Generalized Control Variate Methods for Pricing Asian Options

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Abstract

The conventional control variate method proposed by Kemna and Vorst (1990) to evaluate Asian options under the Black-Scholes model can be interpreted as a particular selection of linear martingale controls. We generalize the constant control parameter into a control process to gain more reduction on variance. By means of an option price approximation, we construct a martingale control variate method, which outperforms the conventional control variate method. It is straightforward to extend such linear control to a nonlinear situation such as the American Asian option problem. From the variance analysis of martingales, the performance of control variate methods depends on the distance between the approximate martingale and the optimal martingale. This measure becomes helpful for the design of control variate methods for complex problems such as Asian option under stochastic volatility models. We demonstrate multiple choices of controls and test them under MC/QMC (Monte Carlo/ Quasi Monte Carlo)- simulations. QMC methods work significantly well after adding a control, the variance reduction ratios increase to 260 times for randomized QMC compared with 60 times for MC simulations with a control.

1 Introduction

Control variate methods have been widely used for computational finance as a mean of variance reduction. Perhaps the most successful application is to evaluate continuous-time arithmetic-average Asian options. Kemna and Vorst [14] employed a control, which is a discounted counterpart geometric-average Asian option payoff less than its price. This method is efficient because the correlation between the arithmetic-average and the geometric-average is high, and the counterpart geometric-average Asian option price has a closed-form solution. Therefore, criteria for constructing control variates are: (i) highly correlated payoff random variable with (ii) closed-form expectation. There are sophisticated techniques such as stratified sampling or Brownian bridge, which are developed to combine with the control variate method in order to enhance variance reduction. See [7] and references therein for more discussions. However, for complex problems such as American style or the case when volatility is random, the criteria for constructing control variates may not be easy to satisfy.

In this paper we investigate the conventional control variate method proposed by Kemna and Vorst

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as a linear control with a constant control parameter. Combining the martingale representation of the control with another control *process*, rather than a parameter, we generalize the conventional control variate so that the controlled variance is reduced to zero in the optimal case. It turns out that the variance induced from the conventional control variate method is a constant projection from a stochastic control parameter space. Using some option price approximation, we reduce the generalized control variate to a martingale control variate. Therefore, Therefore, conventional control variate (or martingale control variate) can be viewed as a special case for generalized control variate by using a constant control parameter (or an approximate control process), respectively.

The martingale control constructed from a more accurate price approximation is more correlated to the original discounted payoff, and its mean is zero. This idea links studies of control variate methods to option price approximations. There are numerous studies on arithmetic-average Asian option prices even under the Black-Scholes model. See [18, 27, 30] for example. Using a recent approximation by Zhang [30], we can construct a new martingale control for the use of variance reduction, which outperforms the conventional control variate in terms of generating smaller standard errors. In particular, these martingale controls are applicable to nonlinear problems such as American Asian options. There is only one uncertainty induced by Brownian motion under the Black-Scholes model. Hence, martingale controls defined on the same uncertainty are expected to reduce the variance. We confirm this with many numerical results in this paper.

In the context of stochastic volatility models, construction of control variate is less known and studied. Clewlow and Carverhill [1] applied a financial intuition of delta hedging to build portfolio possibly with other Greeks hedge as controls in order to evaluate option prices. They found that the variance obtained by this control variate under an one-factor stochastic volatility model [9] can be significantly reduced. By means of the least-squares method [16], Potters, Bouchaud, and Sestovic [21] proposed an optimal hedged Monte Carlo method to price options under the "historical" probability measure. Later, Pochart and Bouchaud [22] extended the same idea to problems that consider shortfall risk and transaction cost constraints. Heath and Platen [8] and Fouque and Han [5] used different approximate European option prices to approximate *delta* portfolio under stochastic volatility models and showed significant variance reduction. Moreover, in [3], Fouque and Han obtained an asymptotic result to characterize the variance of a martingale control variate under stochastic volatility models when the *time scales* of driving volatility processes are well-separated.

Based on the construction of hedging martingales under stochastic volatility models, we further investigate the Asian option pricing problem. We find that using the hedging martingale as a one-step control can only partially reduce uncertainties associated with driving volatility processes. In contrast, the conventional control by the geometric-average counterpart can be shown to reduce the variance of each source of randomness, though the counterpart option price has no closed-form solution. Therefore, we propose a two-step algorithm to overcome this difficulty.

All Monte Carlo methods mentioned so far are fundamentally related to pseudo random sequences. Integration methods using the alternative quasi-random sequences (or called low-discrepancy sequences) have drawn much attention in recent years because its theoretical rate of convergence is $\mathcal{O}(1/n^{1-\varepsilon})$ for all $\varepsilon > 0$ subjected to the dimensionality and the regularity of the integrand. These are the so-called quasi-Monte Carlo (QMC) methods. We refer to [7], [11] and [13] for further discussions on MC/QMC methods in applications of computational finance. We use randomized QMC to deal with a non-smooth call payoff and a high dimension of 384 in these experiments under stochastic volatility models. After adding controls to our estimators, QMC methods are able to reduce variance significantly in all cases. Even in a high dimensional regime, the control starts to play the role of a smoother. It can be seen from variance analysis in Section 4.3 that on average the fluctuation of the control variate is continuous and of small order. In [5], the authors study the smoothing effect by martingale controls. An example of estimating low-biased solutions of an American option by least squares method [16] shows that using QMC, including Niederreiter sequence and Sobol' sequence, actually gives estimates greater than the true option price. However, adding a martingale control for variance reduction can cause all problematic estimates to become low-bias and very close to true values.

The organization of this paper is as follows. In Section 2, we review the conventional control variate method and generalize it in a stochastic framework. By option price approximations, we construct new martingale controls and apply them to American Asian option problems. In Section 3, we introduce multi-factor stochastic volatility models. In Section 4 to evaluate geometric-average Asian options, we apply a singular and regular perturbation method to construct a martingale control variate method, which outperforms an importance sampling studied in [5]. We present a variance analysis for a simplified perturbed volatility model. By a combination of martingale control variate method for the geometric-average Asian option and the conventional control variate method for the arithmetic-average Asian option, we propose a two-step method in Section 5. We present numerical experiments implemented by MC/QMC methods. We test several combinations of control variate methods with randomized QMC methods, including the Sobol' sequence and L'Ecuyer type good lattice points together with the Brownian bridge sampling technique.

2 Control Variate Method: Revisit and a Generalization

We consider the estimation of a mathematical expectation $E\{X\}$ by Monte Carlo simulations, where X is a one-diemnsional square-integrable random variable defined on a probability space. A control variate method is a variance reduction technique which aims to improve the precision of the estimate obtained from plain Monte Carlo simulations [7, 17]. The basic idea of control variate is to choose an appropriate counterpart square-integrable random vector, say $Y \in \Re^d$ being centered at zero, and multiple control parameters $\lambda \in \Re^d$ so that the control variate $X + \lambda^T Y$ is unbiased, $E\{X\} = E\{X + \lambda^T Y\}$, and the variance can be reduced, $Var\{X\} \geq Var\{X + \lambda^T Y\}$. This can be done by choosing (i) any mean-zero random vector Y correlated with X and (ii) the optimal control parameters, deduced from minimizing the variance of the control variate,

$$\lambda^\star = -\Sigma_{YY}^{-1} \Sigma_{XY}, \tag{1}$$

where Σ_{XY} denotes the covariance of X and Y. From a straightforward calculation, one can see that the new variance can not be greater than the original variance:

$$Var\{X + {\lambda^\star}^T Y\} = \left(1 - R^2\right) Var\{X\} \leq Var\{X\}, \tag{2}$$

where $R^2 = \frac{\Sigma_{XY}^T \Sigma_{YY}^{-1} \Sigma_{XY}}{\Sigma_{XX}}$ is a generalization of the squared correlation coefficient or known as the coefficient of determination in regression analysis. In the single control parameter case, i.e. $Y \in \Re$, we have $R = \rho$, the correlation coefficient between random variables X and Y. When random variables X and Y are highly correlated, either positive or negative, R^2 is close to one and a considerable reduction of variance is expected.

Because calculating the optimal control parameter $\lambda^\star$ requires the exact value of $E\{X\}$, in practice one can only use a suboptimal control parameter λ by approximating the right hand side of (1) empirically through plain Monte Carlo simulations. A sensitivity analysis considering the error of variances obtained from the optimal control variate parameter $\lambda^\star$ and its perturbation λ^ε is shown below.

Lemma 1 *Given any constant ε and an identity vector $I = (1, \cdots, 1)^T \in \Re^{d \times 1}$ such that the suboptimal control parameter $\lambda^\varepsilon = \lambda^\star + \varepsilon I$ being a perturbed parameter, the gain of variance from the* *suboptimal control variate is of order ε^2:*

$$Var\left\{X + \lambda^{\varepsilon^T} Y\right\} - \left(1 - R^2\right) Var\{X\} = \varepsilon^2\, Var\{I^T Y\}.$$

Proof: It is easy to calculate the variance of the perturbed control variate

$$\begin{aligned} Var\left\{X + (\lambda^\star + \varepsilon\, I)^T\, Y\right\} &= Var\left\{X + \lambda^{\star^T} Y\right\} + \varepsilon^2 Var\{I^T\, Y\} \\ &= \left(1 - R^2\right) Var\{X\} + \varepsilon^2 Var\{I^T\, Y\}, \end{aligned}$$

where we have used the definition in (1), $Cov(X + \lambda^{\star^T} Y, I^T\, Y) = 0$, and a result in (2).

