Joint Calibration of Market Risk, Credit Risk, and Interest Rate Risk

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Work supported by MOST 102-2918-I-007 -023 -

[2] Opinions expressed herein are entirely those of the authors, rather than those of DBS Bank.

Abstract

Recent literature has highlighted joint movements between the credit default swap (CDS for short) spread and its corresponding option price. Some dynamically consistent frameworks have been proposed for the joint evaluation and estimation of stock options and their CDS spreads in order to integrate both market information.

This paper extends previous studies and provides a new methodology for joint evaluation of stock option prices, CDS spreads and bond prices based on three separate calibration methods. They include (1) a two-step Monte Carlo procedure for calibration to the term structure of implied volatilities, (2) an approximated default intensity rate under the reduced form model for the credit risk calibration, and (3) a closed-form of zero coupon bond price for the interest rate risk calibration. Various innovative combinations of these three calibration methods are proposed to allow a genuine robust and efficient estimation for the joint dynamics of multiple risk factors. Our investigation discloses the importance of cross-market information to fit the implied volatility surfaces by means of a joint dynamic model which includes market risk, credit risk and the interest risk.

Keywords: instantaneous volatility, implied volatility surface, default intensity, interest rate, model calibration, Monte Carlo method, martingale control variate, vulnerable *option*

1 Introduction

With the process of economic globalization, financial derivatives markets such as the stock option market, credit derivative market, and the bond market have experienced an explosive growth in decades. Recent studies have shown that risks associated with these markets are intertwined. It raises a great attention to construct joint dynamics that contains equity risk, credit risk, interest rate risk, and so on in order to evaluate or hedge options under an incomplete market.

The relationship between CDS and equity markets have been well documented in the literature. Bystrom (2005) studied the relationship between the iTraxx CDS index market and the stock market. They found that the CDS spread increases (decreases) with increasing (decreasing) stock price volatilities. Berndt (2007) found significant information flow from the CDS and equity options markets to the equity market. Due to the relation between the credit market and equity market, Carr and Wu (2010) investigated a dynamically consistent framework that evaluates and estimates the stock options and CDS written on the same company. They also allowed both the common movements and independent variations between the equity market and credit market. For high dimensional credit derivatives, Collin-Dufresne et al. (2010) proposed a model fitting the time series of tranche spreads. Due to the information of the default time and the specification of idiosyncratic dynamics, they found that it is important to calibrate the model to match the entire term structure of CDX spread. They proposed a structural model which could jointly price long-dated S&P 500 options and tranche spreads on the five-year CDX index.

A number of studies investigate multiple effects of equity risk, credit risk, and interest rate risk. Norden (2004) analyzed the relationships between CDS, bond and equity markets. They found weekly and daily equity returns were negatively correlated with CDS and bond spread changes. Norden (2009) further discovered that CDS market was significantly more sensitive to the equity market than the bond market, and the sensitivity arose when credit quality drops.

Our goal is to develop a genuine approach to deal with multiple risks. We start from a joint dynamics to incorporate equity risk and credit risk based on the recent work of Carr and Wu (2010). Their work modeled the default intensity by the Cox process and assumed once the default occurs, the stock price went to zero. Before the default, the stock price followed the jump-diffusion process with stochastic volatility. The default intensity and the return variance rate were set to follow a bivariate diffusion with dynamic interactions that matched the stock option implied volatilities and CDS spreads. Note that the performance of the model calibration proposed by Carr and Wu (2010) was not stable[3].

A simplified yet robust calibration procedure is investigated in this paper. We estimate instantaneous volatilities of the equity price using the Fourier transform method (Malliavin and Mancino (2009), Han (2015)), and estimate the default intensity by an approximation. Efficient computation for option prices in European and American styles is crucial for solving the optimization problem of model calibration. A variance reduction method of Monte Carlo simulation, i.e., martingale control variate (Fouque and Han (2007, 2008)) is employed to enhance a fast convergence. Han and Kuo (2017) developed this approach for model calibration to implied volatility surfaces only for multifactor volatility models. This paper continues to derive a methodology that can jointly evaluate the credit spread, the stock option, and the bond price. Through our empirical studies, it confirms empirically that the proposed method is able to effectively reduce the complexity and error of model calibration.

The rest of the paper is organized as follows. We review calibration of three single risk factors in Section 2. Based on Carr and Wu (2010), the joint calibration of market risk and credit risk is considered and the framework of our approach is proposed in Section 3. In Section 4, we investigate the calibration of multiple risk factors. A brief summary and financial implication is provided in Section 5. We conclude in Section 6.

[3] That is, the estimated parameter β is close to zero and contradicts the result proposed by Zhang et al. (2009)

2 Calibration of Single Risk Dynamics

A calibration problem refers to solving for optimal model parameters given a set of traded market data. In this section, market information such as implied volatility surface, treasury and/or corporate bond yields, and CDS spread are utilized. They are primarily associated with equity risk, interest rate risk, and credit risk, respectively. Each financial model will be calibrated according to a single market information. The joint calibration according to multiple market risks will be studied in the next two sections.

For the calibration problem of an implied volatility surface, the two-stage Monte Carlo calibration method proposed by Han and Kuo (2017) is employed. This method consists of two technical parts: Fourier Transform method (Malliavin and Mancino (2009), Han (2015)) for estimating volatility to reduce parameter dimensions, and martingale control variate method (Fouque and Han (2007, 2008), Han and Lai (2010)) for variance reduction to evaluate option prices.

For the calibration of a corporate bond yield or the U.S. treasury yield, we employee the bootstrapping method to construct a zero-coupon bond yield given the Vasicek process as the underlying risk-free interest rate model. As for the calibration of a CDS spread, we first approximate spreads as default intensity processes, then the maximum likelihood method is applied to estimate an exponential OU (Ornstein–Uhlenbeck) model.

2.1 Calibration of Implied Volatility Surface

Han and Kuo (2017) proposed a genuine methodology, termed two-stage Monte Carlo calibration method, under a variety of volatility models. We will apply this method to retrieve the information content of market risk by using a class of one-factor stochastic volatility models in this paper. A brief introduction to this two-stage calibration method is below and more empirical results with comparison to other methods can be found in the next two sections when joint calibration problems are discussed.

Stage 1 (Fourier Transform Method (Malliavin & Mancino (2009), Han (2015)))- Estimate stochastic volatility model parameters under the historical probability measure. This step consists of

i. Estimate the spot volatility time series by the Fourier transform method.

ii. Estimate stochastic volatility model parameters using the maximum likelihood method.

Stage 2 (Martingale Control Variate Method (Fouque & Han (2007))) - Estimate stochastic volatility model parameters under a risk-neutral probability measure. This step aims to estimate some other separated model parameters from option prices.

Note that the problem of model calibration to an implied volatility surface can be very technical. Stage two may require a tremendous effort on high performance computing (Han and Lin (2014)). Complex volatility models such as two-factor Heston model (Christoffersen et al. (2009)), hybrid model and three-factor models (Han and Kuo (2017)), etc. are considered to obtain fast computation with minimum errors of model calibration. In this paper we employ only a basic one-factor stochastic volatility model but focus on incremental effects gained from multiple risks such as credit risk and interest rate risk.

