A Smooth Estimator for MC/QMC Methods in Finance
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Abstract
We investigate the effect of martingale control as a smoother for MC/QMC methods. Numerical results of estimating low-biased solutions of the American put option price under the Black-Scholes model demonstrate the unreliability of using QMC methods. But it can be fixed by considering a martingale control variate estimator. In another example of estimating European option prices under stochastic volatility models, randomized QMC methods improve the variance by a single digit. After adding a martingale control the variance reduction ratios raise up to 700 times for randomized QMC and about 50 times for MC simulations. An analysis to exploit the effect of the smoother is provided.

Keywords: Option pricing; Multi-factor stochastic volatility models; control variate method; Monte Carlo and quasi-Monte Carlo methods.

1 Introduction

The evaluation of financial derivatives are central problems in modern finance. In the seminal work of Black and Scholes [4], the fair price of a European-style derivative, denoted by $P$, can be derived to be a conditional expectation under the risk-neutral probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P}^*)$

$$P(t, S_t) = \mathbb{E}^* \{ e^{-r(T-t)} H(S_T) | \mathcal{F}_t \}$$

(1)

where the underlying risky asset $S_t$ is governed by the geometric Brownian motion

$$dS_t = rS_t dt + \sigma S_t dW_t^*.$$  

(2)

Other notations are defined as follows: $t$ the current time, $T < +\infty$ the maturity, $r$ the risk-free interest rate, $\sigma$ the volatility, $W_t^*$ the standard Brownian motion, $H(x)$ the payoff function satisfying the usual integrability condition. For example if $H(x) = \max \{ x - K, 0 \} \equiv (x - K)^+$ for the strike price $K > 0$ it is a call payoff; if $H(x) = \max \{ K - x, 0 \} \equiv (K - x)^+$ it is a put payoff. A financial contract with the call or put payoff only exercised at the maturity date is called a European call option or a European put option respectively.

From the simulation point of view, it is straightforward to construct the basic Monte Carlo (MC for short) estimator

$$\frac{1}{N} \sum_{i=1}^{N} e^{-rT} H(S_T^{(i)})$$

(3)
where $N$ is the total number of sample paths and $S_T^{(i)}$ denotes the $i$-th independent replication of the random variable $S_T$.

Our main interest in this paper is to improve the accuracy of the estimate obtained from (12) by variance reduction techniques or by Quasi Monte Carlo (QMC for short) method. Motivated from stochastic financial theory, every option contract as defined in (1) and (2) can be perfectly replicated by a hedging portfolio such that in the case of European options

$$P(0, S_0) = e^{-rT}H(S_T) - \mathcal{M}(P; T)$$

(4)

where $\mathcal{M}(P; T)$ is a zero-centered (hedging) martingale

$$\mathcal{M}(P; t) = \int_0^t e^{-r(s)} \frac{\partial P}{\partial x}(s, S_s) \sigma S_s dW_s^*.$$  

(5)

Equation (4) can be understood as a martingale representation by applying Ito’s lemma to the discounted option price process $e^{-rt}P(t, S_t)$ provided that the derivative price function $P(t, x)$ is first differentiable in time $t$ and twice differentiable in the asset price $x$. Ideally if one was able to sample perfectly for $S_T$ and $\mathcal{M}(P; T)$, then one can run a single Monte Carlo simulation on the right hand side of (4) to obtain the derivative price $P(0, S_0)$. In reality, if one was able to calculate $\mathcal{M}(P; T)$ perfectly, the partial derivative $\frac{\partial P}{\partial x}(t, x)$ would be known so that the option price $P(t, x)$ could be known in advance. Therefore Equation (4) is not feasible for a direct computation for the option price. Nevertheless by employing a martingale as a control we can formulate the unbiased control variate estimator

$$\frac{1}{N} \sum_{i=1}^N \left[ e^{-rT}H(S_T^{(i)}) - \mathcal{M}^{(i)}(\tilde{P}; T) \right]$$

(6)

for the option price $P(0, S_0) = \mathbb{E}^{\mathcal{F}_0} \left\{ e^{-rT}H(S_T) - \mathcal{M}(\tilde{P}; T) \right\}$ where the new martingale control $\mathcal{M}(\tilde{P}; T)$ consists of a price approximation $\tilde{P}$ to the actual option price $P$. In financial interpretation $\mathcal{M}(\tilde{P}; T)$ represents the delta hedging portfolio accumulated up to time $T$, so the term $\mathcal{M}(\tilde{P}; T)$ is called the hedging martingale by the price $\tilde{P}$ and the estimator defined in (6) is called the martingale control variate estimator.

