partial}{\partial\sigma} + V_1^\delta x \frac{\partial^2}{\partial x \partial \sigma} + V_2^\varepsilon x^2 \frac{\partial^2}{\partial x^2} + V_3^\varepsilon x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2}{\partial x^2}\right)\right] U_0 \right. \\ &\left. + (T_2 - T_1)\left[V_0^\delta R_0 + V_1^\delta R_1 + V_2^\varepsilon R_2 + V_3^\varepsilon R_3\right]\right\} \left[\frac{\partial U_0}{\partial\sigma}(t,x;T,K;\bar\sigma(z))\right]^{-1} \end{aligned} \tag{48}$$

in the least square sense. Explicit formulas of derivatives on the homogenized compound option price U_0 can be found in [10] and we skip them here.
Remark: We should have lumped together coefficients of these small parameters $\left(V_0^\delta, V_1^\delta, V_2^\varepsilon, V_3^\varepsilon\right)$ in the expression (48). The main reason we present (48) in such a way is to distinguish different roles $\left(V_0^\delta, V_1^\delta, V_2^\varepsilon, V_3^\varepsilon\right)$ play according to different time to maturities they are associated with. For example, the second set of $\left(V_0^\delta, V_1^\delta, V_2^\varepsilon, V_3^\varepsilon\right)$ related to $T_2 - T_1$ contain information of "forward volatility" of the underlying option defined between time T_1 and T_2, because the terminal condition in (38) is given by the solution solved from the forward time T_2 to T_1. While modeling volatility by one stochastic model between time t and T_1 and another stochastic model between T_1 and T_2, each set of $\left(V_0^\delta, V_1^\delta, V_2^\varepsilon, V_3^\varepsilon\right)$ has different values according to the time region they stay. In practice, the first set parameters $\left(V_0^\delta, V_1^\delta, V_2^\varepsilon, V_3^\varepsilon\right)$ is calibrated to the term structure of implied volatility by (40), then the second set parameters is calibrated to the implied compound volatility surface by (48).
For simplicity of discussion in this paper we confine the stochastic volatility model to be always the same from the current time t to the later maturity T_2.

5 Numerical Computation

Numerical PDE computation for compound option prices under the two-factor stochastic volatility model (3) requires to solve two iterative three-dimensional parabolic type PDEs within an unbounded domain. The first PDE solves for the underlying option price $P^{\varepsilon,\delta}$ at the expiry date T_1. It is the same PDE as in (18) with the call function with a strike price K_1 as the terminal condition. The second PDE (18) solves the compound option price. The difficulty of this approach is not just that solving these three-dimensional PDEs are time consuming, but also whether the specification of the full model is validated or not. As presented in Section 3, we alternatively provide an approximated price of compound options. It allows us to take advantage of the close-form solution of homogenized compound options (36) with the constant volatility $\bar{\sigma}(z)$ such that we only need to solve a one-dimensional PDE (38). The group of universal parameters $\left(V_0^\delta, V_1^\delta, V_2^\varepsilon, V_3^\varepsilon\right)$ used to compute price corrections (37, 38) are calibrated to the term structure of implied volatility or the implied compound volatility as discussed in the last section.

We choose a call-on-call option as an example and calculate its price approximation (36, 37, 38). Parameters are chosen as: the effective volatility $\bar{\sigma}(z) = 0.2$, the risk-free interest rate r=0.06, the call-on-call option strike $K_1 = 3$, the underlying call option strike $K_2 = 25$. The current time t is 0, the expiry of the call-on-call is $T_1 = 0.5$, the expiry of the underlying call option is $T_2 = 1.5$. The group parameters are chosen as typical values fitted from implied volatility of S&P 500 [6]: $V_0^\delta = 0.045$, $V_1^\delta = 0.0025$, $V_2^\varepsilon = 0.0047$, $V_3^\varepsilon = 0.000154$. In the top plot of Figure 1, the solid line presents the homogenized price of a call-on-call, i.e. the solution of (36), and the dashed line presents the approximated price of a call-on-call, i.e. the sum of solutions in (36,37, 38). The bottom plot shows contributions of the first correction (37) in solid line and the second correction (38) in dashed line respectively. It is observed that the relative magnitude of these corrections to the homogenized call-on-call can be relatively significant. For instance, around the stock price $S = 27$ the relative magnitude can be as much as 30%.