2.2 Calibration of Bond Yield

A yield is a value to describe a zero coupon bond in the bond market. The definition of bond yield is

$$yield(T) = -\frac{\ln B(0,T)}{T}, \tag{2. 1}$$

and equation (2. 1) is the same as a continuously compounded interest rate of the zero coupon bond. Thus, the zero-coupon bond price $B(0,T)$ with maturity T can be rewritten as

$$B(0,T) = \exp(-yield(T)\times T). \tag{2. 2}$$

A term structure, which is formed by the yield rates and maturities, is called a yield curve. In order to capture the variation of yields, a "short rate" model is in use here to describe bond prices,

$$B(t,T) = E^{*}\left[exp\,(-\int_{t}^{T} r_u\, du)|\,\mathcal{F}_t\right]. \tag{2. 3}$$

Obviously, this is a stochastic model to simulate the behavior of the observed market data, i.e., bond $yield(T)$ stated above. The notation $\mathcal{F}_t$ stands for a natural filtration associated with the driving Brownian motion W_t^* below under the risk-neutral probability measure P^* . If a short-term interest rate r_t is governed by the

mean-reverting process below, the bond price admits a closed-form solution.
The Vasicek (Ornstein-Uhlenbeck) process is presented by

$$dr_t = \alpha(m - r_t)dt + \beta dW_t^* \,. \tag{2. 4}$$

Since this process is Makovian, $B(t,T)$ can be further simplified as a function of time and short-term spot rate, denoted by $P(t, r_t)$. Given this stochastic model above, it is known that the bond price function is of an affine structure:

$$P(t, r) = A(T-t)\, exp(-C(T-t)\, r), \tag{2. 5}$$

where *A* and *C* represent solutions of some ordinary differential equations. See Shreve (2004) for details. The same result can also be obtained by applying the discount factor $D_t = exp\,(-\int_0^t r_u\, du)$ to the bond price such that the martingale property of $D_t B(t,T)$ becomes a vehicle to derive the same result as of equation (2-5) :

$$A(\tau) = exp\left\{-\left[R * \tau - R\frac{1-e^{-\alpha\tau}}{\alpha} + \frac{\beta^2}{4\alpha^3}(1-e^{-\alpha\tau})^2\right]\right\}$$

$$C(\tau) = \frac{1-e^{-\alpha\tau}}{\alpha}$$

$$R = m - \beta^2\Big/(2\alpha^2). \tag{2. 6}$$

Consequently, the zero coupon bond price with Vasicek model is given as below,

$$P(t, r_t | T, r, \alpha, \beta, m) \tag{2. 7}$$

$$= exp\left\{-\left[R(T-t) - (R - r_t)\frac{1-e^{-\alpha(T-t)}}{\alpha} + \frac{\beta^2}{4\alpha^3}(1-e^{-\alpha\tau})^2\right]\right\}.$$

The calibration problem of a bond yield is now formulated as an optimization problem defined by

$$\min_{\alpha,m,\beta} \frac{1}{M}\sum_{t=1}^{M}\left(P_t - P(t, r_t | T, r, \alpha, \beta, m)\right)^2 , \tag{2. 8}$$

where P_t is an observed bond yield from the finacial market.

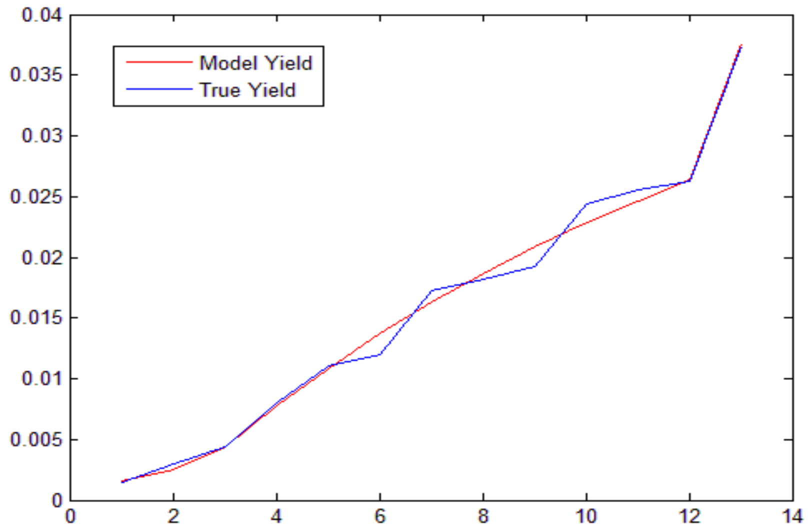


Figure 2.1 The calibration result of U.S treasury yield curve on 2009/01/03

The Table 2.1 and Figure 2.1 describe the parameter estimation and fitting error of U.S. treasury yield curve, respectively.

Table 2.1 Model parameter estimation and fitting error (MSE) of U.S treasury yield

Date	α	β	m	*Mean Square Error*
1/2/09	0.16388	0.006059	0.049268	8.13E-07
1/5/09	0.140681	0.004196	0.056612	8.13E-07
1/6/09	0.133579	0.006804	0.05771	9.11E-07
1/7/09	0.142484	0.004855	0.056315	9.56E-07
1/8/09	0.122745	4.13E-05	0.059196	1.28E-06
1/9/09	0.110318	3.55E-05	0.062264	1.25E-06
1/12/09	0.108405	9.90E-06	0.060788	1.32E-06
1/13/09	0.101333	0.000848	0.062515	1.17E-06
1/14/09	0.097825	1.84E-05	0.06098	1.22E-06
1/15/09	0.103512	0.001551	0.059306	9.24E-07
1/16/09	0.12934	0.005139	0.054483	9.62E-07
1/20/09	0.118646	0.00306	0.059037	9.67E-07
1/21/09	0.132136	0.005906	0.059953	8.07E-07
1/22/09	0.127467	0.007579	0.064133	8.15E-07
1/23/09	0.12982	0.005393	0.063174	8.21E-07
1/26/09	0.127643	1.30E-05	0.063834	1.06E-06
1/27/09	0.124685	-3.11E-05	0.061617	9.67E-07
1/28/09	0.126497	0.000113	0.06402	8.33E-07

1/29/09	0.144915	0.007917	0.064475	7.10E-07
1/30/09	0.142541	0.009479	0.065595	6.86E-07

Due to the fact that coporate bonds are often with coupons, it is necessary to apply a bootstraping method to convert those bond prices to the zero-coupon bond price.

2.3 Calibration of CDS Spread

In the literature of credit risk modeling (Leland (1994)), the reduced form model, also called the intensity model, treats default as an exogenous jump process. This kind of models is particularly suited to model credit spreads and is easy to calibrate CDS data.

The default intensity is the probability of default per year conditional on no earlier default. A good approximation of the risk-neutral default intensity λ_t per year is

$$\lambda_t \approx \frac{C_t}{1-R}, \tag{2. 12}$$

where C_t denotes the CDS spread and R denotes the recovery rate. We have applied the Vasicek model to describe movements of interest rates for bond price evaluation. The same model can also be adapted here by an exponential function in order to avoid the natural negative property of the Vasicek model for modeling default intensity. This model specifies that the instantaneous default intensity follows the stochastic differential equation as follows,

$$e^{h_t} \approx \frac{C_t}{1-R}, \tag{2. 13}$$

$$dh_t = \alpha(m - h_t)dt + \beta dZ_t\,, \tag{2. 14}$$

where Z_t is a Wiener process and α denotes the mean-reversion speed, β denotes the volatility of volatility, and m means the long-run mean. By the Euler discretization (Shreve (2004)), dh_t can be rewritten as equation (2.15)

$$h_{t+1} = \alpha m\Delta t + (1 - \alpha\Delta t)h_t + \beta\sqrt{\Delta t}\ \varepsilon_t\ \ ,\ \varepsilon_t \sim N(0,1) \tag{2. 15}$$

The Maximum likelihood estimation (MLE) is applied to estimate model parameters α, m, β in equation (2. 14). The data used in our experiment is the credit default swap spreads of IBM for one year. Table 2.2 records the results of parameter estimation for one year IBM CDS under the default intensity approximation.