A financial intuition about the effectiveness of the martingale control variate $e^{-rT}H(S_T) - \mathcal{M}(\tilde{P}; T)$ is that if delta trading $\frac{\partial P}{\partial x}(t, x)$ is closest to the actual hedging strategy $\frac{\partial P}{\partial x}(t, x)$, fluctuations of the replicating error will be small so that the variance of the estimator (6) should be small.

Other variance reduction techniques with less or no financial implication include conditional Monte Carlo [19], importance sampling [5, 10], direct sampling [3], etc. We refer to [11] and references therein. All Monte Carlo methods mentioned so far are fundamentally related to pseudo random sequences. As an alternative integral methods using quasi-random sequences (or called low-discrepancy sequences) have drawn lots of attentions in recent years because its theoretical rate of convergence is $O(1/n^{1-\varepsilon})$ for all $\varepsilon > 0$ subjected to the dimensionality and the regularity of the integrand [15]. Despite the regularity of the integrand function corresponding to the payoff $H(S_T)$ is generally poor [11], there are still many applications of using QMC or randomized QMC as a computational tool in finance. Many developed QMC techniques are motivated from financial applications [2, 12, 13, 18]. In next section we give a counterexample of using QMC method to estimate lower bound solutions of the American put option prices. However after combining a martingale control variate with QMC, we find that the accuracy of the America option price estimate is significantly improved.

In this paper, we investigate the effect of martingale control variate for MC/QMC methods. The evaluation of option prices under multifactor stochastic volatility models are also explored. Several
numerical experiments are conducted to compare the variance reduction performance for the martingale control variate method mentioned above with or without randomized QMC methods. The organization of this paper is the following. In Section 2, we give an example that pricing American options by QMC can be infeasible. It is fixed when martingale control variates are considered. In Section 3 we introduce the class of multifactor stochastic volatility model and the construction of martingale control by means of perturbation techniques. Numerical experiments by Monte Carlo method and quasi-Monte Carlo methods are presented in Section 4. We test several combinations of martingale control variate methods with or without QMC methods, including the Sobol’ sequence and L’Ecuyer type good lattice points together with the Brownian bridge sampling technique.

2 Low-Biased Estimate of American Option Price by MC/QMC

The right to early exercise a contingent claim is an important feature for derivative trading. An American option offers its holder, not the seller, the right but not the obligation to exercise the contract any time prior to maturity during its contract life time. Based on the no arbitrage argument, the American option price at time 0, denoted by $P_0$, with maturity $T < \infty$ is considered as an optimal stopping time problem [17] defined by

$$P_0 = \sup_{0 \leq \tau \leq T} \mathbb{E}^* \{ e^{-r\tau} H(S_\tau) | \mathcal{F}_0 \},$$

where $\tau$ denotes a bounded stopping time less than or equal to the maturity $T$. Longstaff and Schwartz [14] took a dynamic programming approach and proposed a least-square regression to estimate the continuation value at each in-the-money asset price state. By comparing the continuation value and the instant exercise payoff, their method exploits a decision rule, denoted by $\tau$, for early exercise along each sample path generated. As $\tau$ being a suboptimal stopping rule, Longstaff-Schwartz’ method induces a low-biased American option price estimate

$$\mathbb{E}^* \{ e^{-r\tau} H(S_\tau) | \mathcal{F}_0 \}.$$ 

By the optional sampling theorem we can use a locally hedging martingale $\mathcal{M}(\tilde{P}; \tau)$ to preserve the low-biased estimate (8) by

$$\mathbb{E}^* \{ e^{-r\tau} H(S_\tau) - \mathcal{M}(\tilde{P}; \tau) | \mathcal{F}_0 \}$$

where $\tilde{P}$ is an approximation of the American option price. By the spirit of hedging martingale discussed in Section 1, we consider a stochastic integral

$$\mathcal{M}(P_E; \tau) = \int_0^\tau e^{-rs} \frac{\partial P_E}{\partial x}(s, S_s) \sigma S_s dW^*_s,$$

where $P_E$ denotes the counterpart European option price. In the case of the American put option, $P_0$ is unknown but its approximation $P_E$ admits a closed-form solution, known as the Black-Scholes formula. Its delta $\frac{\partial P_E}{\partial x}$ is given by

$$\frac{\partial P_E}{\partial x}(t, x; T, K, r, \sigma) = \mathcal{N} \left( \frac{\ln(x/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right) - 1,$$

where $\mathcal{N}(x)$ denotes the cumulative normal integral function.