6 Parameter Reduction

We have described in Section 4 how to estimate the effective volatility $\bar{\sigma}(z)$ from the historical stock prices and how the group parameters $\left(V_0^\delta, V_1^\delta, V_2^\varepsilon, V_3^\varepsilon\right)$

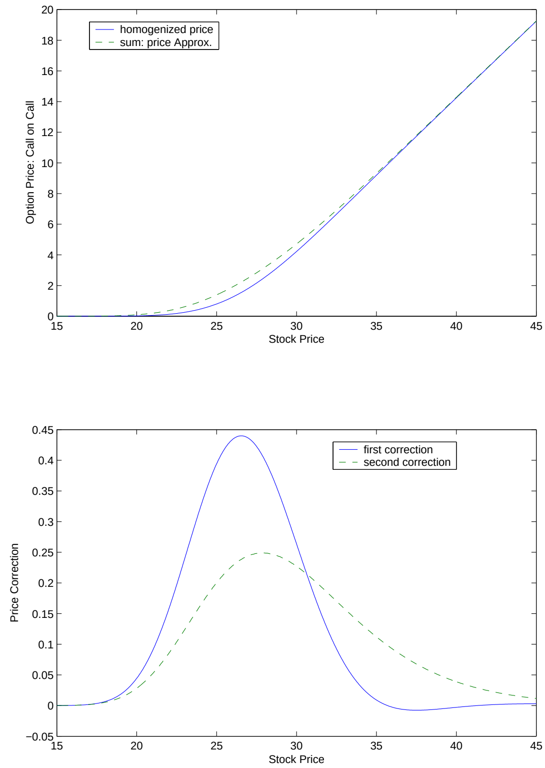

Figure 1: Finite difference numerical solutions of the first order price approximations to a call-on-call option (top plot) and solutions for the corrections to the homogenized call-on-call price (bottom plot).

are calibrated to the implied volatility surface. These parameters are essential to compute price approximations to compound options. It is observed, for instance in (7), that the effective volatility $\bar{\sigma}^2(z)/2$ and V_2^ε are coefficients of second order derivatives in x. Fouque et al. [7] define a corrected volatility $\sigma^\star$:

$$\sigma^\star := \sqrt{\bar{\sigma}^2(z) + 2V_2^\varepsilon(z)}.$$

and derive modified price approximations, $P_0^\star + \tilde{P}_1^\star$, for European options at the corrected volatility $\sigma^\star$. The zero order term $P_0^\star$ solves

$$\begin{cases} \mathcal{L}_{BS}(\sigma^\star)P_0^\star(t,x) = 0 \\ P_0^\star(T,x) = h(x), \end{cases}$$

and the second order term $\tilde{P}_1^\star$ solves

$$\begin{cases} \mathcal{L}_{BS}(\sigma^\star)\tilde{P}_1^\star(t,x) = -\left(\mathcal{A}_\star^\varepsilon + 2\mathcal{B}^\delta\right) P_0^\star(t,x) \\ \tilde{P}_1^\star(T,x) = 0. \end{cases}$$

Since V_2^ε is absorbed to $\bar{\sigma}(z)$, the original differential operator A^ε in (6) is thus reduced to

$$A_\star^\varepsilon := V_3^\varepsilon x \frac{\partial}{\partial x}\left(x^2 \frac{\partial^2}{\partial x^2}\right).$$

Similarly to the explicit solution of $\tilde{P}_1$, we use the homogeneous property in x-derivatives and obtain

$$\tilde{P}_1^\star(t,x;\sigma^\star) = (T-t)\left(\mathcal{A}_\star^\varepsilon + \mathcal{B}^\delta\right) P_0^\star(t,x;\sigma^\star).$$