Table 2.2 Estimated parameters of IBM CDS between 2009/01/02 and 2009/01/27

Date	α	β	m		Date	α	β	m
2009/01/02	0.52357	1.47435	-3.46252		2009/01/15	0.91338	1.46588	-4.78015
2009/01/05	0.53000	1.47434	-3.48999		2009/01/16	0.91415	1.46587	-4.77372
2009/01/06	0.60854	1.47087	-3.71706		2009/01/20	0.85175	1.46698	-4.62080
2009/01/07	0.52743	1.46814	-3.46044		2009/01/21	0.81481	1.46737	-4.51729
2009/01/08	0.58120	1.46848	-3.74387		2009/01/22	0.73614	1.46885	-4.27549
2009/01/09	0.67900	1.46368	-3.97002		2009/01/23	0.62551	1.47142	-3.84344
2009/01/12	0.66251	1.46091	-4.09061		2009/01/26	0.54253	1.47271	-3.40711
2009/01/13	0.69012	1.46099	-4.18420		2009/01/27	0.55440	1.47271	-3.46380
2009/01/14	0.84328	1.46495	-4.62751					

3 Joint Calibration of Market Risk and Credit Risk

Some empirical results suggested that the stock market and CDS market are correlated. Zhang (2009) found that the volatility of the equity could predict 48% of the variation of CDS spread. Carr and Wu (2010) documented evidences that the variance of an equity return was positively related to its credit spread of the same company. Hence, they postulated the following linear equation:

$$\lambda_t = \beta_{corr} v_t + z_t \,, \tag{3.1}$$

where λ_t denoted the default intensity, β_{corr} was the correlation of the default intensity λ_t and variance v_t, and z_t was a stochastic process to capture the independent shock.

However, the estimated parameter β_{corr} in equation (3.1) contradicts with other empirical results. This motivates us to propose an alternative to extend the study by Carr and Wu (2010) so that a genuine methodology to solve for calibration problems under multiple market risks becomes possible. To achieve this goal, we first consider a parameter estimation problem by (1) applying the Fourier transform method (Malliavin and Mancino (2009), Han (2015)) for estimating the volatility of the equity price, and (2) applying an approximation method for estimating the default intensity.

Then we further consider a vulnerable option evaluation problem by an efficient Monte Carlo simulation with martingale control variate method (Fouque and Han (2007,2008)).

3.1 Correlation between Stock Market and Credit Market

The market data used in our experiment is one year IBM CDS spread and stock price from 2007/01/02 to 2008/12/31 from Thomson Reuters. Motivated by Zhang et al. (2009), we consider the correlation between stock volatility and default intensity as a different approach from Carr and Wu (2010). Table 3.1 displays the result of the correlation between variance/volatility of IBM's stock prices and default intensities. When considering the default risk, this section will be divided into two cases: Case 1 is the correlation of variance and default intensity and Case 2 is the correlation of volatility and default intensity.

Table 3.1 Correlation between stock variance/volatility and default intensity of IBM

IBM 1year CDS	
Case 1	-0.18314
Case 2	-0.1494

3.2 Joint Dynamics: Evaluation and Estimation

We provide a methodology that allows both the volatility and default intensity are stochastic, and model their joint dynamics under the risk-neutral probability. No interest rate is considered. A two-step Monte Carlo procedure is employed for calibration to the implied volatility surfaces. Approximated default intensity approach under the reduced form model is employed for credit risk calibration. Combinations of these calibration methods allow a robust and efficient estimation for the joint dynamics of risk factors from the equity market and the credit market.

Recall the Vasicek model in the previous section, the value of the vulnerable call option at time t can be written as

$$C(t,S_t) = E^* \left[exp\left(-\int_t^T (r_s + \lambda_s) ds \right) * H(S_T) | F_t \right]. \quad (3.2)$$

Given the interest rate *r(t)* being deterministic assumption, the call option price is

$$C(t,S_t) = B(t,T) E^* \left[exp\left(-\int_t^T \lambda_s \, ds \right) * H(S_T) | F_t \right]. \quad (3.3)$$

Based on equation (3.1), we have

$$C(t,S_t) = B(t,T) E^* \left[exp\left(-\int_t^T z_s \, ds \right) | F_t \right] E^* \left[exp\left(-\beta_{corr} \int_t^T \upsilon_s \, ds \right) * H(S_T) | F_t \right]$$

$$= B(t,T) * exp\left(\frac{\sigma_Z{}^2 (T-t)^3}{6} \right) E^* \left[exp\left(-\beta_{corr} \int_t^T \upsilon_s \, ds \right) * H(S_T) | F_t \right] \quad (3.4)$$

where $B(t,T)$ denotes the discount factor and $H(S_T)$ is the payoff function at T.

Two models for the linear relationship between the default intensity and stock variance/volatility are considered below. Case 1 follows Carr and Wu (2010) by taking stock variance into account:

$$\lambda_t = \beta_{corr} \upsilon_t + z_t \,,$$

$$z_t \sim N(0, \sigma_Z{}^2 t), \quad (3.5)$$

We propose to take stock volatility into account by investigating the second case defined by

$$\lambda_t = \beta_{corr} \sigma_t + z_t \,,$$

$$z_t \sim N(0, \sigma_Z{}^2 t), \quad (3.6)$$

where λ_t denotes the default intensity, β_{corr} is the correlation of the default intensity λ_t and variance υ_t, or β_{corr} is the correlation of the default intensity and volatility σ_t , z_t is a stochastic process to capture the independent factor.

We use the OLS regression method to estimate β_{corr} and σ_Z. The market data used in this regression is IBM Company CDS spread and stock price from Thomson Reuters. The data period is from January 1, 2007 to December 31, 2008. Table 3.2 shows the result of regression.

Table 3.2 The results of regression for Case 1 and Case 2

	β_{corr}	σ_Z	adjusted R^2
Variance (Case 1)	0.011812	0.006118	0.11465

Volatility (Case 2)	0.014621	0.004773	0.46036

Note that the estimated parameters of case 1 and case 2 have little differences, but the adjusted R^2 are different obviously. It is observed from this dataset that case 2 performs better than case 1.

Based on some parameters estimated by Fourier transform method under the historical probability measure, we construct an objective function of the option price estimator with the rest unknown parameters. By using MATLAB function “fminsearch.m", we can solve for the following equation

$$\min_{\Theta} \sum_{i=1}^{n} \frac{1}{n} [ImpVol^{model}(\Theta) - ImpVol^{market}]^2 \tag{3. 7}$$

Where Θ denotes the parameter set, $ImpVol^{model}(\Theta)$ denotes the implied volatility calculated from the model, and $ImpVol^{market}$ denotes the implied volatility observed from the real data on market.

The market data used in our experiment are IBM stock price and IBM call option which are truncated with the moneyness from 0.9 to 1.1, and in order to make sure the results are consistent, the period we chose both are from 2009/01/02 to 2009/01/27. To reduce the complexity, the option is assumed no dividend payout. Table 3.3 presents the results of parameters (α, β, m) estimation under historical probability measure

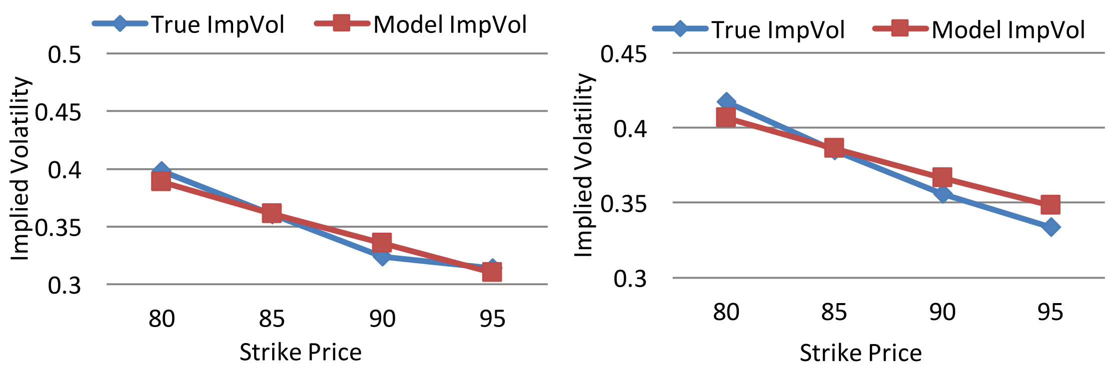
Table 3. 3 Parameters (α, β, m) under historical probability measure