Note that the simple variance analysis shown in (1) and (2) does not depend on the variance of multiple controls. They become significant when we study sensitivity analysis over multiple control parameters as shown in Lemma 1. The variance of the sum of multiple controls is not often emphasized in use of control variate methods, but we will see in Section 5 that it becomes important for a further reduction of variance when we compare two different sets of controls (one-step versus two-step control variates).
In computational finance a well-known example of using the control variate method is the evaluation of a continuous-time and arithmetic-average Asian options under the Black-Scholes model. Based on the risk-neutral pricing theory [24], the fair price of the Asian option, denoted by P_A, is equal to the conditional expectation under the risk-neutral probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, I\!P^\star\right)$

$$P_A(t, S_t, A_t) = I\!E^\star \left\{e^{-r(T-t)} H(A_T) | \mathcal{F}_t\right\} \tag{3}$$

where the underlying risky asset S_t is governed by a geometric Brownian motion defined as

$$dS_t = rS_t dt + \sigma S_t dW_t^\star, \tag{4}$$

the arithmetic-average price process A_t is defined by $A_t = \frac{1}{T} \int_0^t S_s\, ds$. Other notations are defined as follows: t is the current time, $T < +\infty$ is the maturity, r is the risk-free interest rate, σ is volatility, $W_t^\star$ is the standard Brownian motion, $H(x)$ is the payoff function satisfying the usual integrability condition. For example if $H(x) = \max\{x - K, 0\} \equiv (x - K)^+$ for the strike price $K > 0$, it is a call payoff; if $H(x) = \max\{K - x, 0\} \equiv (K - x)^+$, it is a put payoff.
The control variate method proposed by Kemna and Vorst [14] to evaluate P_A defined in (3) introduces a counterpart geometric-average price process $G_t = \exp(\frac{1}{T} \int_0^t \ln S_s\, ds)$ and a geometric-average Asian option price P_G defined by

$$P_G(t, S_t, G_t) = I\!E^\star \left\{e^{-r(T-t)} H(G_T) | \mathcal{F}_t\right\} \tag{5}$$

such that a control variate for estimating P_A is

$$e^{-r(T-t)} H(A_T) + \lambda \left(e^{-r(T-t)} H(G_T) - P_G(t, S_t, G_t)\right). \tag{6}$$

The success of this control variate is attributed to at least two facts:
• The control

$$e^{-r(T-t)} H(G_T) - P_G(t, S_t, G_t) \tag{7}$$

is highly correlated to the original discounted payoff variable $e^{-r(T-t)} H(A_T)$. This is confirmed by empirical tests. See for instance [7, 14] that correlations between the control variate (7) and $e^{-r(T-t)} H(A_T)$

are close to 1 in many examples. From Lemma 1 a small error in the control parameter does not affect the variance reduction much. Therefore, it is also practical to simply use a constant control parameter.
• The counterpart geometric-average Asian option price P_G admits a Black-Scholes type closed-form solution and the random variable G_T is defined on the same probability space as A_T so that Monte Carlo simulations for the control variate become easy to implement.

2.1 Generalized Control Variate

By an application of Ito's lemma, the control in (7) has the following martingale representation

$$e^{-rT}H(G_T) - P_G(0, S_0, G_0) = \mathcal{M}\left(\frac{\partial P_G}{\partial x}; T\right) \tag{8}$$

where the process $\mathcal{M}(\Delta; t)$ is a zero-centered martingale with

$$\mathcal{M}(\Delta; t) = \int_0^t e^{-rs}\Delta\left(s, S_s = x, G_s\right)\sigma S_s dW_s^\star. \tag{9}$$

The conventional control variate method utilizes a control parameter λ to obtain the minimized variance which is not zero.
We now introduce a generalized control variate by assuming that the control parameter λ is a $\mathcal{F}_t$-adapted process, denoted by λ_t, so that a new control variate becomes

$$e^{-r\,T}H(A_T) + \mathcal{M}\left(\lambda.\,\frac{\partial P_G}{\partial x}; T\right). \tag{10}$$

Its variance conditional at time zero is equal to

$$\begin{aligned} & Var\left\{e^{-r\,T}H(A_T) - P_A(0, S_0, A_0) + \mathcal{M}\left(\lambda.\,\frac{\partial P_G}{\partial x}; T\right)\right\} \\ &= Var\left\{\mathcal{M}\left(\frac{\partial P_A}{\partial x}; T\right) + \mathcal{M}\left(\lambda.\,\frac{\partial P_G}{\partial x}; T\right)\right\} \\ &= Var\left\{\mathcal{M}\left(\frac{\partial P_A}{\partial x} + \lambda.\,\frac{\partial P_G}{\partial x}; T\right)\right\} \\ &= \int_0^T e^{-2rs}\sigma^2 \mathbb{E}^\star\left\{\left(\frac{\partial P_A}{\partial x}\left(s, S_s, A_s\right) + \lambda_s \frac{\partial P_G}{\partial x}\left(s, S_s, G_s\right)\right)^2 S_s^2\right\} ds. \end{aligned} \tag{11}$$

In this calculation, we have used the linearity of stochastic integrals, Ito's isometry, and Fubini's theorem. To eliminate the variance, the optimal control *process*, rather than a parameter, is chosen as

$$\lambda_t^\star = -\frac{\partial P_A}{\partial x}\left(t, S_t, A_t\right) / \frac{\partial P_G}{\partial x}\left(t, S_t, G_t\right), a.s. \tag{12}$$

The optimal control process requires the exact value of P_A or its delta, i.e., $\frac{\partial P_A}{\partial x}$ in order to eliminate variance. Thus the generalized control variate can be viewed as in a dynamic setting, compared with the conventional control one as in a static setting. Note that the conventional control variate method can only reduce the variance proportionally as shown in (1) and (2), even if the optimal $\lambda^\star$ is used.
We show next that the optimal control parameter $\lambda^\star$ is simply a projection of the optimal control process over the real line $\Re$, i.e, the conventional control variate method is a special case of the generalized one.

Lemma 2 *For pricing Asian options by conventional control variate method, the optimally reduced variance in (2) can be obtained by solving the following minimization problem*

$$\left(1-\rho^{\star^2}\right) Var\left\{e^{-rT}H(A_T)\right\} = \min_{\lambda\in\Re} Var\left\{e^{-rT}H(A_T) + \mathcal{M}(\lambda.\frac{\partial P_G}{\partial x};T)\right\}.$$

Proof: From (11) the constant minimizer solving the variance above is

$$\lambda^\star = -\frac{Cov\left(\mathcal{M}(\frac{\partial P_A}{\partial x};T), \mathcal{M}(\frac{\partial P_G}{\partial x};T)\right)}{Var\left(\mathcal{M}(\frac{\partial P_G}{\partial x};T)\right)}$$

such that the optimally reduced variance is

$$I\!E^\star\left\{\int_0^T e^{-2rs}\left(\frac{\partial P_A}{\partial x}(s,S_s,A_s) + \lambda^\star\frac{\partial P_G}{\partial x}(s,S_s,G_s)\right)^2\sigma^2 S_s^2 ds\right\} = \left(1-\rho^{\star^2}\right) Var\left\{e^{-rT}H(A_T)\right\},$$

where $\rho^\star$ denotes the correlation of $e^{-rT}H(A_T)$ and the control $\mathcal{M}(\frac{\partial P_G}{\partial x};T)$.
Because the geometric-average control $\mathcal{M}(\frac{\partial P_G}{\partial x};T)$ is not perfectly correlated with the counterpart arithmetic-average payoff $e^{-rT}H(A_T)$, the variance of the control variate (6) can only be optimally reduced by the factor $1-\rho^{\star^2}$ with the choice of a constant control parameter $\lambda^\star$.

2.2 Construction of More Linear Controls

A practical way of implementing the generalized control variate method for variance reduction is to approximate the arithmetic-average Asian option price function P_A in the optimal control process (12). Let $\tilde{P}$ be a price approximation to P_A. A suboptimal control is deduced:

$$\tilde{\lambda}_t = \frac{\partial \tilde{P}}{\partial x}(t,S_t,A_t)/\frac{\partial P_G}{\partial x}(t,S_t,G_t). \tag{13}$$

Substituting $\tilde{\lambda}_t$ into (10), one can readily observe that this generalized control variate is reduced to $e^{-rT}H(A_T) - \mathcal{M}\left(\frac{\partial\tilde{P}}{\partial x};T\right)$, known as the martingale control variate. Hence the martingale control $\mathcal{M}\left(\frac{\partial\tilde{P}}{\partial x};T\right)$ can be thought of as a generalization of the conventional control variate by employing an approximate control process (13). One can of course incorporate another control parameter in front of the martingale control to gain additional variance reduction. By this construction, we are able to enlarge the class of control variate by martingale controls, in which approximate option price or its delta must be used.
There has been many studies on approximating arithmetic-average Asian option prices in recent decades. Our goal is not comparing variance induced from price approximations in these studies. Instead, we only demonstrate the construction of a generalized control variate as follows. We employ a price approximation with a closed form proposed by Zhang [30] based on a perturbation expansion on the singularity at the diffusion coefficient. For an Asian call option with the strike price K, we consider only the leading order expansion in [30] as

$$P_Z(t,S_t,A_t) = \frac{S_t}{T}f(\xi_t,T-t), \tag{14}$$

Table 1: Comparisons of three control variates to estimate the arithmetic-average Asian call option prices with a fixed strike price while constant volatility varies from 10% to 70 %. Other model parameters are chosen by $S_0 = 65, K = 55, r = 0.06, T = 1$ year. In column 2, CV denotes the conventional control as in (7); in column 3, $\mathcal{M}(\frac{\partial P_G}{\partial x}; T)$ denotes the equivalent martingale representation as in (8); in column 4, $\mathcal{M}(\frac{\partial P_Z}{\partial x}; T)$ denotes the new control as in (16). Sample means and standard errors in parenthesis are shown in pair. Monte Carlo simulations are implemented under the sample size $N = 10000$ and the discretized time step 200.