As a matter of fact that the difference or the error, denoted by E, between these two price approximations, $P_0 + \tilde{P}_1$ and $P_0^\star + \tilde{P}_1^\star$, is of $\mathcal{O}\left(\max\{\varepsilon,\delta\}\right)$ when the payoff h is smooth with bounded derivatives. By operating the Black-Scholes partial differential operator (21) at the homogenized volatility $\bar{\sigma}(z)$ to the error E, we observe that the source term is indeed of $\mathcal{O}\left(\varepsilon+\delta\right)$. Therefore, we deduce the desired error estimation. Combining this result with (5), we obtain

$$\begin{aligned} &P^{\varepsilon,\delta}(t,x,y,z;T,h) - \left(P_0^\star(t,x;T,h;\sigma^\star) + \tilde{P}_1^\star(t,x;T,h;\sigma^\star)\right) \\ &= \mathcal{O}(\max\{\varepsilon,\delta\}). \end{aligned} \tag{49}$$

Moreover, using the modified price approximation $P_0^\star + \tilde{P}_1^\star$, the approximation of the implied volatility skew

$$I^\star \approx b_0 + b_1(T-t) + \{m_0 + m_1(T-t)\}\frac{\log(K/x)}{T-t}$$

can be used as a regression [7] to estimate the new group parameters $\left(\sigma^\star, V_0^\delta, V_1^\delta, V_3^\varepsilon\right)$, where

$$
\begin{array}{ll}
\sigma^\star = b_0 + m_0\left(r - \frac{b_0^2}{2}\right), & V_0^\delta = b_1 + m_1\left(r - \frac{b_0^2}{2}\right), \\
V_1^\delta = m_1 b_0^2, & V_3^\varepsilon = m_0 b_0^3.
\end{array}
$$

Note that all parameters including $\sigma^\star$ can be calibrated to the term structure of implied volatility. As a consequence, the compound option price formula we derive in (30), (31) and (32) can be modified such that the zero order term $U_0^\star$ solves

$$
\boxed{\begin{array}{l}
\mathcal{L}_{BS}(\sigma^\star) U_0^\star(t,x;T_1,T_2,h_1,h_2;\sigma^\star) = 0 \\
U_0^\star(T_1,x;T_1,T_2,h_1,h_2;\sigma^\star) = h_1\left(P_0^\star(T_1,x;T_2,h_2;\sigma^\star)\right)
\end{array}}
$$

the first correction $\tilde{U}_1^\star$ has an explicit solution

$$
\boxed{\tilde{U}_1^\star(t,x;T_1,T_2,h_1,h_2;\sigma^\star) = (T_1 - t)\left(\mathcal{A}_\star^\varepsilon + \mathcal{B}^\delta\right) U_0^\star(t,x;\sigma^\star)}
$$

and the second correction $\tilde{U}_2^\star$ solves the other one-dimensional Black-Scoles PDE with nontrivial terminal condition

$$
\boxed{\begin{array}{l}
\mathcal{L}_{BS}(\sigma^\star) \tilde{U}_2^\star(t,x;T_1,T_2,h_1,h_2;\sigma^\star) = 0 \\
\tilde{U}_2^\star(T_1,x;T_1,T_2,h_1,h_2;\sigma^\star) = h_1'\left(P_0^\star(T_1,x;\sigma^\star)\right) \tilde{P}_1^\star(T_1,x;T_2,h_2;\sigma^\star) \\
= (T_2 - T_1) h_1'\left(P_0^\star(T_1,x;\sigma^\star)\right)\left(\mathcal{A}_\star^\varepsilon + \mathcal{B}^\delta\right) P_0^\star(T_1,x;\sigma^\star)
\end{array}}
$$

These modifications of compound option price approximations require less efforts on calibration (from $\left(\bar{\sigma}(z), V_0^\delta, V_1^\delta, V_2^\varepsilon, V_3^\varepsilon\right)$ to $\left(\sigma^\star, V_0^\delta, V_1^\delta, V_3^\varepsilon\right)$) as well as numerical PDE computation (from A^ε to $A_\star^\varepsilon$).
When payoffs h_1 and h_2 are smooth with bounded derivatives, these modified price approximations of compound options, $U_0^\star + \tilde{U}_1^\star + \tilde{U}_2^\star$, still keep the same accuracy as shown in Theorem 3.1 such as

$$
\begin{aligned}
&\left| U^{\varepsilon,\delta}(t,x,y,z;T_1,T_2,h_1,h_2) - \left(U_0^\star + \tilde{U}_1^\star + \tilde{U}_2^\star\right)(t,x;T_1,T_2,h_1,h_2;\sigma^\star)\right| \\
&\leq C \max\{\varepsilon, \delta, \sqrt{\varepsilon\delta}\} \qquad (50)
\end{aligned}
$$