Date	α	β	m		Date	α	β	m
2009/01/02	10.640	2.202	-1.518		2009/01/15	11.581	2.276	-1.509
2009/01/05	10.734	2.177	-1.514		2009/01/16	10.869	2.231	-1.507
2009/01/06	11.483	2.233	-1.514		2009/01/20	10.180	2.178	-1.510
2009/01/07	11.327	2.211	-1.509		2009/01/21	11.326	2.312	-1.514
2009/01/08	11.010	2.190	-1.510		2009/01/22	11.611	2.294	-1.507
2009/01/09	11.252	2.224	-1.507		2009/01/23	10.400	2.140	-1.499
2009/01/12	11.384	2.278	-1.511		2009/01/26	10.993	2.211	-1.501
2009/01/13	11.212	2.269	-1.510		2009/01/27			
2009/01/14	10.979	2.208	-1.506					

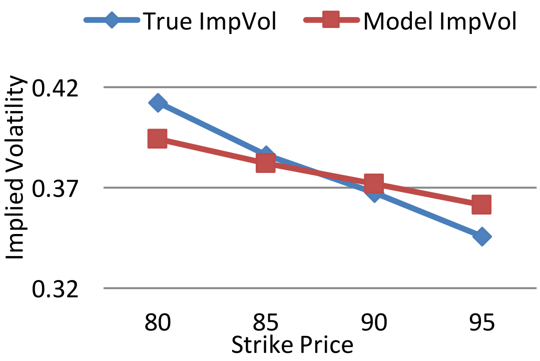
According to the Two-Step Monte Carlo approach, the rest unknown model parameters are estimated under a risk-neutral probability measure. Table 3.4 describes the results of the parameters estimation under risk-neutral probability measure with different time to maturities on 2009/01/02 and 2009/01/05. Consequently, we show the result of implied volatility surface fitting in Figure 3. 1.

Table 3. 4 Parameters (m^*, ρ^*) estimated under risk-neutral probability measure on 2009/01/02 and 2009/01/05

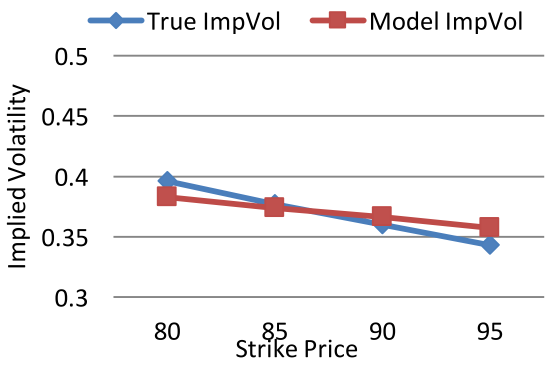
Time to Maturity (years) 2009/01/02	m^*	ρ^*		Time to Maturity (years) 2009/01/05	m^*	ρ^*
0.041667	-2.11374	-0.97511		0.033333	-1.78038	-0.98743
0.138889	-1.97593	-0.95114		0.130556	-1.8378	-0.99591
0.294444	-1.99972	-0.74166		0.286111	-1.8592	-0.93879
0.547222	-2.04297	-0.93639		0.538889	-2	-0.7175
1.052778	-2.06252	-0.73294		1.044444	-2.02514	-0.71292
2.083333	-2.08125	-0.74703		2.075	-2.12594	-0.65819



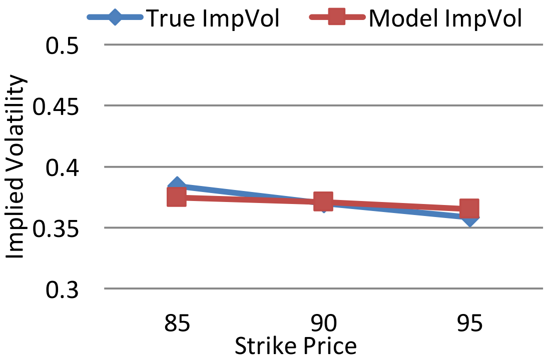
(a) Time to maturity = 0.04167 years (b) Time to maturity = 0.13889 years



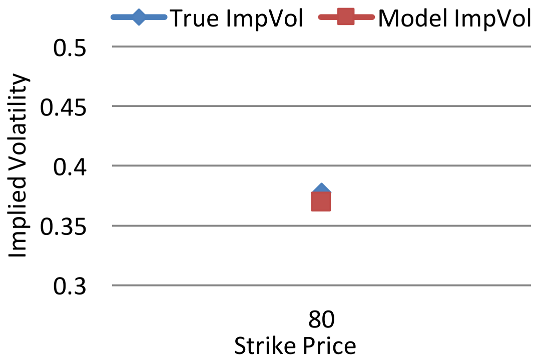
(c) Time to maturity = 0.2945 years



(d) Time to maturity = 0.5473 years



(e) Time to maturity = 1.0528 years



(f) Time to maturity = 2.0834 years

Figure 3.1 The implied volatility fitting results of IBM stock option on 2009/01/02; the six figures represent six different time to maturity

Final results of the mean square error of calibration are showed in Table 3.5
We observe that these parameters are stable and that means that the parameters are not volatile when the market changes slightly. Besides, the mean square error of the calibration result is relatively small.

Table 3. 5 Total mean square error of implied volatility surface from 2009/01/02 to 2009/01/27

Date	*Total MSE*
2009/01/02	1.82E-05
2009/01/05	1.26E-05
2009/01/06	1.60E-05
2009/01/07	4.53E-05
2009/01/08	8.57E-05

2009/01/09	5.12E-05
2009/01/12	3.35E-04
2009/01/13	9.65E-05
2009/01/14	7.94E-06
2009/01/15	5.98E-06
2009/01/16	8.43E-06
2009/01/20	2.22E-05
2009/01/21	3.70E-05
2009/01/22	1.03E-05
2009/01/23	8.44E-06
2009/01/26	1.60E-05
2009/01/27	2.02E-05

To observe the effect of default risk in option price, we compare the calibration result of default-free option with the result of vulnerable option. We discuss the calibration result for the two following cases.

Case 1: Correlation between Variance and Default Intensity

For call options, we assume there is a correlation between variance and default intensity as seen in section 3.1. The data we employed in our experiment is IBM call option from 2009/01/02 to 2009/01/27. The outcome of the comparison is described in Table 3.7 and the ratio of the improvement is in Figure 3.4. We observe that the calibration result considering credit risk is much better than the default-free model.

Table 3. 7 The comparison of fitting result between default-free option and vulnerable option considering the correlation between variance and default intensity.

Date	*Default-free Option*	*Vulnerable Option*
2009/01/02	1.82E-05	5.67E-06 (68.92%)
2009/01/05	1.26E-05	1.74E-05 (-37.94%)
2009/01/06	1.60E-05	1.13E-05 (29.52%)
2009/01/07	4.53E-05	5.01E-05 (-10.54%)
2009/01/08	8.57E-05	7.89E-05 (7.87%)
2009/01/09	5.12E-05	4.56E-05 (10.78%)
2009/01/12	3.35E-04	3.25E-04 (3.01%)
2009/01/13	9.65E-05	9.57E-05 (0.90%)
2009/01/14	7.94E-06	5.26E-06 (33.73%)
2009/01/15	5.98E-06	5.32E-06 (11.07%)

2009/01/16	8.43E-06	4.07E-06	(51.77%)
2009/01/20	2.22E-05	7.74E-06	(65.04%)
2009/01/21	3.70E-05	1.65E-05	(55.28%)
2009/01/22	1.03E-05	7.37E-06	(28.35%)
2009/01/23	8.44E-06	4.60E-06	(45.50%)
2009/01/26	1.60E-05	4.62E-06	(71.07%)
2009/01/27	2.02E-05	5.56E-06	(73.89%)