As an example we consider a pricing problem at time 0 for the American put option with parameters
Table 1: Comparisons of low-biased estimates (Column 2-5) and the actual American option prices (Column 6). MC denotes the basic Monte Carlo estimates. MC+CV denotes the control variate estimates with the hedging martingale $\mathcal{M}(P_E; \tau)$ being the additive control. Standard errors are shown in the parenthesis. QMC and QMC+CV denote calculations of Equation 8 and 9 using quasi sequences respectively.

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>MC</th>
<th>MC+CV</th>
<th>QMC</th>
<th>QMC+CV</th>
<th>$P_0$ (true)</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>20.7368 (0.2353)</td>
<td>20.6876 (0.0124)</td>
<td>20.9435</td>
<td>20.6626</td>
<td>21.6059</td>
</tr>
<tr>
<td>85</td>
<td>17.3596 (0.2244)</td>
<td>17.3586 (0.0134)</td>
<td>17.8031</td>
<td>17.3321</td>
<td>18.0374</td>
</tr>
<tr>
<td>90</td>
<td>14.3871 (0.2125)</td>
<td>14.3930 (0.0139)</td>
<td>15.0429</td>
<td>14.4030</td>
<td>14.9187</td>
</tr>
<tr>
<td>95</td>
<td>11.8719 (0.1995)</td>
<td>11.8434 (0.0148)</td>
<td>12.6472</td>
<td>11.8795</td>
<td>12.2314</td>
</tr>
<tr>
<td>100</td>
<td>9.8529 (0.1881)</td>
<td>9.6898 (0.0157)</td>
<td>10.5380</td>
<td>9.6942</td>
<td>9.946</td>
</tr>
<tr>
<td>105</td>
<td>7.9586 (0.1684)</td>
<td>7.8029 (0.0154)</td>
<td>8.7117</td>
<td>7.8351</td>
<td>8.0281</td>
</tr>
<tr>
<td>110</td>
<td>6.2166 (0.1518)</td>
<td>6.2606 (0.0150)</td>
<td>7.1663</td>
<td>6.2949</td>
<td>6.4352</td>
</tr>
<tr>
<td>115</td>
<td>5.0815 (0.1367)</td>
<td>5.0081 (0.0144)</td>
<td>5.8568</td>
<td>5.0221</td>
<td>5.1265</td>
</tr>
<tr>
<td>120</td>
<td>4.0885 (0.1245)</td>
<td>3.9389 (0.0146)</td>
<td>4.7480</td>
<td>3.9699</td>
<td>4.0611</td>
</tr>
</tbody>
</table>

$K = 100, r = 0.06, T = 0.5, \text{ and } \sigma = 0.4$. Numerical results of the low-biased estimates by MC/QMC with or without hedging martingales are demonstrated in Table 1. The first column illustrates a set of different initial asset price $S_0$. The true American option prices corresponding to $S_0$ are given in Column 6, depicted from Table 1 of [17]. Monte Carlo simulations are implemented by sample size $N = 5000$ and time step size (Euler discretization) $\Delta t = 0.01$. Column 2 and Column 3 illustrate low-biased estimates and their standard errors (in parenthesis) obtained from MC estimator defined in (8) and MC+CV estimator defined in (9) respectively. We observe that those estimates are indeed below the true prices and the standard errors are significantly reduced after adding the martingale control $\mathcal{M}(P_E; \tau)$. For QMC methods we use 5000 Niederreiter sequences of dimension 100. In column 4 we see clearly that in most situations, except $S_0 = 80$ and 85, low-biased QMC estimates are unreasonably greater than the true American prices. The striking part is that after adding hedging martingales, low-biased QMC+CV estimates shown in Column 5 are indeed below the true price. And these QMC+CV estimates are within 99% conference interval of the correspondent MC+CV estimates. These results strongly indicate that the hedging martingale plays the role of a smoother for MC/QMC methods. Because the complexity of American option pricing problems is high, we stress the smooth effect of hedging martingales by considering European option pricing problems under multifactor stochastic volatility models.