σ	CV	$\mathcal{M}(\frac{P_G}{\partial x}; T)$	$\mathcal{M}(\frac{\partial P_Z}{\partial x}; T)$
10	11.2920 (0.00052)	11.2910 (0.00051)	11.2920 (0.00019)
20	11.4010 (0.0018)	11.4020 (0.0021)	11.4010 (0.0013)
30	11.8950 (0.0038)	11.8930 (0.0046)	11.8960 (0.0031)
40	12.6820 (0.0066)	12.6840 (0.0081)	12.6790 (0.0053)
50	13.6230 (0.012)	13.6070 (0.012)	13.6280 (0.0079)
60	14.6670 (0.017)	14.6270 (0.017)	14.6870 (0.011)
70	15.7910 (0.024)	15.7310 (0.026)	15.7600 (0.014)

where the variable $\xi_t = \frac{T(K-A_t)}{S_t} e^{-r(T-t)} - \frac{1-e^{-r(T-t)}}{r}$ and the function $f(\xi, \tau) = -\xi \mathcal{N}(-\xi/\sqrt{2\tau}) + \sqrt{\frac{\tau}{\pi}} e^{-\xi^2/(4\tau)}$. The standard normal cumulative distribution function is denoted by $\mathcal{N}(\cdot)$. Thus, we construct a stochastic control process

$$\lambda_t = -\frac{\partial P_Z}{\partial x}(t, S_t, A_t) \Big/ \frac{\partial P_G}{\partial x}(t, S_t, G_t), \tag{15}$$

then form a martingale control

$$\mathcal{M}\left(\frac{\partial P_Z}{\partial x}; t\right) = \int_0^t e^{-rs} \frac{\partial P_Z}{\partial x}(s, S_s = x, A_s)\, \sigma S_s dW_s^{\star}. \tag{16}$$

In Table 1 we illustrate effects of three controls, including the conventional control of the geometric-average Asian option shown in (7), its equivalent martingale control $\mathcal{M}(\frac{\partial P_G}{\partial x}; T)$ as in (8), and another martingale control by Zhang's approximation $\mathcal{M}(\frac{\partial P_Z}{\partial x}; T)$ as in (16). Note that the last martingale control should be understood as a generalized control variate (10) using a stochastic control process (15) rather than a constant control parameter.

Because the three corresponding estimators are unbiased, we shall focus on standard errors. Since the first two controls are equivalent, it is observed that standard errors are roughly of the same order. The last control by $\mathcal{M}(\frac{\partial P_Z}{\partial x}; T)$ produces the smallest standard errors as expected. Their empirical control parameters ($\lambda^{\star}$) are calculated as 1.0090, 1.0011, and 0.9929 respectively. The three computing times are 0.23, 2.36, and 1.25 seconds respectively executed under a PC Pentium-4 2.4 GHz CPU with Matlab. Due to the calculation of stochastic integrals, martingale control variate require more computing time because the control is additive. But we will see in next section that when a noninear situation appears such as in the American Asian option pricing problem, the computational costs become roughly the same.

We have proposed a way to construct martingale control $\mathcal{M}(\frac{\partial \tilde{P}}{\partial x}; T)$ where $\tilde{P}$ is any price approximation to P_A. For instance if $\tilde{P} = P_G$, we recover the conventional control variate from (8) because the discounted process $e^{-rt} P_G(t, S_t, G_t)$ is a martingale. When we choose the approximation $\tilde{P} = P_Z$, the representation of the equivalent process-type control is no longer suitable for the use as a conventional control. This is because $e^{-rt} P_Z(t, S_t, A_t)$ is not a martingale which can be seen from a probability representation of the transformed function

$$f(\xi,\tau) = E\left\{max\left(-\varphi_T, 0\right) \mid \varphi_t\right\}, \tag{17}$$

where the process φ_t is governed by the stochastic differential equation

$$\begin{aligned} d\varphi_t &= \frac{\sigma}{r}(1 - e^{-r(T-t)})dW_t, \\ \varphi_0 &= \frac{T\,K}{S_0}e^{-r\,T} - \frac{1 - e^{-r\,T}}{r}, \end{aligned}$$

and W_t is a Brownian motion. Note that this W_t can be defined on a different probability space than $W_t^\star$ in (4). This fact implies that martingale controls are more general than conventional controls where concrete correlated processes must be postulated.

So far we have discussed only the European Asian options. We have reviewed and expanded the conventional control variate method [14] to martingale control variates through an approximation to the optimal control processs. One can readily observe that these controls are in a linear form. Next, we apply these martingale controls in a nonlinear form to the situation where early exercise is permitted.

2.3 Pricing American Asian Options: Nonlinear Control

In this section, we consider the American Asian option pricing problem under the Black-Scholes model. We study the price of an American Asian call option with a fixed strike price K as an example but other payoffs can be treated similarly. The American Asian call option price is a solution of an optimal stopping time problem

$$P_{AA}(t, S_t, A_t) = ess \sup_{t \le \tau \le T} I\!\!E^\star\left\{e^{-r\,(\tau - t)}(A_\tau - K)^+ | \mathcal{F}_t\right\}, \tag{18}$$

with τ being a bounded stopping time between the current time t and the maturity T. To solve an optimal stopping problem by simulation, known as a primal approach, is challenging even though there exist methods such as least squares method[16], etc. Because the primal approach is less related to control variate, we will focus directly on the dual approach to solve the problem (18). Using the dual formulation [4, 23], we have an equivalent expression at time zero:

$$P_{AA}(0, S_0, A_0) = \inf_{\mathcal{M} \in H_0^1} I\!\!E^\star\left\{\sup_{0 \le t \le T}\left((A_t - K)^+ - \mathcal{M}_t\right) | \mathcal{F}_0\right\}, \tag{19}$$

where martingales in the space H_0^1 are uniformly integrable and centered at zero at time zero. Hence, given a suitable martingale, a high-biased estimate of the American Asian price P_{AA} can be obtained. It turns out that the optimal martingale $\mathcal{M}_t^\star = \mathcal{M}(\frac{\partial P_{AA}}{\partial x}; t)$ gives the pathwise equality [4]

$$P_{AA}(0, S_0, A_0) = \sup_{0 \le t \le T}\left((A_t - K)^+ - \mathcal{M}_t^\star\right).$$

The price and variance error analysis are also given in [4] . Heuristically, we like to propose some sub-optimal martingales using P_{AA} price approximation such as its European counterpart approximation P_G or P_Z.

In Table 2 we illustrate the effect of two martingale controls, including the European geometric-average Asian option counterpart $\mathcal{M}(\frac{\partial P_G}{\partial x}; t)$ and Zhang's approximation $\mathcal{M}(\frac{\partial P_Z}{\partial x}; t)$, with various strike prices. Notice that the standard errors, shown in parenthesis and MAD (the mean absolute deviation from

Table 2: Comparisons of two martingale control variates in estimating American arithmetic-average Asian call option price when the fixed strike price K takes value of 45, 50 and 55. Other model parameters are $S_0 = 55, r = 0.1, \sigma = 40\%, T = 1$ year. In column 2, $\mathcal{M}(\frac{\partial P_G}{\partial x}; t)$ denotes the the martingale control as in (8); and MAD in column 3 is computed based on results in column 2. In column 4, $\mathcal{M}(\frac{\partial P_Z}{\partial x}; t)$ denotes the martingale control as in (16) and column 5 shows the associated MAD. Sample means and standard errors in parenthesis are shown in columns 2 and 4. Monte Carlo simulations are implemented under the sample size $N = 5000$ and the discretized time step 500.

K	$\mathcal{M}(\frac{\partial P_G}{\partial x}; t)$	MAD	$\mathcal{M}(\frac{\partial P_Z}{\partial x}; t)$	MAD
45	8.3598 (0.0113)	0.2451	8.3671 (0.0051)	0.1455
50	5.5622 (0.0111)	0.2536	5.5675 (0.0063)	0.1880
55	3.5361 (0.0113)	0.7600	3.5330 (0.0065)	0.7355

the mean) are important factors to indicate how much the high-biased estimate deviated from the true price. See [4] for the relationship between price bound and variance error, and [23] the definition of MAD. We observe that all standard errors and MADs from Zhang's approximation are smaller than the typical approximation by the geometric-average Asian option. These numerical results also support that $\mathcal{M}(\frac{\partial P_Z}{\partial x}; t)$ is a better control than $\mathcal{M}(\frac{\partial P_G}{\partial x}; t)$.

Note that $\mathcal{M}(\frac{\partial P_G}{\partial x}; t) = e^{-rt}H(G_t) - P_G(0, S_0, G_0; t)$, where $P_G(0, S_0, G_0; t)$ denotes the price of a geometric-average Asian option price with the maturity t rather than T. One can in principle replace this martingale control by its process representation, but this does not reduce any computational burden. In pricing European Asian options, the martingale at maturity T, $\mathcal{M}(\frac{\partial P_G}{\partial x}; T)$, is only needed for the linear control. The computational complexity for $e^{-rT}H(G_T) - P_G(0, S_0, G_0; T)$ is much less than $\mathcal{M}(\frac{\partial P_G}{\partial x}; T)$, where the delta must be computed along each simulated path. In pricing American Asian options, the martingale control process $\left\{\mathcal{M}(\frac{\partial P_G}{\partial x}; t)\right\}_{0 \leq t \leq T}$ is required to estimate the high-biased solution. Therefore, the computational complexity for $\mathcal{M}(\frac{\partial P_G}{\partial x}; t)$ and $e^{-rt}H(G_t) - P_G(0, S_0, G_0; t)$ are roughly the same.

Currently only one random source is encountered, namely the Brownian motion $W_t^\star$. General martingale controls are in the form of stochastic integrals with respect to that Brownian motion. Next we introduce multi-factor stochastic volatility models, where multiple Brownian motions are involved. It then becomes hard to choose from many controls. We shall introduce multi-factor stochastic volatility model and the perturbation technique in Section 3. In Section 4, the singular and regular perturbation technique is applied to simplify the choice of martingales for the geometric-average Asian options. In Section 5, we consider using generalized control variates in estimating arithmetic-average Asian options. A combination of the conventional control variate method and the martingale control turns out to be a much better choice than applying martingale controls alone.