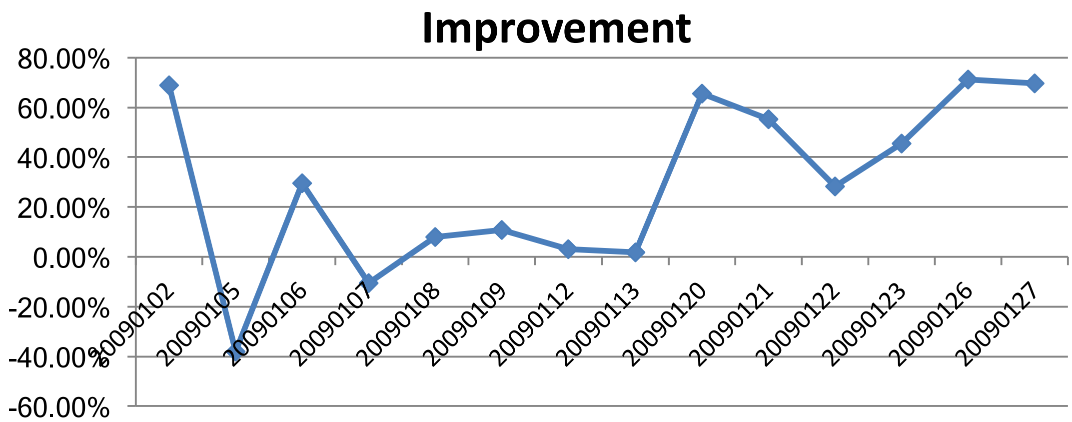


Figure 3. 4 The improvement of fitting result compared to default-free option

Case 2: Correlation between Volatility and Default Intensity

For call options, the outcome of the comparison is presented in Table 3.8 and the ratio of the improvement is in Figure 3.5. The data we employed in our experiment is IBM call option from 2009/01/02 to 2009/01/27. We can observe that the calibration result considering credit risk is much better than the default-free model.

Table 3. 8 The fitting result of default-free option and vulnerable option in Case 2.

Date	*Default-free Option*	*Vulnerable Option*	
2009/01/02	1.82E-05	6.48E-06	(64.46%)
2009/01/05	1.26E-05	1.17E-05	(7.28%)
2009/01/06	1.60E-05	1.35E-05	(15.39%)
2009/01/07	4.53E-05	4.77E-05	(-5.36%)
2009/01/08	8.57E-05	7.57E-05	(11.60%)
2009/01/09	5.12E-05	4.63E-05	(9.55%)
2009/01/12	3.35E-04	3.14E-04	(6.44%)

2009/01/13	9.65E-05	9.35E-05	(3.11%)
2009/01/14	7.94E-06	5.84E-06	(26.39%)
2009/01/15	5.98E-06	2.83E-06	(52.65%)
2009/01/16	8.43E-06	2.12E-06	(74.85%)
2009/01/20	2.22E-05	6.58E-06	(70.30%)
2009/01/21	3.70E-05	1.63E-05	(55.81%)
2009/01/22	1.03E-05	5.17E-06	(49.68%)
2009/01/23	8.44E-06	2.78E-06	(67.07%)
2009/01/26	1.60E-05	4.74E-06	(70.36%)
2009/01/27	2.02E-05	7.91E-06	(62.83%)

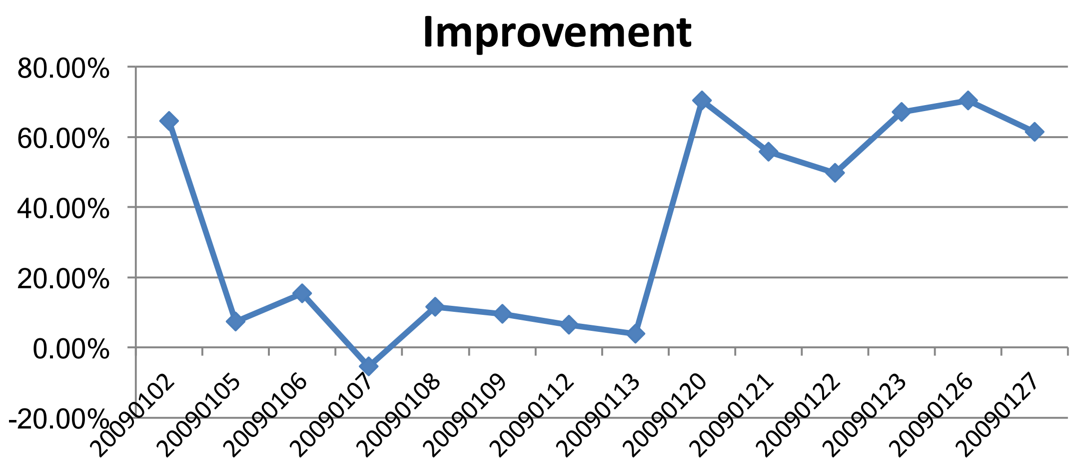


Figure 3. 5 The improvement of fitting result compared to default-free option

Empirical results reveal that Case 2 fits better to implied volatility surfaces. We find that instead of using the approach Carr and Wu (2010), the correlation between volatility and default intensity is more adopted for fitting implied volatility surfaces.

4 Joint Calibration of Multiple Risks

The aim of this section is to extend risk models studied in the previous section. Several joint calibration problems among multiple risks are considered. For example, the combination of market risk and interest rate risk, the combination of market risk, interest rate risk and credit risk, and the combination of market risk and corporate bond risk for put options. It can be seen that our proposed methodology can be applied to those joint calibration problems in a robust and efficient way so that the cross-market information can be revealed in Section 5.

4.1 Market Risk and Interest Rate Risk

In this section, we propose a new method to incorporate the interest rate risk to our approach but exclude the credit risk. The interest rate process r_t is allowed to be stochastic and the joint model is specified as follows,

$$\frac{dS_t}{S_t} = rdt + \sigma_t dW_{1t}$$

$$\sigma_t = e^{\frac{y_t}{2}},$$

$$dy_t = \alpha(m - y_t)dt + \beta dW_{2t} ,$$

$$dr_t = \alpha_r(m_r - r_t)dt + \beta_r dW_{3t}$$

$$\langle W_1 , W_2 \rangle_t = \rho t ,$$

where S_t is the asset price at time *t*, σ_t denotes the volatility process, y_t denotes the driving volatility process, W_{1t} , W_{2t} and W_{3t} are standard Brownian motions, α and α_r denote mean-reversion speeds, β and β_r denote volatilities of volatility, and *m* and m_r denote long-run means. We assume the correlation of W_{1t} and W_{2t} is ρt.

The comparison for call option with interest rate risk and without interest rate risk illustrates in Table 4.1 and the ratio of improvement is in Figure 4.1. The data we employed in our experiment is also IBM call option from 2009/01/02 to 2009/01/27, 3 months U.S. treasury yield, and all maturity treasury yields. It is readily observed that the calibration result with interest risk is better than that without the interest rate risk.

Table 4.1 The comparison of fitting result between option without interest rate risk and option with interest rate risk

Date	*Option without interest rate risk*	*Option with three months treasury*		*Option with all treasuries*	
2009/01/02	1.82E-05	1.22E-05	(33.06%)	7.61E-06	(58.29%)
2009/01/05	1.26E-05	1.14E-05	(9.68%)	1.30E-05	(-2.97%)
2009/01/06	1.60E-05	1.68E-05	(-5.10%)	1.22E-05	(23.72%)
2009/01/07	4.53E-05	5.17E-05	(-14.19%)	4.90E-05	(-8.14%)
2009/01/08	8.57E-05	8.15E-05	(4.86%)	8.04E-05	(6.13%)
2009/01/09	5.12E-05	5.38E-05	(-5.24%)	4.40E-05	(14.02%)
2009/01/12	3.35E-04	3.16E-04	(5.72%)	3.33E-04	(0.56%)
2009/01/13	9.65E-05	9.79E-05	(-1.46%)	9.06E-05	(6.17%)
2009/01/14	7.94E-06	7.51E-06	(5.39%)	3.85E-06	(51.49%)
2009/01/15	5.98E-06	5.11E-06	(14.47%)	2.89E-06	(51.61%)
2009/01/16	8.43E-06	5.13E-06	(39.19%)	3.66E-06	(56.60%)
2009/01/20	2.22E-05	5.04E-06	(77.24%)	9.16E-06	(58.66%)
2009/01/21	3.70E-05	2.66E-05	(27.98%)	1.77E-05	(52.19%)
2009/01/22	1.03E-05	7.22E-06	(29.81%)	5.30E-06	(48.49%)
2009/01/23	8.44E-06	4.99E-06	(40.96%)	2.45E-06	(71.01%)
2009/01/26	1.60E-05	6.35E-06	(60.28%)	5.05E-06	(68.41%)
2009/01/27	2.02E-05	6.13E-06	(71.22%)	4.26E-06	(79.98%)