The following statement explains the accuracy result. Applying a Taylor expansion on the payoff (16) and using the modified price approximation

(49), an analog accuracy result like (25) can be obtained:

$$h_1\left(P^{\varepsilon,\delta}(T_1, S_{T_1}, Y_{T_1}, Z_{T_1}; T_2, h_2)\right) = h_1\left(P_0^\star(T_1, S_{T_1}; T_2, h_2; \sigma^\star)\right)$$
$$+h_1^{'}\left(P_0^\star(T_1, S_{T_1}; T_2, h_2; \sigma^\star)\right)\tilde{P}_1^\star(T_1, S_{T_1}; T_2, h_2; \sigma^\star) + \mathcal{O}(\max\{\varepsilon, \delta\}).$$

Then we can straightforwardly generalize the argument in Section 3.2 to obtain the desired accuracy result (50).

7 Conclusion

We present an asymptotic analysis to derive price approximations to compound options under the two-factor stochastic volatility model (3). For instance, the first-order price approximation to a call-on-call option requires to apply Geske's formula (36), calculate explicit derivatives (37), and solve a one-dimensional PDE (38). These reduce significantly computational efforts in comparison with solving two iterative three-dimensional PDEs under the full specification of the stochastic volatility model. Accuracy results are provided. Moreover, the group parameters needed in our computational procedure are calibrated to the term structure of implied volatility.

A compound option defined as an option on another option are naturally sensitive to the forward changes in volatility, which is embedded in the underlying option. With our perturbation techniques, we explain that up to the first order price approximation compound options are weakly dependent on forward volatility in the sense that the dependence on volatility level in the payoff of a compound option can be replaced by a constant volatility.

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References

[1] S. Alizadeh, M. Brandt, and F. Diebold, "Range-based estimation of stochastic volatility models," Journal of Finance, 57, 1047-1091, 2002.

[2] D. Breeden, E. Y. Ou, and J. E. Zhang, "Hedging volatility risk," preprint, New York University, 2000.

[3] P. Carr and D. Madan, “Towards a Theory of Volatility Trading”, In Volatility: New Estimation Techniques for Pricing Derivatives(London: Risk Publications).

[4] M. Chernov, R. Gallant, E. Ghysels, and G. Tauchen, “Alternative models for stock price dynamics,” Preprint, Columbia University, January 2001.

[5] J.-P. Fouque and C.-H. Han, “Option Pricing with Nonsmooth Payoffs under Stochastic Volatility ,” in preparation.

[6] J.-P. Fouque, G. Papanicolaou, R. Sircar, and K. Solna, “Multiscales Stochastic Volatility Asymptotics,” SIAM Journal on Multiscale Modeling and Simulation 2(1), 2003 (22-42).

[7] J.-P. Fouque, G. Papanicolaou, R. Sircar, and K. Solna, “Timing The Smile,” Wilmott magzine(March 2004).

[8] R. Geske, ”The Valuation of Compound Options”, J. Financial Economics, 7(1979a), 63-81.

[9] R. Geske and H. E. Johnson, “The American put option valued analytically,” Journal of Finance, 1984.

[10] C.-H. Han, ”Singular Perturbation on Non-Smooth Boundary Problems in Finance,” Dissertation, 2003.

[11] S. Howison, A. Rafailidis, and H. Rasmussen, “On The Pricing and Hedging of Volatility Derivatives,” OFRC Working Papers Series, Oxford University, 2003.

[12] G. Molina, C.H. Han and J.P. Fouque, “ MCMC Estimation of Multiscale Stochastic Volatility Models,” Preprint, 2003.