3 Monte Carlo Pricing under Multi-factor Stochastic Volatility Models

Under a risk-neutral probability measure $I\!\!P^\star$ parametrized by the combined market price volatility premium (Λ_1, Λ_2), we consider the following multi-factor stochastic volatility models defined by

$$\begin{aligned}
dS_t &= rS_t dt + \sigma_t S_t dW_t^{(0)*}, \\
\sigma_t &= f(Y_t, Z_t), \\
dY_t &= \left[\frac{1}{\varepsilon} c_1(Y_t) + \frac{g_1(Y_t)}{\sqrt{\varepsilon}} \Lambda_1(Y_t, Z_t)\right] dt + \frac{g_1(Y_t)}{\sqrt{\varepsilon}} \left(\rho_1 dW_t^{(0)*} + \sqrt{1-\rho_1^2} dW_t^{(1)*}\right), \\
dZ_t &= \left[\delta c_2(Z_t) + \sqrt{\delta} g_2(Z_t) \Lambda_2(Y_t, Z_t)\right] dt \\
&\quad + \sqrt{\delta} g_2(Z_t) \left(\rho_2 dW_t^{(0)*} + \rho_{12} dW_t^{(1)*} + \sqrt{1-\rho_2^2-\rho_{12}^2} dW_t^{(2)*}\right),
\end{aligned} \tag{20}$$

where S_t is the underlying asset price process with a constant risk-free interest rate r. Its random volatility σ_t is driven by two stochastic processes Y_t and Z_t varying on the *time scales* ε and $1/\delta$, respectively. The vector $\left(W_t^{(0)*}, W_t^{(1)*}, W_t^{(2)*}\right)$ consists of three independent standard Brownian motions. Instant correlation coefficients ρ_1, ρ_2, and ρ_{12} satisfy $|\rho_1| < 1$ and $|\rho_2^2 + \rho_{12}^2| < 1$. The volatility function f is assumed to be smooth and bounded. Coefficient functions of processes Y_t and Z_t, namely (c_1, g_1, Λ_1) and (c_2, g_2, Λ_2) are assumed to be smooth such that they satisfy the existence and uniqueness conditions for the strong solutions of stochastic differential equations. Mean-reverting processes such as Ornstein-Uhlenbeck (OU) processes or square-root processes are typical examples of modeling driving volatility processes [6, 9]. Under this setup, the joint process (S_t, Y_t, Z_t) is Markovian.
Given the multi-factor stochastic volatility model (20) under $I\!\!P^\star$, the price of a plain European option with the integrable payoff function H and expiry T is defined by

$$P^{\varepsilon,\delta}(t, x, y, z) = I\!\!E^\star{}_{t,x,y,z}\left\{e^{-r(T-t)} H(S_T)\right\},$$

where $I\!\!E^\star{}_{t,x,y,z}$ is a short notation for the expectation with respect to $I\!\!P^\star$ conditioning on the current states $S_t = x, Y_t = y, Z_t = z$. A basic Monte Carlo simulation approximates the option price $P^{\varepsilon,\delta}(0, S_0, Y_0, Z_0)$ at time 0 by the sample mean

$$\frac{1}{N}\sum_{i=1}^{N} e^{-rT} H(S_T^{(i)}) \tag{21}$$

where N is the total number of sample paths and $S_T^{(i)}$ denotes the i-th simulated stock price at time T. Variance reduction techniques are particularly important to accelerate the computing efficiency of the basic Monte Carlo pricing estimator (21). Next, we briefly review the construction of a generic algorithm, i.e. martingale control variate method proposed and analyzed by Fouque and Han [3, 5].

3.1 Construction of Martingale Control Variates

Assuming that the European option price $P^{\varepsilon,\delta}(t, S_t, Y_t, Z_t)$ is twice differentiable in state space and once differentiable in time, we apply Ito's lemma to its discounted price $e^{-rt}P^{\varepsilon,\delta}$, then integrate from time 0 to the maturity T. One can use the pricing partial differential equation to cancel out non-martingale terms, and use the fact that $P^{\varepsilon,\delta}(T,S_T,Y_T,Z_T)=H(S_T)$ to obtain the following martingale representation

$$P^{\varepsilon,\delta}(0,S_0,Y_0,Z_0)=e^{-rT}H(S_T)-\mathcal{M}_0(P_x^{\varepsilon,\delta})-\frac{1}{\sqrt{\varepsilon}}\mathcal{M}_1(P_y^{\varepsilon,\delta})-\sqrt{\delta}\mathcal{M}_2(P_z^{\varepsilon,\delta}), \tag{22}$$

where subscripts denote partial derivatives and centered martingales are given by

$$\mathcal{M}_0(P_x^{\varepsilon,\delta})=\int_0^T e^{-rs}\frac{\partial P^{\varepsilon,\delta}}{\partial x}(s,S_s,Y_s,Z_s)f(Y_s,Z_s)S_s dW_s^{(0)*}, \tag{23}$$

$$\mathcal{M}_1(P_y^{\varepsilon,\delta})=\int_0^T e^{-rs}\frac{\partial P^{\varepsilon,\delta}}{\partial y}(s,S_s,Y_s,Z_s)g_1(Y_s)d\tilde{W}_s^{(1)*}, \tag{24}$$

$$\mathcal{M}_2(P_z^{\varepsilon,\delta})=\int_0^T e^{-rs}\frac{\partial P^{\varepsilon,\delta}}{\partial z}(s,S_s,Y_s,Z_s)g_2(Z_s)d\tilde{W}_s^{(2)*}, \tag{25}$$

where the Brownian motions are

$$\tilde{W}_s^{(1)*}=\rho_1 W_s^{(0)*}+\sqrt{1-\rho_1^2}W_s^{(1)*},$$
$$\tilde{W}_s^{(2)*}=\rho_2 W_s^{(0)*}+\rho_{12}W_s^{(1)*}+\sqrt{1-\rho_1^2-\rho_{12}^2}W_s^{(2)*}.$$

These martingales $\mathcal{M}_0,\mathcal{M}_1$, and $\mathcal{M}_3$ play the role of "perfect" controls of Monte Carlo simulations. Namely, if the martingales (23), (24), and (25) can be exactly computed, then one can generate one sample path to evaluate the option price through (22). Unfortunately the gradient $\left(P_x^{\varepsilon,\delta},P_y^{\varepsilon,\delta},P_z^{\varepsilon,\delta}\right)$ of the option price appearing in the martingales is not known because the option price $P^{\varepsilon,\delta}$ itself is exactly what we want to estimate. However, one can choose an approximate option price to substitute $P^{\varepsilon,\delta}$ used in the martingales (23), (24) and (25), and still retain their martingale properties.
When time scales ε and $1/\delta$ are well separated; namely, $0<\varepsilon,\delta\ll 1$, the *zeroth* order approximation of the Black-Scholes type can be found in [6]

$$P^{\varepsilon,\delta}(t,x,y,z)\approx P_{BS}(t,x;\bar{\sigma}(z)) \tag{26}$$

with its point-wise accuracy of order $\mathcal{O}\left(\sqrt{\varepsilon},\sqrt{\delta}\right)$ for continuous piecewise payoffs. The homogenized price $P_{BS}(t,x;\bar{\sigma}(z))$ solves the Black-Scholes partial differential equation with the constant volatility $\bar{\sigma}(z)$ and the terminal condition $P_{BS}(T,x)=H(x)$. Note that this approximation $P_{BS}(t,x;\bar{\sigma}(z))$ is y-variable independent, where y is the initial value of the fast varying process Y_t. The z-dependent *effective volatility* $\bar{\sigma}(z)$ is defined as the square root of an average of the variance function f^2 with respect to a limiting distribution of Y_t:

$$\bar{\sigma}^2(z)=\int f^2(y,z)d\Phi(y)=\left\langle f^2(y,z)\right\rangle. \tag{27}$$

Here $\Phi(y)$ denotes the invariant distribution of the fast varying process Y_t where volatility premium Λ_1 is zero, because in the drift term of the dY_t equation in (20), $\frac{1}{\varepsilon}$ term dominates $\frac{1}{\sqrt{\varepsilon}}$ term when $\varepsilon\ll 1$. We use the bracket $<\cdot>$ to represent such average. In the OU case, we let $c_1(y)=m_1-y$ and $g_1(y)=\nu_1\sqrt{2}$ with $\Lambda_1=0$ such that $1/\varepsilon$ is the rate of mean reversion, m_1 is the long-run mean, and ν_1 is the long-run standard deviation. Its invariant distribution Φ is normal with mean m_1 and variance ν_1^2. Please refer to [6] for detailed discussions.
Because the approximate option price $P_{BS}(t,x;\bar{\sigma}(z))$ is independent of y, the term $\mathcal{M}_1(P_{BS_y})$ becomes

zero. In (22), the last term associated with $\mathcal{M}_2(P_{BS_z})$ is small of order $\sqrt{\delta}$. Therefore, we can neglect this term as well. We then select the stochastic integral $\mathcal{M}_0(P_{BS_x})$ as the major control for the estimator (21) and formulate the following martingale control variate estimator:

$$\frac{1}{N}\sum_{i=1}^{N}\left[e^{-rT}H(S_T^{(i)}) - \mathcal{M}_0^{(i)}(P_{BS_x})\right]. \tag{28}$$

This is the approach taken in [5] where the proposed martingale control variate method is numerically superior to an importance sampling method in [2] for pricing European options. An asymptotic analysis of the martingale control variate, shown in Theorem 1 [3], guarantees that:
under OU-type processes for modeling (Y_t, Z_t) in (20) with $0 < \varepsilon, \delta \ll 1$, the variance of the martingale control variate for European options is small of order ε and δ; namely

$$\text{Var}\ \left(e^{-rT}H(S_T) - \mathcal{M}_0(P_{BS_x})\right) = \mathcal{O}(\max\{\varepsilon, \delta\}). \tag{29}$$

Moreover the financial interpretation of the martingale control term

$$\mathcal{M}_0(P_{BS_x}) = \int_0^T e^{-rs}\frac{\partial P_{BS}}{\partial x}(s, S_s; \bar{\sigma}(Z_s))f(Y_s, Z_s)S_s dW_s^{(0)*}$$

corresponds to the cumulative cost of a *delta* hedging strategy. This martingale control variate method can be easily extended to hitting time problems such as barrier options and optimal stopping time problems such as American options. Numerical results and variance analysis are discussed in [3].