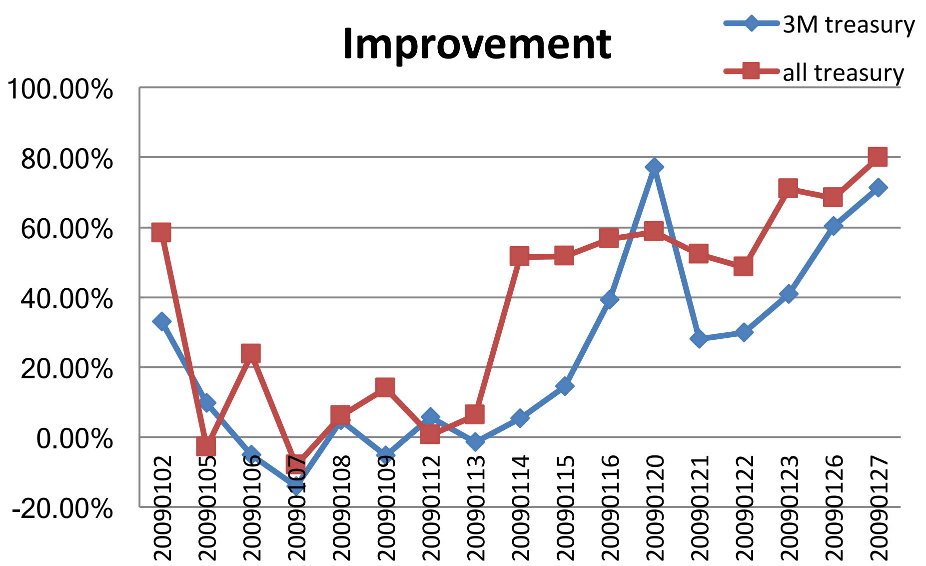


Figure 4.1 The improvement of fitting result compared to option without considering interest rate risk

4.2 Market Risk, Credit Risk and Interest Rate Risk

Based on the empirical results in section 3.3, we observe that both the default risk and interest rate risk are significant to reduce errors for fitting the implied volatility surfaces. To an extend, we incorporate an additional credit risk. The newly proposed model that considers market risk, interest rate risk, and credit risk is specified as follows.

$$\frac{dS_t}{S_t} = r_t dt + \sigma_t dW_{1t}$$

$$\sigma_t = e^{\frac{y_t}{2}},$$

$$dy_t = \alpha(m - y_t)dt + \beta dW_{2t} \ ,$$

$$dr_t = \alpha_r(m_r - r_t)dt + \beta_r dW_{3t}$$

$$dh_t = \tilde{\alpha}(\tilde{m} - h_t)dt + \tilde{\beta} dW_{4t} \ ,$$

$$e^{h_t} \approx \frac{C_t}{1-R} \ ,$$

$$\langle W_1 , W_2 \rangle_t = \rho t.$$

Most symbols are the same as before while C_t denotes the credit spread, W_{4t} is a standard Brownian motion independent to others, $\tilde{\alpha}$, $\tilde{\beta}, \tilde{m}$ denote the mean-reverting speed, volatility of volatility, and long-run mean, respectively. We will see that this joint dynamics is capable of retrieving all risks under the same framework and it produces the minimum modelling errors.

As seen in section 3.2, the correlation structure for market risk (variance or volatility) and credit risk (default intensity) was discussed. We will be using the "default-free option without interest rate risk," that is market risk, only as a benchmark.

Case 1: Correlation between Variance and Default Intensity

Based on Carr and Wu (2010), we consider the correlation between variance and default intensity. The outcome of the comparison is recorded in Table 4.1 and the ratio of improvement is demonstrated in Figure 4.2. The data set includes IBM call option price from 2009/01/02 to 2009/01/27, U.S treasury yield for 3 months and all maturities, and CDS spreads during the same time period.

Table 4.2 The comparison of fitting result compared to default-free Option without interest rate risk

Date	*Default- free Option without interest rate risk*	*Vulnerable Option with 3- month treasury*	*Vulnerable Option with Treasury yield for all maturities*
2009/01/02	1.82E-05	1.14E-05 (37.51%)	3.74E-06 (79.49%)
2009/01/05	1.26E-05	1.20E-05 (4.44%)	1.18E-05 (6.34%)
2009/01/06	1.60E-05	1.37E-05 (14.33%)	1.25E-05 (21.76%)
2009/01/07	4.53E-05	5.41E-05 (-19.40%)	4.54E-05 (-0.29%)
2009/01/08	8.57E-05	8.07E-05 (5.77%)	7.90E-05 (7.78%)
2009/01/09	5.12E-05	4.67E-05 (8.67%)	5.05E-05 (1.23%)
2009/01/12	3.35E-04	3.27E-04 (2.45%)	3.21E-04 (4.18%)
2009/01/13	9.65E-05	9.25E-05 (4.14%)	9.48E-05 (1.74%)
2009/01/14	7.94E-06	9.12E-06 (-14.93%)	6.31E-06 (20.55%)
2009/01/15	5.98E-06	5.16E-06 (13.64%)	3.15E-06 (47.31%)
2009/01/16	8.43E-06	3.35E-06 (60.24%)	2.47E-06 (70.77%)
2009/01/20	2.22E-05	7.10E-06 (67.95%)	5.58E-06 (74.81%)
2009/01/21	3.70E-05	2.69E-05 (27.19%)	2.05E-05 (44.59%)
2009/01/22	1.03E-05	7.46E-06 (27.48%)	5.13E-06 (50.14%)
2009/01/23	8.44E-06	7.59E-06 (10.17%)	2.69E-06 (68.14%)
2009/01/26	1.60E-05	6.46E-06 (59.60%)	3.87E-06 (75.79%)
2009/01/27	2.02E-05	7.04E-06 (66.95%)	7.04E-06 (66.93%)

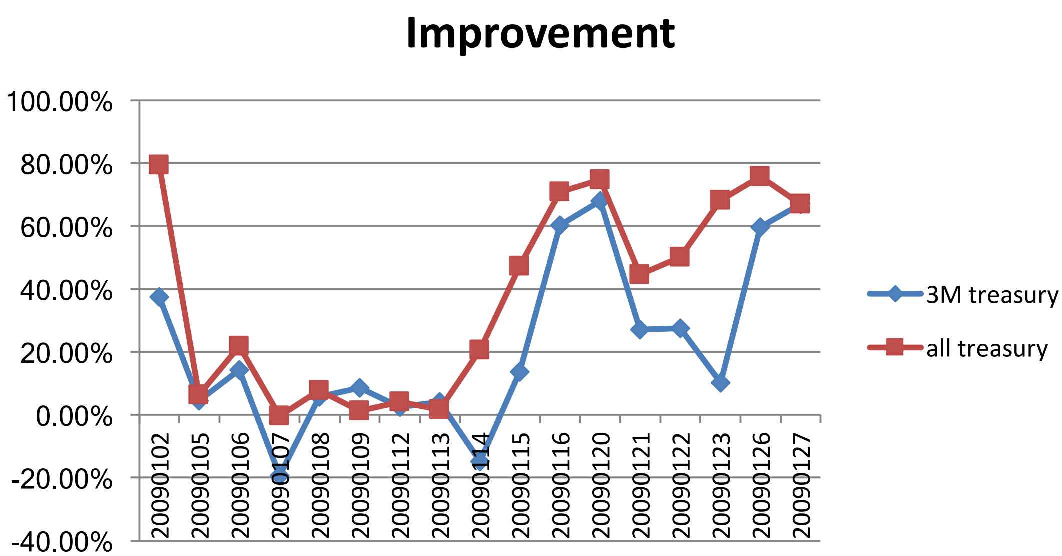


Figure 4.2 The improvement of fitting result compared to default-free option without interest rate risk

Case 2: Correlation between Volatility and Default Intensity

Considering the correlation between volatility and default intensity, the outcome of the comparison for call options is given in Table 4.3 and the ratio of improvement is illustrated in Figure 4.3. The data we employed is the same as in Case 1.