3 Monte Carlo Pricing under Multi-factor Stochastic Volatility Models

Stochastic volatility models have been an important class of diffusions as an extend of the Black-Scholes models. Stochastic volatility models are convenient to capture some stylized facts appearing either on historical data or implied (or derivatives) data. See [1, 8] and references therein. Under stochastic volatility models, closed-form solutions for typical option pricing problems barely exist even for European options. Numerical methods become essential. In this section we describe multi-factor stochastic volatility models and the construction of approximate hedging martingales by means of perturbation techniques.

Under a risk-neutral probability measure $\mathbb{P}^*$ parametrized by the combined market price volatility
premium \((\Lambda_1, \Lambda_2)\), multi-factor stochastic volatility models are defined by
\[
\begin{align*}
    dS_t &= rS_t dt + \sigma_t S_t dW^{(0)*}_t, \\
    \sigma_t &= f(Y_t, Z_t), \\
    dY_t &= \left[ \frac{1}{\varepsilon} c_1(Y_t) + \frac{g_1(Y_t)}{\sqrt{\varepsilon}} \Lambda_1(Y_t, Z_t) \right] dt + \frac{g_1(Y_t)}{\sqrt{\varepsilon}} \left( \rho_1 dW^{(0)*}_t + \sqrt{1 - \rho_1^2} dW^{(1)*}_t \right), \\
    dZ_t &= \left[ \delta c_2(Z_t) + \sqrt{\delta g_2(Z_t)} \Lambda_2(Y_t, Z_t) \right] dt + \sqrt{\delta g_2(Z_t)} \left( \rho_2 dW^{(0)*}_t + \rho_{12} dW^{(1)*}_t + \sqrt{1 - \rho_2^2 - \rho_{12}^2} dW^{(2)*}_t \right),
\end{align*}
\]
where \(S_t\) is the underlying asset price process with a constant risk-free interest rate \(r\). Its random volatility \(\sigma_t\) is driven by two stochastic processes \(Y_t\) and \(Z_t\) varying on different \textit{time scales} \(\varepsilon\) and \(1/\delta\), respectively. The vector \((W^{(0)*}_t, W^{(1)*}_t, W^{(2)*}_t)\) consists of three independent standard Brownian motions. Instant correlation coefficients \(\rho_1, \rho_2,\) and \(\rho_{12}\) satisfy \(|\rho_1| < 1\) and \(|\rho_2^2 + \rho_{12}^2| < 1\). The volatility function \(f\) is assumed to be smooth and bounded. Coefficient functions of processes \(Y_t\) and \(Z_t\), namely \((c_1, g_1, \Lambda_1)\) and \((c_2, g_2, \Lambda_2)\) are assumed to be smooth such that they satisfy the existence and uniqueness conditions for stochastic differential equations. Mean-reverting processes such as Ornstein-Uhlenbeck (OU) processes or square-root processes are typical examples to model driving volatility processes \([1, 9]\). Under this setup, the joint process \((S_t, Y_t, Z_t)\) is Markovian.

Given the multi-factor stochastic volatility model (11), the price of a plain European option with the integrable payoff function \(H\) and expiry \(T\) is defined by
\[
    P^{\varepsilon, \delta}(t, x, y, z) = \mathbb{E}^*_{x, y, z} \left\{ e^{-r(T-t)} H(S_T) \right\},
\]
where \(\mathbb{E}^*_{x, y, z}\) is a short notation for the expectation with respect to \(\mathbb{P}^*\) conditioning on the current states \(S_t = x, Y_t = y, Z_t = z\). A basic Monte Carlo simulation estimates the option price \(P^{\varepsilon, \delta}(0, S_0, Y_0, Z_0)\) at time 0 by the sample mean
\[
    \frac{1}{N} \sum_{i=1}^{N} e^{-rT} H(S_T^{(i)})
\]
where \(N\) is the total number of sample paths and \(S_T^{(i)}\) denotes the \(i\)-th simulated stock price at time \(T\). Variance reduction techniques are particularly important to accelerate the computing efficiency of the basic Monte Carlo pricing estimator (12). Next we briefly review the construction of a generic algorithm, i.e. martingale control variate method, recently proposed and analyzed by Fouque and Han \([6, 7]\).