4 Variance Reduction for Asian Options: Geometric-Average Case

A general form of payoffs for geometric-average Asian options (GAO) consists of a fixed strike K_2, a floating strike S_T, a coefficient K_1, and a geometric-average of stock prices. For example, the price at time t of a GAO call option is defined by

$$I\!E^\star\left\{e^{-r(T-t)}(G_T - K_1\, S_T - K_2)^+ \mid \mathcal{F}_t\right\},$$

where $\mathcal{F}_t$ denotes the filtration generated by the process $(S_s, Y_s, Z_s)_{0\le s\le t}$. The random variable G_T denotes the geometric average of stock prices up to time T

$$G_T = \exp\left(\frac{1}{T}\int_0^T \ln S_t dt\right).$$

We introduce the running sum process $L_t = \int_0^t \ln S_u du$ whose differential form is

$$dL_t = \ln S_t dt, \tag{30}$$

such that the joint dynamics (S_t, Y_t, Z_t, L_t) is Markovian. Hence we denote the call price of GAO by

$$P_G^{\varepsilon,\delta}(t, x, y, z, L) = I\!E^\star{}_{t,x,y,z,L}\left\{e^{-r(T-t)}\left(\exp\left(\frac{L_T}{T}\right) - K_1\, S_T - K_2\right)^+\right\}. \tag{31}$$

We assume $(S_t = x, Y_t = y, Z_t = z, L_t = L)$. A basic Monte Carlo simulation consists in generating N independent trajectories governed by equations (20) and (30), and averaging the discounted sample payoffs in order to obtain an unbiased GAO price estimator:

$$P_G^{\varepsilon,\delta} \approx P_G^{MC} = \frac{e^{-r(T-t)}}{N}\sum_{k=1}^{N}\left(\exp\left(\frac{L_T^{(k)}}{T}\right) - K_1\, S_T^{(k)} - K_2\right)^+. \tag{32}$$

4.1 Martingale Control Variate for Geometric-Average Asian Options

The construction of martingale control variates for GAO price is similar to the procedure presented in Section 3. We first apply Ito's lemma to the discounted GAO price and then integrate it with respect to the time variable. Therefore, the following martingale representation is obtained

$$
\begin{aligned}
P_G^{\varepsilon,\delta}(0, S_0, Y_0, Z_0, L_0) = e^{-rT}\left(\exp\left(\frac{L_T}{T}\right) - K_1\, S_T - K_2\right)^+ - \mathcal{M}_0(P_{G_x}^{\varepsilon,\delta}) - \frac{1}{\sqrt{\varepsilon}}\mathcal{M}_1(P_{G_y}^{\varepsilon,\delta}) \\
- \sqrt{\delta}\mathcal{M}_2(P_{G_z}^{\varepsilon,\delta}),
\end{aligned}
\tag{33}
$$

where stochastic integrals $\mathcal{M}_0, \mathcal{M}_1$ and $\mathcal{M}_2$ are defined similarly as in (23) - (25) but with $P_G^{\varepsilon,\delta}$ in these martingales instead. The martingale control variate estimator for the GAO price is then formulated by

$$
P_G^{\varepsilon,\delta}(0, S_0, Y_0, Z_0, L_0) \approx \frac{1}{N}\sum_{k=1}^{N}\left[e^{-rT}\left(\exp\left(\frac{L_T^{(k)}}{T}\right) - K_1\, S_T^{(k)} - K_2\right)^+ - \mathcal{M}_0^{(k)}(P_{BS_x}^{G})\right], \tag{34}
$$

provided that the "homogenized" GAO price P_{BS}^G, as an approximation to the true GAO price, can be easily calculated. For the case of fixed-strike GAOs. i.e. $K_1 = 0$, it is shown in [2] that $P_{BS}^G(t, x, L; \bar{\sigma}(z))$ has a closed-form solution

$$
\begin{aligned}
&P_{BS}^G(t, x, L; \bar{\sigma}(z)) \\
&= \exp\left(\frac{L - t\ln x}{T} + \ln x + R(t, T, z)\right)\mathcal{N}(d_1(x, z, L)) - Ke^{-r(T-t)}\mathcal{N}(d_2(x, z, L)),
\end{aligned}
\tag{35}
$$

where

$$
\begin{aligned}
R(t, T, z) &= \left(r - \frac{\bar{\sigma}^2(z)}{2}\right)\frac{(T-t)^2}{2T} + \bar{\sigma}^2(z)\,\frac{(T-t)^3}{6T^2} - r(T-t), \\
d_1(x, z, L) &= \frac{T\ln(x/K) + L - t\ln x + (r - \bar{\sigma}^2(z)/2)(T-t)^2/2 + \bar{\sigma}^2(z)\frac{(T-t)^3}{3T}}{\bar{\sigma}(z)\sqrt{\frac{(T-t)^3}{3}}} \\
d_2(x, z, L) &= d_1(x, z, L) - \bar{\sigma}(z)\sqrt{\frac{(T-t)^3}{3T^2}}.
\end{aligned}
\tag{36}
$$

The probabilistic representation of the homogenized GAO price $P_{BS}^G(t, x, L; \bar{\sigma}(z))$ is

$$
P_{BS}^G(t, \bar{S}_t = x, \bar{L}_t = L; \bar{\sigma}(z)) = \tilde{I\!E}\left\{e^{-r(T-t)}\left(e^{\bar{L}_T/T} - K\right)^+ \mid \bar{S}_t = x, \bar{L}_t = L\right\}, \tag{37}
$$

where $\bar{S}_t$ and $\bar{L}_t$ are governed by

$$
\begin{aligned}
d\bar{S}_t &= r\bar{S}_t dt + \bar{\sigma}(z)\bar{S}_t d\tilde{W}_t, \\
d\bar{L}_t &= \ln \bar{S}_t dt,
\end{aligned}
\tag{38}
$$

respectively and $Z_t = z$. Let $\tilde{W}_t$ denote a Brownian motion under a probability measure $\tilde{P}$, under which the conditional expectation is defined. These derivation can also be found in [2].

For other GAO payoffs such as call or put of the floating strike, i.e. $K_2 = 0$, one can derive similar results. We omit these cases to limit the length of this paper.

Table 3: Parameters used in the two-factor stochastic volatility model (20).

r	m_f	m_s	ν_f	ν_s	ρ_1	ρ_2	ρ_{12}	Λ_f	Λ_s	$f(y,z)$
10%	-0.8	-0.6	0.7	1	-0.2	-0.2	0	0	0	$\exp(y+z)$

Table 4: Initial conditions and Asian call option parameters.

$\$S_0$	Y_0	Z_0	L_0	$\$K_2 = K$	T years
100	-1	-0.5	0	110	1

4.2 Numerical Results for Pricing GAO

We present numerical results from Monte Carlo simulations to evaluate fixed-strike GAO prices in this section. Parameters in our model are shown in Table 3. Other values (initials conditions and option parameters) are given in Table 4. Parameters chosen in these tables are exactly the same as ones used in [2] for the purpose of comparing efficiency. Sample paths in (34) are simulated based on the Euler scheme to discretize equations (20) and (30) with time step $\Delta t = 0.005$. The stochastic integral $\mathcal{M}_0$ is approximated by a Riemann sum, and the number of total paths are 5000. As demonstrated in [2], results of variance reduction ratios obtained from an importance sampling technique versus the basic Monte Carlo method are now listed in the third column of Table 5. The variance reduction ratios obtained from the martingale control variate method are listed in the last column of Table 5. We find that the martingale control variate method outperforms the importance sampling method in all cases.

4.3 Variance Analysis of Perturbed Volatility

Based on the fact that random volatility is fluctuating around its long-run mean, we analyze the variance of a simplified model which is helpful to explain the effect of martingale control. Let's assume that under the risk-neutral probability measure, $S_t^{\varepsilon,\delta}$ is a risky asset defined by

$$dS_t^{\varepsilon,\delta} = rS_t^{\varepsilon,\delta}dt + \sigma_t^{\varepsilon,\delta}S_t^{\varepsilon,\delta}dW_t^*, \tag{39}$$

where the perturbed volatility is $\sigma_t^{\varepsilon,\delta} = \bar{\sigma} + \sqrt{\varepsilon}g_t + \sqrt{\delta}h_t$, $\bar{\sigma} > 0$ denotes the effective volatility, ε and δ are small parameters, and perturbed functions $\{g_t, h_t\}_{0\leq t\leq T}$ are assumed to be deterministic and

Table 5: Comparison of variance reduction ratios for various time scales ε and δ. V^{MC} denotes the sample variance obtained from the basic Monte Carlo method. $V^{IS}(\tilde{P}_G)$ denotes the sample variance computed by an importance sampling with the first-order price approximation $\tilde{P}_G$. This technique and its several numerical variance reduction ratios $V^{MC}/V^{IS}(\tilde{P}_G)$ can be found in [2]. $V^{MC+CV}(P_{BS}^G)$ denotes the sample variance computed by the martingale control variate method with the zeroth-order GAO price approximation P_{BS}^G in (35).

ε	δ	$V^{MC}/V^{IS}(\tilde{P}_G)$	$V^{MC}/V^{MC+CV}(P_{BS}^G)$
1/100	0.05	7.6320	26.0610
1/75	0.1	5.6264	24.2428
1/50	0.5	5.3007	12.2437
1/25	1	3.9444	10.4226

bounded such that $\sigma_t^{\varepsilon,\delta} > 0, 0 \leq t \leq T$. A geometric-average Asian option is defined by

$$P_G^{\varepsilon,\delta}\left(t, S_t^{\varepsilon,\delta}\right) = I\!E^{\star}{}_{t,S_t^{\varepsilon,\delta}}\left\{e^{-r(T-t)}H(L_T^{\varepsilon,\delta})\right\}, \tag{40}$$

where we denote the running sum process $L_T^{\varepsilon,\delta} = \frac{1}{T}\int_0^T \ln S_t^{\varepsilon,\delta} dt$ and assume that the function H is smooth and bounded.