Table 4.3 The comparison of fitting result compared to default-free option without interest rate risk

Date	*Default-free Option without interest rate risk*	*Vulnerable Option with 3- month treasury*		*Vulnerable Option with Treasury yield for all maturities*	
2009/01/02	1.82E-05	7.85E-06	(56.98%)	4.40E-06	(75.90%)
2009/01/05	1.26E-05	1.19E-05	(5.07%)	1.02E-05	(18.98%)
2009/01/06	1.60E-05	1.13E-05	(29.20%)	1.16E-05	(27.45%)
2009/01/07	4.53E-05	4.64E-05	(-2.42%)	4.72E-05	(-4.28%)
2009/01/08	8.57E-05	7.42E-05	(13.44%)	7.81E-05	(8.87%)
2009/01/09	5.12E-05	5.40E-05	(-5.65%)	4.79E-05	(6.44%)
2009/01/12	3.35E-04	3.39E-04	(-1.18%)	3.31E-04	(1.14%)
2009/01/13	9.65E-05	9.10E-05	(5.68%)	9.26E-05	(4.07%)
2009/01/14	7.94E-06	9.87E-06	(-24.33%)	3.47E-06	(56.28%)
2009/01/15	5.98E-06	4.56E-06	(23.70%)	2.27E-06	(61.95%)
2009/01/16	8.43E-06	3.67E-06	(56.47%)	2.32E-06	(72.48%)
2009/01/20	2.22E-05	6.32E-06	(71.47%)	3.95E-06	(82.16%)

2009/01/21	3.70E-05	2.04E-05	(44.91%)	1.64E-05	(55.63%)
2009/01/22	1.03E-05	2.99E-06	(70.90%)	4.75E-06	(53.79%)
2009/01/23	8.44E-06	4.10E-06	(51.47%)	2.10E-06	(75.14%)
2009/01/26	1.60E-05	5.14E-06	(67.84%)	4.32E-06	(72.93%)
2009/01/27	2.02E-05	6.32E-06	(70.31%)	5.31E-06	(75.06%)

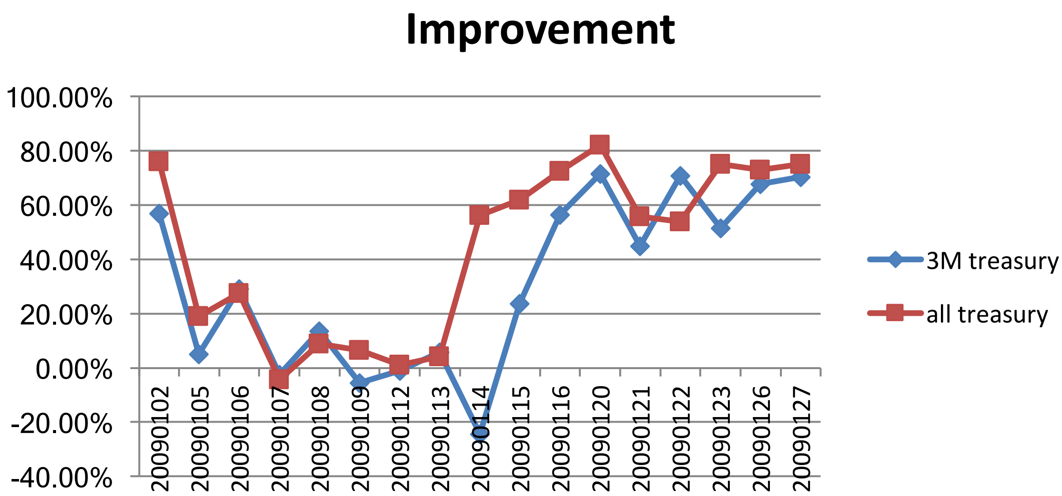


Figure 4.3 The improvement of fitting result compared to default-free option without interest rate risk

We can observe that in both cases, the calibration result with the interest risk is better than that without the interest rate risk. Therefore, we conclude that the model with interest rate risk and default risk is more suitable for fitting the implied volatility surface.

4.3 Case of Put Option: Joint Calibration to Stock Option Prices and Corporate Bond Yields

We extend the pricing model to American put options. Its price with multiple risks including market risk, credit risk, and interest rate risk, can be given by

$$P(t,S_t) = \sup_{t\leq\tau\leq T} E^*\left[exp\left(-\int_t^\tau (r_s+\lambda_s)\right) ds\times(K-S_\tau)^+|F_t\right], \qquad (4.1)$$

One key problem in pricing American-style options is the decision of optimal stopping (or executing) time. The basic Monte Carlo simulation method is often used for pricing vanilla options; however, deciding the optimal stopping time leads to a problem of simulations on simulations. That costs tremendous computing loading. To resolve this, we incorporate the least squares method (Longstaff and Schwartz (2001)) that approximates a lower bound of American put option price, and the duality approach (Rogers (2002)) that constructs an upper bound of the option price. Then, the martingale control method (Fouque and Han (2008)) is employed to enhance the accuracy of lower and upper bound prices so that a theoretical price of the American put options is defined as the average of these two bound prices. We design such a scheme for model calibration to implied volatility surface in the context of put options in American style such as IBM put options. The calibration result of the IBM put options illustrates in Table 4.4.

Table 4.4 The comparison of fitting result between no default risk put option and vulnerable put option of IBM

Date	*Default-free Option*	*Vulnerable Option*	
2009/01/02	2.16257E-05	0.000126572	(-485.28%)
2009/01/05	2.41115E-05	0.01005205	(-41589.78%)
2009/01/06	1.28297E-05	5.06165E-05	(-41589.78%)
2009/01/07	0.0001735	0.006641747	(-3728.09%)
2009/01/08	6.75783E-05	6.45998E-05	(4.41%)
2009/01/09	4.12474E-05	0.00010857	(-163.22%)
2009/01/12	0.000182168	0.000330275	(-81.30%)
2009/01/13	0.000167379	0.016699456	(-9877.03%)
2009/01/14	0.02783	0.02143	(23.00%)
2009/01/15	5.4E-06	3.4E-05	(-529.63%)
2009/01/16	0.00075	5.3E-04	(29.33%)
2009/01/20	1.67664E-05	5.48288E-06	(67.30%)
2009/01/21	4.8975E-05	1.98379E-06	(95.95%)
2009/01/22	9.9924E-06	1.14315E-06	(88.56%)
2009/01/23	7.7345E-06	5.52401E-06	(28.58%)
2009/01/26	1.10578E-05	9.64401E-06	(12.79%)
2009/01/27	8.77815E-05	3.00506E-05	(65.77%)

Although the literature denoted the CDS can reflect the default risk in the market immediately, the joint calibration result of put option and CDS spread does not perform well. Therefore, to investigate the risk factor of put options, we involve the corporate bond yield as the combination of default risk and interest rate risk and use the method in section 2 to calibrate the corporate bond yield. A joint calibration method for American put option and corporate bond yield is,

$$\frac{dS_t}{S_t} = r_t dt + \sigma_t dW_{1t}$$

$$\sigma_t = e^{\frac{y_t}{2}},$$

$$dy_t = \alpha(m - y_t)dt + \beta dW_{2t},$$

$$dF_t = \alpha_F(m_F - r_t)dt + \beta_r dW_{3t}$$

$$\langle W_1, W_2 \rangle_t = \rho t,$$

where S_t is the asset price at time t, λ_t is the default intensity at time t, F_t is the corporate bond yield without coupon at time t, W_{1t}, W_{2t} are standard Brownian motions, α, α_F denote the mean-reversion speed, β, β_r denote the volatility of volatility, and m, m_F are the long-run mean. We assume the correlation of W_{1t} and W_{2t} is ρt. The joint calibration results of the IBM put option and IBM bond yield are given in Table 4.5.