### 3.1 Construction of Martingale Control Variates

Assuming that the European option price \(P^{\varepsilon, \delta}(t, S_t, Y_t, Z_t)\) is twice differentiable in state space and once differentiable in time, we apply Ito’s lemma to its discounted price \(e^{-rt}P^{\varepsilon, \delta}\), then integrate from time 0 to the maturity \(T\). After canceling out the pricing partial differential equation in some non-martingale terms, the following martingale representation can be obtained \([6]\)
\[
    P^{\varepsilon, \delta}(0, S_0, Y_0, Z_0) = e^{-rT} H(S_T) - \mathcal{M}_0(P^{\varepsilon, \delta}) - \frac{1}{\sqrt{\varepsilon}} \mathcal{M}_1(P^{\varepsilon, \delta}) - \sqrt{\delta} \mathcal{M}_2(P^{\varepsilon, \delta}),
\]
where centered martingales are given by
\[ \mathcal{M}_0(\varepsilon, \delta) = \int_0^T e^{-rs} \frac{\partial P^{\varepsilon, \delta}}{\partial x}(s, S_s, Y_s, Z_s)f(Y_s, Z_s)S_s dW_s^{(0)*}, \]  
\[ \mathcal{M}_1(\varepsilon, \delta) = \int_0^T e^{-rs} \frac{\partial P^{\varepsilon, \delta}}{\partial y}(s, S_s, Y_s, Z_s)g_1(Y_s)d\tilde{W}^{(1)*}_s, \]  
\[ \mathcal{M}_2(\varepsilon, \delta) = \int_0^T e^{-rs} \frac{\partial P^{\varepsilon, \delta}}{\partial z}(s, S_s, Y_s, Z_s)g_2(Z_s)d\tilde{W}^{(2)*}_s, \]

where the Brownian motions are
\[ \tilde{W}^{(1)*}_s = \rho_1 W^{(0)*}_s + \sqrt{1 - \rho_1^2} W^{(1)*}_s, \]
\[ \tilde{W}^{(2)*}_s = \rho_2 W^{(0)*}_s + \rho_{12} W^{(1)*}_s + \sqrt{1 - \rho_1^2 - \rho_{12}^2} W^{(2)*}_s. \]

These martingales play the role of “perfect” controls for Monte Carlo simulations. Namely, if the martingales (14), (15), and (16) can be exactly computed, then one can just generate one sample path to evaluate the option price through Equation (13). Unfortunately the gradient components \( \left( \frac{\partial P^{\varepsilon, \delta}}{\partial x}, \frac{\partial P^{\varepsilon, \delta}}{\partial y}, \frac{\partial P^{\varepsilon, \delta}}{\partial z} \right) \) of the option price appearing in the martingales is not possibly known in advance when the option price \( P^{\varepsilon, \delta} \) itself is exactly what we want to estimate. However, one can choose an approximate option price to substitute \( P^{\varepsilon, \delta} \) used in the martingales (14, 15, 16) and still retain martingale properties.

By an application of singular and regular perturbation techniques, the first order approximation derived in [9] is
\[ P^{\varepsilon, \delta}(t, x, y, z) \approx P_{BS}(t, x; \tilde{\sigma}(z)) \]  
where \( P_{BS}(t, x; \tilde{\sigma}(z)) \) denotes the solution of the Black-Scholes partial differential equation with the constant volatility \( \tilde{\sigma}(z) \) and the terminal condition \( P_{BS}(T, x) = H(x) \). The \( z \)-dependent effective volatility \( \tilde{\sigma}(z) \) is defined as the square root of an averaging of the variance function \( f^2 \) with respect to a limiting distribution of \( Y_t \):

\[ \tilde{\sigma}^2(z) = \int f^2(y, z)d\Phi(y) = \langle f^2(y, z) \rangle, \]

where \( \Phi(y) \) denotes the invariant distribution of the fast varying process \( Y_t \) while setting the volatility premium \( \Lambda_1 \) as zero. We use the bracket to represent such average. In the OU case, we choose that \( c_1(y) = m_1 - y \) and \( g_1(y) = \nu_1 \sqrt{2} \) with \( \Lambda_1 = 0 \) such that \( 1/\varepsilon \) is the rate of mean reversion, \( m_1 \) is the long run mean, and \( \nu_1 \) is the long run standard deviation. Its invariant distribution \( \Phi \) is normal with mean \( m_1 \) and variance \( \nu_2^2 \). We refer to [9] for detailed discussions of such models.