Theorem 3 *(**Variance Analysis**) Given conditions described above, for any fixed initial state $(0, S_0^{\varepsilon,\delta})$, there exist $\varepsilon > 0$ and $\delta > 0$ small enough and a positive constant C such that*

$$Var\left(e^{-rT}H\left(L_T^{\varepsilon,\delta}\right) - \mathcal{M}_0(P_{BS_x}^G)\right) \leq C\max\{\varepsilon, \delta\},$$

where P_{BS}^G is defined as in (37) and (38) except that the homogenized volatility $\bar{\sigma}$ is chosen as constant.
Proof: (For simplicity, we remove subscripts under the expectation and use $I\!E^{\star}$ hereafter in this theorem.) Taking the pathwise derivative [7] for option price $P_G^{\varepsilon,\delta}$ with respect to the stock price, we can apply the chain rule to obtain

$$\begin{aligned}\frac{\partial P_G^{\varepsilon,\delta}}{\partial S_t^{\varepsilon,\delta}}(t, S_t^{\varepsilon,\delta}, L_t^{\varepsilon,\delta}) &= I\!E^{\star}\left\{e^{-r(T-t)}H'\left(L_T^{\varepsilon,\delta}\right)\int_t^T \frac{\partial \ln S_s^{\varepsilon,\delta}}{\partial S_t^{\varepsilon,\delta}} ds \mid \mathcal{F}_t\right\} \\ &= \frac{T-t}{S_t^{\varepsilon,\delta}T} I\!E^{\star}\left\{e^{-r(T-t)}H'\left(L_T^{\varepsilon,\delta}\right) \mid \mathcal{F}_t\right\}. \end{aligned} \tag{41}$$

Similarly, given the state variable $S_t^{\varepsilon,\delta}$ and $L_t^{\varepsilon,\delta}$, the *delta* of the geometric-average Asian option with the constant volatility $\bar{\sigma}$ has the following decomposition

$$\frac{\partial P_{BS}^G}{\partial \bar{S}_t}(t, \bar{S}_t = S_t^{\varepsilon,\delta}, \bar{L}_t = L_t^{\varepsilon,\delta}) = \frac{T-t}{\bar{S}_t T} I\!E^{\star}\left\{e^{-r(T-t)}H'\left(\bar{L}_T\right) \mid \mathcal{F}_t\right\}, \tag{42}$$

where the constant-volatility stock price $\bar{S}_t$ satisfies

$$d\bar{S}_t = r\bar{S}_t dt + \bar{\sigma}\bar{S}_t dW_t^*,$$

and we denote running sum as $\bar{L}_t = \int_0^t \ln \bar{S}_s ds$. Conditional on the driving volatility processes, the absolute difference between $\frac{\partial P_G^{\varepsilon,\delta}}{\partial S_t^{\varepsilon,\delta}}$ and $\frac{\partial P_{BS}^G}{\partial \bar{S}_t}$ is equal to

$$\begin{aligned}&\frac{\partial P_G^{\varepsilon,\delta}}{\partial S_t^{\varepsilon,\delta}}(t, S_t^{\varepsilon,\delta}, L_t^{\varepsilon,\delta}) - \frac{\partial P_{BS}^G}{\partial \bar{S}_t}(t, \bar{S}_t = S_t^{\varepsilon,\delta}, \bar{L}_t = L_t^{\varepsilon,\delta}) \\ &= \frac{T-t}{S_t^{\varepsilon,\delta}T} I\!E^{\star}\left\{e^{-r(T-t)}H''\left(\hat{L}_T^{\varepsilon,\delta}\right)\left(L_T^{\varepsilon,\delta} - \bar{L}_T\right) \mid \mathcal{F}_t\right\}, \end{aligned} \tag{43}$$

which is obtained by applying the Mean-Value Theorem. The inner difference conditional on $\mathcal{F}_t$ is

$$\begin{aligned} L_T^{\varepsilon,\delta} - \hat{L}_T &= \frac{1}{T}\int_t^T \left(\ln S_t^{\varepsilon,\delta} - \ln \bar{S}_t\right) dt \\ &= \frac{1}{T}\left[\int_t^T \int_t^u \frac{\bar{\sigma}^2 - \sigma_s^{\varepsilon,\delta 2}}{2} ds\, du - \int_t^T \int_t^u \varepsilon\, g_s + \delta\, h_s\, dW_s^*\, du\right]. \end{aligned} \tag{44}$$

The variance of the control variate is

$$\begin{aligned}
& Var\left(e^{-rT} H\left(L_T^{\varepsilon,\delta}\right) - \mathcal{M}_0(P^G_{BS_x})\right) \qquad (45)\\
&= Var\left(\int_0^T e^{-rs}\left(\frac{\partial P_G^{\varepsilon,\delta}}{\partial x} - \frac{\partial P_{BS}^G}{\partial x}\right)(s, S_s^{\varepsilon,\delta}, L_s^{\varepsilon,\delta})\sigma^{\varepsilon,\delta} S_s^{\varepsilon,\delta} dW_s^*\right)\\
&= I\!\!E^\star\left\{\int_0^T e^{-2rs}\left(\frac{\partial P_G^{\varepsilon,\delta}}{\partial x} - \frac{\partial P_{BS}^G}{\partial x}\right)^2 (s, S_s^{\varepsilon,\delta}, L_s^{\varepsilon,\delta})\sigma^{\varepsilon,\delta^2} S_s^{\varepsilon,\delta^2} ds\right\}\\
&\leq C\int_0^T I\!\!E^\star\left\{\left(I\!\!E^\star\left\{e^{-r(T-t)} H''\left(\hat{L}_T^{\varepsilon,\delta}\right)\left(L_T^{\varepsilon,\delta} - \bar{L}_T\right) \mid \mathcal{F}_s\right\}\right)^2 \mid \mathcal{F}_0\right\} ds. \qquad (46)
\end{aligned}$$

Here we substitute (43) and (44) into (45) and C is some constant because $\sigma^{\varepsilon,\delta}$ is assumed to be bounded. The nested expectation defined in (46) can be bounded above by

$$\begin{aligned}
& C I\!\!E^\star\left\{\left(L_T^{\varepsilon,\delta} - \bar{L}_T\right)^2 \mid \mathcal{F}_0\right\}\\
&\leq C\left(\int_0^T\int_0^t I\!\!E^\star\left\{\left(\frac{\bar{\sigma}^2 - \sigma_s^{\varepsilon,\delta^2}}{2}\right)^2 \mid \mathcal{F}_0\right\} ds\, dt + \int_0^T I\!\!E^\star\left\{\left(\int_0^t \varepsilon\, g_s + \delta\, h_s\, dW_s^*\right)^2 \mid \mathcal{F}_0\right\} dt\right)\\
&\leq C\max\{\varepsilon, \sqrt{\varepsilon\,\delta}, \delta\}.
\end{aligned}$$

We use the integrability of g and h and Ito isometry property to obtain the estimate. The notation C denotes some constant independent of parameters ε and δ. We therefore conclude that the variance of martingale control variate is of $\mathcal{O}(\varepsilon, \delta)$. An argument to treat the general multi-factor model (20) will be studied in a separate paper.

5 Variance Reduction for Asian Options: Arithmetic-Average Case

Similarly to the GAO case, it is convenient to introduce a running sum process $I_t = \int_0^t S_u du$, or its differential form, $dI_t = S_t dt$, such that the joint dynamics (S_t, Y_t, Z_t, I_t) defined in (20) is Markovian. Under the risk-neutral probability measure $I\!\!P^\star$ the price of an arithmetic-average Asian call option is given by

$$P_A^{\varepsilon,\delta}(t,x,y,z,I) = I\!\!E^*_{t,x,y,z,I}\left\{e^{-r(T-t)}\left(\frac{I_T}{T} - K_1\, S_T - K_2\right)^+\right\},$$

conditioning on $S_t = x, Y_t = y, Z_t = z, I_t = I$. We use this type of options as typical examples when we discuss variance reduction of Monte Carlo simulations. A basic Monte Carlo simulation for pricing arithmetic-average Asian options (AAO in short) with N replications is given by

$$P_A^{\varepsilon,\delta} \approx P_A^{MC} = \frac{e^{-r(T-t)}}{N}\sum_{k=1}^N\left(\frac{I_T^{(k)}}{T} - K_1\, S_T^{(k)} - K_2\right)^+. \qquad (47)$$