Table 4.5 The comparison of fitting result between no default risk put option and put option with default risk and interest rate risk

Date	*Default-free Option*	*Option with corporate bond yield*	
2009/01/02	2.16257E-05	2.57E-05	(-18.64%)
2009/01/05	2.41115E-05	1.60E-05	(33.76%)
2009/01/06	1.28297E-05	1.22E-05	(5.15%)
2009/01/07	0.0001735	1.13E-04	(34.74%)
2009/01/08	6.75783E-05	4.01E-05	(40.72%)
2009/01/09	4.12474E-05	1.02E-04	(-148.00%)
2009/01/12	0.000182168	1.46E-04	(20.08%)
2009/01/13	0.000167379	1.88E-04	(-12.95%)
2009/01/14	0.02783	2.62E-02	(5.78%)
2009/01/15	5.4E-06	1.61E-06	(70.10%)
2009/01/16	0.00075	3.62E-04	(51.61%)
2009/01/20	1.67664E-05	5.97E-06	(64.40%)
2009/01/21	4.8975E-05	5.24E-06	(89.31%)

2009/01/22	9.9924E-06	2.71E-06	(72.93%)
2009/01/23	7.7345E-06	1.74E-06	(77.55%)
2009/01/26	1.10578E-05	4.84E-06	(56.25%)
2009/01/27	8.77815E-05	1.96E-06	(81.76%)

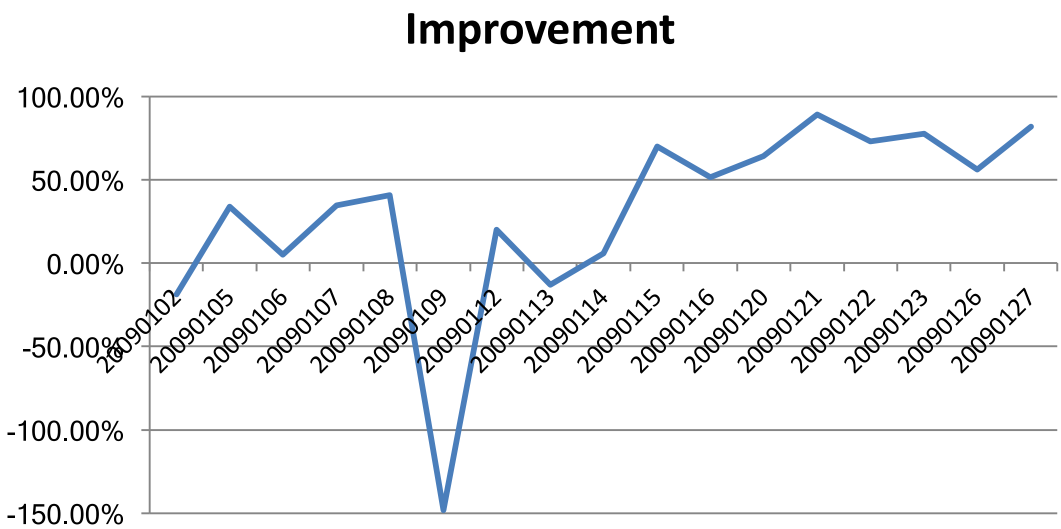


Figure 4.4 The improvement of fitting result compared to default-free Option without interest rate risk

5 Financial Implication: Option prices contain cross-market information

In this section, we consider whether the cross-market information can provide better calibration result in fitting implied volatility surface and make a complete comparison between each extended model and the simplest baseline model, that considers only the market risk without any default risk and any interest rate risk.

By taking the average of the improvement of mean square error in section 3.2, 4.1 and 4.2, we acquire the following tables. Table 5.1 below is meant to describe the improvement considering only the credit risk for IBM call option and Table 5.2 explains the improvement considering only the interest rate risk for IBM call option

Table 5.1 The improvement and the significance of mean difference considering only the credit risk for IBM call options

IBM call	*Variance*	*Volatility*
Improvement	29.89%	37.79%

Table 5.2 The improvement and the significance of mean difference considering only the interest rate risk for IBM call option

IBM call	*3 month treasury yield*	*All treasury yields*
Improvement	23.17%	37.43%

From Table 5.1, we observe the approach that assuming the correlation of variance and default intensity (Case 1) does not fit the option prices as good as the method we proposed (Case 2). Moreover, the Table 5.2 illustrates the U.S. treasury yield for all maturities performs better than the 3 month treasury yield does. That is, the U.S. treasury yields for all maturities do contain more information than the shorter period interest rate. Thus, when calibrating the implied volatility surface, we can use the proposed methodology instead of Carr and Wu (2010) to get the better fitting results.

To make a complete comparison between those models we proposed, we present all the results in Table 5.3 and find that with interest rate risk and credit risk, Case 2 provides a larger improvement than Case 1. In addition, from Table 5.3, we can make a conclusion that the best model is integrating the information of considering the correlation of volatility and default intensity and the information of the U.S. treasury yield for all maturities.

To examine the robustness of the results, a paired-sample t test is conducted to evaluate whether the models proposed provide the smaller mean square error. The results indicate that errors for the base model is significantly greater than errors for the proposed models. In Table 5.3, *** represents 99% significance level, ** represents 95% significance level and * means 90% significance level.

Table 5.3 The complete calibration result of all models in this paper

IBM call option				*IBM put option*
Credit Risk / ***Interest Rate Risk***		*Variance (Case 1)*	*Volatility (Case 2)*	*Corporate Bond*
		29.89%***	37.79%***	30.86%
3 month	23.17%***	22.13%***	31.4%***	
All Treasury	37.43%***	37.72%***	**43.76%*****	

Financial Implications

In financial literature, information contents of option prices are often discussed. For example, see Norden and Weber (2004), Berndt and Ostrovnaya (2007) and their references therein. Based on the previous comparison results, we raise the following financial questions and try to answer them from the perspective of model calibration. Note that the base pricing model contains only the market risk, i.e. the stock price risk and its volatility risk.

(1) Do option prices contain the information of the default risk?

(2) Do option prices contain the information of the interest rate risk?

(3) Do option prices contain the information of both the default and interest rate risks?

Our answers to these hypothetical questions are based on the improvement performance of the joint model calibration. That is, when the joint model induces smaller mean square errors to fit the implied volatility surfaces, we say that the information content of option prices include those risks considered within the joint model. For example, from Table 5.1 we observe that significant improvements 29.89% and 37.79% are obtained by considering additional default risk modeled by its correlation with the variance or volatility, respectively. These results provide a positive answer to Question (1) asked above. Similarly, based on improvement results from Tables 5.2 and 5.3, our answers to Questions (2) and (3) are both positive. We conclude that the joint model provides the best fit to the implied volatility surface.

This finding confirms that given the dataset of IBM stock prices, its CDS spreads or corporate bond prices, and treasury yields, stock option prices indeed contain cross market information, including the market risk, interest rate risk, and credit risk.

6 Conclusion

This paper provides a new methodology for joint model calibration of market risk, credit risk and interest rate risk. This methodology consists of
(1) a two-step Monte Carlo procedure for calibration to the term structure of implied volatility surfaces,
(2) a closed-form of bond yield under Vasicek model to calibrate the treasury yield and corporate bond yield,
(3) an approximate default intensity approach under the reduced form model for credit risk calibration.

Various combinations of these calibration methods allow a robust and efficient estimation for the joint dynamics of risk factors from the equity market, the credit market and the bond market.

The empirical performance confirms the accuracy of capturing the implied volatility surface by the two-stage Monte Carlo calibration method which includes Fourier transform method and martingale control variate. By comparing the two-step method with some well-known methods of describing the implied volatility such as time-varying LMMR, we observe that the calibration result of considering the default risk is more accuracy than the time-varying LMMR.

In addition, considering the credit risk from the company makes the model more complete and the joint dynamics fits more accurately to the market implied volatility. It implies that researchers may predict the default probability of the company from the option data. We leave this problem as a future research.

From the perspective of joint calibration, we propose a consistent model that can integrate three markets, rather than having separate models for each market. Based on improving performances demonstrated in Section 5, incorporating the information content from the credit market and the bond market can significantly reduce the fitting errors of the implied volatility surfaces. When a whole joint model is considered, our improvement increases up to 40%.

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