Note that the approximate option price \( P_{BS}(t, x; \tilde{\sigma}(z)) \) is independent of the variable \( y \) such that the term \( \mathcal{M}_1(P_{BS}) \) diminishes. Since the approximate martingale \( \mathcal{M}^{(i)}_2(P_{BS}) \) for (16) is small of order \( \sqrt{\delta} \). Intuitively we can neglect this term as well. We then select the stochastic integral \( \mathcal{M}_0(P_{BS}) \) as a control for the estimator (12) and formulate the following martingale control variate estimator:

\[ \frac{1}{N} \sum_{i=1}^{N} \left[ e^{\gamma T}H(S^{(i)}_T) - \mathcal{M}^{(i)}_0(P_{BS}) \right]. \]

This is the approach taken by Fouque and Han [6], in which the proposed martingale control variate method is numerically superior to an importance sampling method in [5] for pricing European options. Under OU-type processes to model \( (Y_t, Z_t) \) in (11) with \( 0 < \varepsilon, \delta \ll 1 \), the variance of the martingale control variate for European options is small of order \( \varepsilon \) and \( \delta \). This asymptotic result is shown in [7].
3.2 Variance and Error Analysis for MC/QMC

Based on the fact that random volatility is fluctuating around the long run mean of driving volatilities, we consider a simplified model which is helpful to explain the effect of martingale control as a smoother for MC/QMC methods. We assume that under the risk-neutral probability measure $S_t^\varepsilon$ is a perturbed risky asset price defined by

$$dS_t^\varepsilon = rS_t^\varepsilon \ dt + \sigma_t^\varepsilon S_t^\varepsilon \ dW_t^\ast,$$

where the perturbed volatility around the constant $\sigma$ is $\sigma_t^\varepsilon = \sigma + \varepsilon g_t$. We assume that $g_t$ is a deterministic and bounded function such that $\sigma_t^\varepsilon$ is positive and bounded away from 0. A European option, say a call with the strike $K$, is defined by

$$P^\varepsilon (t, S_t^\varepsilon; \sigma^\varepsilon) = \mathbb{E}_{t,S_t^\varepsilon}^\ast \left\{ e^{-r(T-t)} (S_T^\varepsilon - K)^+ \right\}$$

(21)

Lemma 1 Let $\mathcal{N}(x)$ be the cumulative integral function,

$$d_1(t, x; \sigma) = \frac{\ln(x/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}},$$

and the averaged volatility $\bar{\sigma}^\varepsilon = \sqrt{\frac{1}{T-t} \int_t^T (\sigma_s^\varepsilon)^2 \ ds}$. Then there exists a constant $C$ such that

$$\mathcal{N}(d_1(t, x; \bar{\sigma}^\varepsilon)) - \mathcal{N}(d_1(t, x; \sigma)) < C\varepsilon$$

for any $\varepsilon > 0, 0 \leq t < T, x > 0$.

Proof: We observe that on $0 \leq t < T$ and $x > 0 \mathcal{N}(d_1(t, x; \sigma))$ is a smooth function in $\sigma$ and its partial derivative $\frac{\partial \mathcal{N}(d_1(t, x; \sigma))}{\partial \sigma}$ is uniformly bounded. By the mean value theorem, there exists a $\sigma'$ between $\sigma$ and $\bar{\sigma}^\varepsilon$ such that

$$\mathcal{N}(d_1(t, x; \bar{\sigma}^\varepsilon)) - \mathcal{N}(d_1(t, x; \sigma)) = \frac{\partial \mathcal{N}(d_1(t, x; \sigma'))}{\partial \sigma} (\bar{\sigma}^\varepsilon - \sigma)$$

$$< C \frac{(\bar{\sigma}^\varepsilon)^2 - \sigma^2}{\bar{\sigma}^\varepsilon + \sigma}$$

$$< C\varepsilon.$$

Here we abuse the notation $C$ to indicate the existence of some constant.

Theorem 2 (Variance Analysis) Suppose that the payoff function $H$ is a call and the volatility processes $\sigma_t^\varepsilon$ is defined above. For any fixed initial state $(0, S_0^\varepsilon = x)$, there exist $\varepsilon > 0$ small enough and a positive constant $C$ such that

$$\text{Var} \left( e^{-rT} (S_T^\varepsilon - K)^+ - \mathcal{M}(P_{BS}; T) \right) \leq C\varepsilon.$$

Proof: Using the martingale representation

$$e^{-rT}(S_T^\varepsilon - K)^+ - P^\varepsilon(0, S_0^\varepsilon) = \int_0^T e^{-r_t} \frac{\partial P^\varepsilon}{\partial x}(t, S_t^\varepsilon; \bar{\sigma}^\varepsilon) \sigma_t^\varepsilon S_t^\varepsilon \ dW_t^\ast$$