5.1 One-Step versus Two-Step Control Variate Methods

It becomes straightforward to derive the martingale control variate method by directly applying the martingale representation theorem to AAOs such that

$$P_A^{\varepsilon,\delta}(0,S_0,Y_0,Z_0,I_0) = e^{-rT}\left(\exp\left(\frac{I_T}{T}\right) - K_1\,S_T - K_2\right)^+ - \mathcal{M}_0(P_{A_x}^{\varepsilon,\delta};T) - \frac{1}{\sqrt{\varepsilon}}\mathcal{M}_1(P_{A_y}^{\varepsilon,\delta};T) - \sqrt{\delta}\mathcal{M}_2(P_{A_z}^{\varepsilon,\delta};T), \tag{48}$$

where stochastic integrals $\mathcal{M}_0, \mathcal{M}_1$ and $\mathcal{M}_2$ are defined the same as before. Similar to GAO cases, one can formulate an one-step martingale control variate estimator

$$P_A^{\varepsilon,\delta}(0,S_0,Y_0,Z_0,I_0) \approx \frac{1}{N}\sum_{k=1}^{N}\left[e^{-rT}\left(\frac{I_T^{(k)}}{T} - K_1\,S_T^{(k)} - K_2\right)^+ - \mathcal{M}_0^{(k)}(P_{BS_x}^A)\right], \tag{49}$$

with P_{BS}^A as the homogenized AAO price under some constant volatility, which does not have a closed-form solution. Intuitively, one may choose a price approximation such as P_G in (5) or P_Z in (14) to substitute for the homogenized AAO price P_{BS}^A. Notice that these homogenized price approximations are independent of the y-variable. We assume there is no correlation between these Brownian motions, namely all ρ's in (20) are zero. Let $\tilde{P}$ be a well-chosen homogenized price approximation such as P_G or P_Z. Then we can use the martingale control $\mathcal{M}_0(\tilde{P}_x;T)$ instead. The variance of such one-step control variate is

$$\begin{aligned} &Var\left(e^{-rT}\left(\frac{I_T}{T} - K_1\,S_T - K_2\right)^+ - \mathcal{M}_0(\tilde{P}_x;T)\right) \\ &= \left\langle \mathcal{M}_0(P_{A_x}^{\varepsilon,\delta} - \tilde{P}_x;T)\right\rangle_Q + \frac{1}{\varepsilon}\left\langle \mathcal{M}_1(P_{A_y}^{\varepsilon,\delta};T)\right\rangle_Q + \delta\left\langle \mathcal{M}_2(P_{A_z}^{\varepsilon,\delta};T)\right\rangle_Q, \end{aligned} \tag{50}$$

where $\langle\cdot\rangle_Q$ denotes the expectation of a quadratic variation. Notice that the second term above has a large coefficient $\frac{1}{\varepsilon}$ because ε is small. This term is completely not affected by the martingale control. The third term is negligible because δ is small.
To further reduce the one-step controlled variance (50), we would like to find a x, y, z-dependent price approximation so that each quadratic variation can be reduced. The conventional control

$$e^{-r(T-t)}\left(\frac{L_T}{T} - K_1\,S_T - K_2\right)^+ - P_G^{\varepsilon,\delta}(0,S_0,Y_0,Z_0) \tag{51}$$

has such property. This can be seen from the variance of the control variate

$$\begin{aligned} &Var\left(e^{-rT}\left(\frac{I_T}{T} - K_1\,S_T - K_2\right)^+ - \lambda\left(e^{-rT}\left(\frac{L_T}{T} - K_1\,S_T - K_2\right)^+ - P_G^{\varepsilon,\delta}(0,S_0,Y_0,Z_0)\right)\right) \\ &= \left\langle \mathcal{M}_0(P_A^{\varepsilon,\delta} - \lambda P_G^{\varepsilon,\delta};T)\right\rangle_Q + \frac{1}{\varepsilon}\left\langle \mathcal{M}_1(P_A^{\varepsilon,\delta} - \lambda P_G^{\varepsilon,\delta};T)\right\rangle_Q + \delta\left\langle \mathcal{M}_2(P_A^{\varepsilon,\delta} - \lambda P_G^{\varepsilon,\delta};T)\right\rangle_Q. \end{aligned} \tag{52}$$

Comparing with the previous variance (50) reduced by one martingale control, the conventional control variate apparently has the potential to reduce the expectation of each quadratic variation, in particular the large order term $\frac{1}{\varepsilon}\left\langle \mathcal{M}_1(P_A^{\varepsilon,\delta};T)\right\rangle_Q$. The drawback of using the conventional control (51) is that the GAO price does not have a closed-form solution. Therefore, it is reasonable to use a two-step

algorithm for variance reduction:
Step 1: Estimate $P_G^{\varepsilon,\delta}(0,S_0,Y_0,Z_0)$ by a martingale control $\mathcal{M}_0(P_{BS}^G;T)$ such that

$$P_G^{\varepsilon,\delta}(0,S_0,Y_0,Z_0) = I\!E^\star\left\{e^{-rT}\left(\frac{L_T^{(k)}}{T} - K_1\,S_T^{(k)} - K_2\right)^+ - \mathcal{M}_0(P_{BS}^G;T)\right\}.$$

Step 2: Estimate $P_G^{\varepsilon,\delta}(0,S_0,Y_0,Z_0)$ by the conventional control (51) such that

$$\begin{aligned} P_A^{\varepsilon,\delta}(0,S_0,Y_0,Z_0) =& I\!E^\star\left\{e^{-rT}\left(\frac{I_T}{T} - K_1\,S_T - K_2\right)^+\right.\\ &\left. -\lambda\left(e^{-rT}\left(\frac{L_T}{T} - K_1\,S_T - K_2\right)^+ - P_G^{\varepsilon,\delta}(0,S_0,Y_0,Z_0)\right)\right\}. \end{aligned}$$

The variance analysis for this two-step control variate method is the following. In Step 1, it is implied from Theorem 3 that the variance is small of $\mathcal{O}(\varepsilon,\delta)$. In Step 2, it is expected from (50) and (52) that the variance of conventional control variate is smaller than a martingale control variate. Using results in Table 6-13 from the next section, we confirm that the variance induced by the two-step method is much smaller than the variance induced by the one-step method, and the two-step method only increases a small amount of computing time than the one-step method.
We should compare with the other two-step control variate method proposed in [2]. These two control variates only differ in methods to estimate the price $P_G^{\varepsilon,\delta}$, namely Step 1. In [2], an importance sampling is developed while in this paper, a martingale control variate is proposed and analyzed in Section 4. In addition to the accuracy gain by the martingale control variate as shown in Table 5, we would like to note that the martingale control variate method is implemented under the same probability measure as the estimate implemented in Step 2. However, the importance sampling in Step 1 must be implemented under some equivalent probability measure so that the pricing model are different from the original one as in (20). Therefore, there is an additional programing advantage of our newly proposed two-step method.

5.2 Numerical Results for Pricing AAO

In this section, we will compare efficiencies of pricing Asian call option using two different control variates developed above (namely, the one-step and two-step control variate method), combined with Monte Carlo and quasi-Monte Carlo (QMC) methods. The C++ on Unix is our programming language in the following examples. The pseudo-random number generator used is $ran2()$ in [28]. The approximation method given in [29] is used to generate the standard normal random variate. This method achieves 16 digits accuracy as claimed. In the QMC method, a quasi-random sequence or low discrepancy sequence (LDS) is used, instead of a pseudo-random sequence. An LDS is usually more uniformly distributed over $[0,\,1]^s$ than a pseudo-random sequence does. There are two classes of low-discrepancy sequences. One is called the digital net sequences, such as Halton's sequence, Sobol's sequence, Faure's sequence, and Niederreiter's $(t,s)-$sequence, etc. A QMC method using this kind of LDS has convergence rate $O(\frac{(\log N)^s}{N})$ when estimating the following integral

$$\mu = \int_{C^s} f(\mathbf{x})d\mathbf{x}, \tag{53}$$

where $C^s = [0,\,1]^s$ is the s-dimensional unit hypercube, and $f(\mathbf{x})$ is at least integrable. The direction numbers from [12] are used so that our implementation of Sobol sequence can generate points of

dimension as high as 1111.
The other class is the integration lattice rule points. This type of LDS is especially efficient for estimating multivariate integrals with periodic and smooth integrands, and it has convergence rate $O(\frac{(\log N)^{\alpha s}}{N^{\alpha}})$, where $\alpha > 1$ is a parameter related to the smoothness of the integrand. The monograph by Niederreiter [19] gives detailed information on digital net sequences and lattice rule points, while the monographs by Hua and Wang [10], and Sloan and Joe [25] describe lattice rules.
L'Ecuyer also made contributions to lattice rules based on linear congruential generator. One of the features of this type of lattice rule points (referred to L'Ecuyer's type lattice rule points, LTLRP, thereafter) is that it is easy to generate high dimensional LTLRP point sets with convergence rate comparable to digital net sequences. Details of LTLRP can be found in [15] and references therein. We will apply the LTLRP as well since our problem is high dimensional. Besides the above LDS, we also apply the Brownian bridge (BB) sampling technique to our test problems. Detailed information about Brownian bridge sampling can be found in [7].
To compare the efficiencies of different methods, we need a benchmark for fair comparison. If the exact value of the quantity to be estimated can be found, then we use the absolute error or relative error for comparison. Otherwise, we use the unbiased sample variance $\widehat{\sigma}_n^2$ for comparison. For LDS sequences, we define the unbiased sample variance $\widehat{\sigma}_n^2$ by introducing random shift as follows. Assume that we estimate

$$\mu = E[f(X)],$$

where X is an $s-$dimensional random vector. Let $\{\mathbf{x}_i\}_{i=1}^m \subset I^s = [0,1]^s$ be a finite LDS sequence and $\{\mathbf{r}_j\}_{j=1}^n \subset I^s$ be a finite sequence of random vectors. For each fixed j, we have a sequence $\{\mathbf{y}_i^{(j)}\}_{i=1}^m$, where $\mathbf{y}_i^{(j)} = \mathbf{x}_i + \mathbf{r}_j$. Such a sequence still has the same convergence rate as $\{\mathbf{x}_i\}_{i=1}^m$, which is proved in [26]. Denote

$$\mu_j = \frac{1}{m}\sum_{i=1}^{m} f(\mathbf{y}_i^{(j)}),$$

and

$$\widehat{\mu}_n = \frac{1}{n}\sum_{j=1}^{n} \mu_j.$$

The unbiased sample variance is (assuming $n > 1$)

$$\widehat{\sigma}_n^2 = \frac{\sum_{j=1}^n (\mu_j - \widehat{\mu}_n)^2}{n-1} = \frac{n\sum_{j=1}^n \mu_j^2 - \left(\sum_{j=1}^n \mu_j\right)^2}{(n-1)n}.$$

And the variance of the MC method is calculated as usual based on the pseudo- random point set $\{\mathbf{x}_i + \mathbf{r}_j\}$ with mn points. Thus, we can easily construct the confidence interval for a given confidence level. The variance reduction ratio of a QMC method over the MC method is defined by

$$\text{variance reduction ratio} = \frac{\text{The sample variance of the MC method}}{\text{The sample variance of the randomized QMC method}}. \tag{54}$$

In our comparisons, the sample sizes for MC method are 10240, 20480, 40960, 81920, 163840, and 327680, respectively; and those for Sobol' sequence related methods are 1024, 2048, 4096, 8192, 16384, and 32768, respectively, each with 10 random shifts; and the sample sizes for LTLRP related methods are 1021, 2039, 4093, 8191, 16381, and 32749, respectively, and again, each with 10 random shifts. We divide the time interval $[0, T]$ into $m = 128$ subintervals. In the following Tables, column 1 contains the numbers of points. Numbers without parentheses in the MC column are option values, and numbers within parentheses in the same column are the corresponding standard errors. Numbers in the QMC

Table 6: Comparison of simulated Asian option values and variance reduction ratios by one-step method for $1/\varepsilon = 75.0$, $\delta = 0.1$.