(22)
we obtain
\[ \text{Var} \left\{ e^{-rT} (S_T^\varepsilon - K)^+ - \mathcal{M}(P_{BS}; T) \right\} \]
\[ = \text{Var} \left\{ \int_0^T e^{-rt} \left( \frac{\partial P^\varepsilon}{\partial x}(t, S_t^\varepsilon; \bar{\sigma}^\varepsilon) - \frac{\partial P_{BS}}{\partial x}(t, S_t^\varepsilon; \sigma) \right) \sigma_t^\varepsilon S_t dW_t^* \right\} \]
\[ = \mathbb{E}^* \left\{ \int_0^T (\mathcal{N}(d_1(t, S_t^\varepsilon; \bar{\sigma}^\varepsilon)) - \mathcal{N}(d_1(t, S_t^\varepsilon; \sigma)))^2 (\sigma_t^\varepsilon)^2 (e^{-rt} S_t^\varepsilon)^2 \, dt \right\}. \]

The last equation is obtained from the Ito’s Isometry Theorem. By Lemma 1 and the fact that \( \mathbb{E}^* \left\{ (e^{-rS_t^\varepsilon})^2 \right\} \) is bounded [7]. Then we get
\[ \text{Var} \left\{ e^{-rT} (S_T^\varepsilon - K)^+ - \mathcal{M}(P_{BS}; T) \right\} \leq C \varepsilon. \]

Remark: The financial interpretation of the martingale control term
\[ \mathcal{M}(P_{BS}; T) = \int_0^T e^{-rs} \frac{\partial P_{BS}}{\partial x}(s, S_s^\varepsilon; \sigma) \sigma_s^\varepsilon S_s dW_s^* \] \hspace{1cm} (23)

corresponds to the cumulative cost of a \textit{delta} hedging strategy. This martingale control variate method can be easily applied to hitting time problems like barrier options and optimal stopping time problems like American options as discussed in [7].

From Equation (22) we see
\[ e^{-rT} (S_T^\varepsilon - K)^+ - \mathcal{M}(P_{BS}; T) - P^\varepsilon(0, S_0^\varepsilon) = \int_0^T e^{-rt} \left( \frac{\partial P^\varepsilon}{\partial x}(t, S_t^\varepsilon; \bar{\sigma}^\varepsilon) - \frac{\partial P_{BS}}{\partial x}(t, S_t^\varepsilon; \sigma) \right) \sigma_t^\varepsilon S_t dW_t^*. \] \hspace{1cm} (24)

This equation is very helpful to consider an error analysis for the QMC estimate of
\[ \mathbb{E}^* \left\{ e^{-rT} (S_T^\varepsilon - K)^+ - \mathcal{M}(P_{BS}; T) \right\}. \]

We use Proinov bound [15] as the foundation of error analysis for martingale control variate estimator because this bound primarily requires the continuity of the integrand. In the implementation of QMC methods, we first discretize the stochastic integral (23) by Euler scheme as a Reimann sum of normal variables, then define an integral problem over a hypercube space. The integrand is a continuous function. Moreover since the integrand always contains the delta error \( \frac{\partial P^\varepsilon}{\partial x}(t, S_t^\varepsilon; \bar{\sigma}^\varepsilon) - \frac{\partial P_{BS}}{\partial x}(t, S_t^\varepsilon; \sigma) \) which is of \( O(\varepsilon) \) as shown in Lemma 1. We therefore assure that the error of QMC estimates is small of \( O(\varepsilon) \).

4 Numerical Results

In this section, we will compare efficiencies for pricing European call options using control variates technique developed in the previous section, combined with Monte Carlo and quasi-Monte Carlo methods.