N	MC	MC+CV	Sobol'	Sobol+CV	Sobol+BB	Sobol+CV+BB
1024	7.770(0.1508)	24.2	2.1	28.2	1.9	6.5
2048	7.659(0.1063)	22.9	2.7	62.5	2.8	4.5
4096	7.784(0.0759)	22.9	4.7	76.5	1.2	6.3
8192	7.687(0.0528)	22.7	3.4	53.9	1.3	8.6
16384	7.702(0.0374)	22.9	1.3	55.2	0.7	5.9
32768	7.700(0.0265)	22.9	5.8	74.0	1.1	10.9

columns are variance reduction ratios.
An arithmetic-average Asian call option with a fixed strike is considered. The payoff variable is $\max(\frac{1}{T}\int_0^T S_t dt - K, 0) = \left(\frac{1}{T}\int_0^T S_t dt - K\right)^+$. We take input parameters defined in Table 3 and 4 as follows: $S_0 = \$100$, $K = \$110$, $r = 0.1 = 10\%$, $T = 1$ year, $m_1 = -0.8$, $m_2 = -0.6$, $\nu_1 = 0.7$, $\nu_2 = 1.0$, $\rho_1 = \rho_2 = -0.2$, $\rho_{12} = 0.0$, $y_0 = -1.0$, $z_0 = -0.5$, $1/\varepsilon = 75.0$, $\delta = 0.1$. In the following tables we demonstrate numerics of various variance reductions based on two different control variate methods. In Tables 6 and 7, the one-step martingale control $\mathcal{M}_0(P_G; T)$ is used to construct the control variate estimator (49), where we use P_G rather than P^A_{BS}. We observe that the variance reduction ratio is about 23 for pseudo-random sequences by using this control variate technique. Without a control, the variance reduction ratios for Sobol' sequence vary from 1.3 to 5.8 and those for LTLRP are from 1.2 to 5.5; the Brownian bridge (BB in short) sampling does not help much to improve the efficiency for the Sobol sequence, and it is a little bit better for the LTLRP case. In BB sampling, we try two different assigments of the coordinates of quasi-random points to the three state variables: in the first way we assign the first 128 coordinates to the first variable, the second 128 coordinates to the second variable and the last 128 coordinates to the last variable. In the second way, we assign the coordinates alternatively to the three variables: coordinates in $(3i+1)$th, $(3i+2)$th and $(3i+3)$th positions are assigned to the first, the second and the third variable, respectively. The results by these two different assigments show very little difference in magnitude for this SV model. We believe that is because the BB sampling method itself does not have much advantage for this complicated SV model. See [20] for similar observations. When combined with control variate, the variance reduction ratios for both QMC sequences are increased: for the Sobol' sequence they vary from 28.2 to 76.5, and those for L'Ecuyer type lattice rule points are from 55.6 to 137.2.
The CPU time used in the simulations are listed in Tables 8 and 9, from which we observe that simulation times used in the same group of methods (with or without control variates) are essentially the same magnitude, and a method with control variates normally takes twice amount of time of the same method without control variates. It should be pointed out that the CPU time is an approximate measure since there were also many other jobs running when our programs were running on the main frame machine. The situations are very similar for other cases. For example, Tables 12 and 13 list simulation times for two-step control variate methods. We find that the difference in computing times between one-step and two-step control variate methods is small in general.

In Tables 10 and 11, the two-step method is used to construct the control variate estimator. We observe that the variance reduction ratio is about 60 for pseudo-random sequences by using the control variate technique. Without a control, variance reduction ratios for Sobol' sequence vary from 1.3 to 5.8 and those for LTLRP are from 1.2 to 5.5; the BB sampling actually worsen the efficiency

Table 7: Comparison of simulated Asian option values and variance reduction ratios by one-step method for $1/\varepsilon = 75.0$, $\delta = 0.1$ (continued).

N	LTLRP	LTLRP+CV	LTLRP+BB	LTLRP+CV+BB
1021	2.8	98.7	3.9	80.3
2039	1.2	55.6	2.3	99.3
4093	1.9	42.6	2.0	37.6
8191	5.5	137.2	3.6	102.8
16381	3.1	44.7	6.9	60.2
32749	2.8	69.7	2.1	29.5

Table 8: Comparison of time (in seconds) used in simulating Asian option values by one-step method for $1/\varepsilon = 75.0$, $\delta = 0.1$.

N	MC	MC+CV	Sobol'	Sobol+CV	Sobol+BB	Sobol+CV+BB
1024	8	15	8	14	7	14
2048	14	30	15	28	13	29
4096	27	59	25	55	25	57
8192	56	119	50	115	51	114
16384	123	238	97	230	105	225
32768	241	482	195	449	210	455

Table 9: Comparison of time (in seconds) used in simulating Asian option values by one-step method for $1/\varepsilon = 75.0$, $\delta = 0.1$ (continued).

N	LTLRP	LTLRP+CV	LTLRP+BB	LTLRP+CV+BB
1021	6	14	6	15
2039	11	26	13	28
4093	23	54	27	58
8191	46	111	55	117
16381	93	242	109	238
32749	186	439	220	468

Table 10: Comparison of simulated Asian option values and variance reduction ratios by two-step method for $1/\varepsilon = 75.0$, $\delta = 0.1$.

N	MC	MC+CV	Sobol'	Sobol+CV	Sobol+BB	Sobol+CV+BB
1024	7.770(0.1508)	63.4	2.1	135.8	1.9	109.8
2048	7.659(0.1063)	59.5	2.7	86.5	2.8	98.6
4096	7.784(0.0759)	60.2	4.7	201.5	1.2	94.4
8192	7.687(0.0528)	61.8	3.4	83.6	1.3	89.1
16384	7.702(0.0374)	61.0	1.3	37.7	0.7	47.7
32768	7.700(0.0265)	60.8	5.8	100.7	1.1	37.7

Table 11: Comparison of simulated Asian option values and variance reduction ratios two-step method for $1/\varepsilon = 75.0$, $\delta = 0.1$ (continued).

N	LTLRP	LTLRP+CV	LTLRP+BB	LTLRP+CV+BB
1021	2.8	135.1	3.9	93.4
2039	1.2	78.9	2.3	149.7
4093	1.9	91.3	2.0	139.3
8191	5.5	265.2	3.6	209.6
16381	3.1	139.1	6.9	323.1
32749	2.8	181.7	2.1	115.6

for the Sobol sequence, and it is a little bit better for the LTLRP case. When combined with control variate, the variance reduction ratios for the Sobol' sequence vary from 37.7 to 201.5, and those for L'Ecuyer type lattice rule points are from 78.9 to 265.2.

From Table 6 to Table 11 we see that indeed the two-step method performs better than the martingale control variate method. This is because the two-step method is able to eliminate more quadratic variations as explained in the previous section.
Notice that this pricing problem under a stochastic volatility model is a high dimensional problem for QMC methods. In our test, it has dimension $3\times128=384$, the multiplication of 128 time discretization and 3 state variables (S_t, Y_t, Z_t). We see in these tables that after adding controls QMC sequences

Table 12: Comparison of time (in seconds) used in simulating Asian option values with two-step control variate method for $1/\varepsilon = 75.0$, $\delta = 0.1$.

N	MC	MC+CV	Sobol'	Sobol'+CV	Sobol'+BB	Sobol'+CV+BB
1024	10	17	6	15	7	16
2048	19	31	12	28	14	34
4096	40	63	25	62	29	65
8192	77	126	50	119	58	129
16384	127	254	97	254	115	248
32768	240	505	201	496	234	504

Table 13: Comparison of time (in seconds) used in simulating Asian option values with two-step control variate method for $1/\varepsilon = 75.0$, $\delta = 0.1$ (continued).

N	LTLRP	LTLRP+CV	LTLRP+BB	LTLRP+CV+BB
1021	6	14	7	15
2039	13	31	15	33
4093	25	63	29	65
8191	53	127	60	131
16381	103	248	121	259
32749	207	491	238	519

such as Sobol' and LTLPR achieve significant reduction on variance. This shows the superiority over the MC methods. We think that this is a strong evidence that the controls act as smoothers for the QMC methods. At least for GAO cases, Theorem 3 helps to explain this phenomenon because it states that on average a (continuous) control variate is small order of ε or δ. A detailed account for analyzing this smoothing effect is left as future research.
The study of computing Greeks such as delta is not presented here. Please refer [5] on computing delta by control variate methods and pathwise differentiation for European options.

6 Conclusion

In this paper, we study the option pricing problems for Asian options by simulation methods. We revisit the conventional control variate method and give a new interpretation of this method. A class of martingale controls is proposed and can be generalized to nonlinear situation such as calculating high-biased solutions of American Asian options. It is common to evaluate the effectiveness of controls by measuring expectations of their quadratic variations. This is helpful for designing control variates of general problems. We propose a two-step control variate method for pricing Asian options under a class of multi-factor stochastic volatility models. Numerically, we implement control variate methods with Monte Carlo and quasi-Monte Carlo methods. For stochastic volatility models, our tests show that quasi-Monte Carlo methods with control variates are much more efficient than Monte Carlo methods even in high dimensional regimes.

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