We assume that the underlying asset \( S \) is given by (11). In our computations, we use C++ on Unix as our programming language. The pseudo random number generator we used is \textit{ran}2() in [18]. In our comparisons, the sample sizes for MC method are 10240, 20480, 40960, 81920, 163840, and 327680, respectively; and those for Sobol’ sequence related methods are 1024, 2048, 4096, 8192, 16384, and 32768, respectively, each with 10 random shifts; and the sample sizes for L’Ecuyer’s type lattice rule points (LTLRP for short) related methods are 1021, 2039, 4093, 8191, 16381, and 32749, respectively, and again, each with 10 random shifts.
In the following examples, we divide the time interval \([0, T]\) into \(m = 128\) subintervals. In Table 2, the first column labeled as \(N\) indicates the number of Monte Carlo simulations or the Quasi-Monte Carlo points. The second column labeled as MC indicates the option price estimates (standard errors in the parenthesis) based on the basic MC estimator (12). All rest columns record variance reduction ratios obtained from many specific MC/QMC methods in comparison with the basic MC estimates. For example, the third column labeled as MC+CV indicates the variance reduction ratios as the squares of the standard errors in the second column versus the standard errors obtained from the martingale control variate estimation (19). The fourth column labeled as Sobol’ indicates the variance reduction ratios as the squares of the standard errors in the second column versus the standard errors obtained from the estimation (12) by randomized Sobol’ sequence.

As an example we consider a European call option with payoff \(H(S_T) = \max(S_T - K, 0) = (S_T - K)^+\), where \(K\) is the strike price. We take input variables and parameters as follows: \(S_0 = 55\), \(K = 50\), \(r = 0.1 = 10\%\), \(T = 1\) year, \(m_1 = m_2 = -0.8\), \(\nu_1 = 0.5\), \(\nu_2 = 0.8\), \(\rho_1 = \rho_2 = -0.2\), \(\rho_{12} = 0\), \(y_0 = z_0 = -1.0\), where we specify driving volatility processes as \(c_1(y) = m_1 - y\), \(c_2(z) = m_2 - z\), \(\Lambda_1(y, z) = \Lambda_2(y, z) = 0\), \(g_1(y) = \nu_1\), \(g_2(z) = \nu_2\). For \(\varepsilon\) and \(\delta\), we take \(\varepsilon = 1/50\), \(\delta = 0.5\). The results are listed in Tables 2 and 3, where MC+CV stands for Monte Carlo method using control variate technique, Sobol+BB means the quasi-Monte Carlo method using Sobol’ sequence with Brownian bridge sampling technique, LTLRP for QMC method using L’Ecuyer type lattice rule points, etc. As seen the Table, any specific MC/QMC method generates more accurate numerics than the basic MC estimator does.

Table 2: Comparison of simulated European call option values and variance reduction ratios for \(\varepsilon = 1/50\), \(\delta = 0.5\).

<table>
<thead>
<tr>
<th>(N)</th>
<th>MC</th>
<th>MC+CV</th>
<th>Sobol’</th>
<th>Sobol+CV</th>
<th>Sobol+BB</th>
<th>Sobol+CV+BB</th>
</tr>
</thead>
<tbody>
<tr>
<td>1024</td>
<td>11.839(0.126)</td>
<td>45.8</td>
<td>5.0</td>
<td>339.3</td>
<td>2.6</td>
<td>129.3</td>
</tr>
<tr>
<td>2048</td>
<td>11.837(0.090)</td>
<td>48.0</td>
<td>2.3</td>
<td>304.8</td>
<td>4.0</td>
<td>138.4</td>
</tr>
<tr>
<td>4096</td>
<td>11.862(0.064)</td>
<td>48.4</td>
<td>1.8</td>
<td>124.8</td>
<td>2.5</td>
<td>158.4</td>
</tr>
<tr>
<td>8192</td>
<td>11.804(0.045)</td>
<td>47.4</td>
<td>2.3</td>
<td>124.0</td>
<td>2.9</td>
<td>148.4</td>
</tr>
<tr>
<td>16384</td>
<td>11.816(0.032)</td>
<td>47.1</td>
<td>1.4</td>
<td>176.1</td>
<td>7.7</td>
<td>115.5</td>
</tr>
<tr>
<td>32768</td>
<td>11.857(0.022)</td>
<td>48.1</td>
<td>1.7</td>
<td>235.9</td>
<td>4.5</td>
<td>479.9</td>
</tr>
</tbody>
</table>

Table 2: Comparison of simulated European call option values and variance reduction ratios for \(\varepsilon = 1/50\), \(\delta = 0.5\) (continued).

<table>
<thead>
<tr>
<th>(N)</th>
<th>LTLRP</th>
<th>LTLRP+CV</th>
<th>LTLRP+BB</th>
<th>LTLRP+CV+BB</th>
</tr>
</thead>
<tbody>
<tr>
<td>1021</td>
<td>2.0</td>
<td>75.5</td>
<td>7.3</td>
<td>687.9</td>
</tr>
<tr>
<td>2039</td>
<td>3.1</td>
<td>135.1</td>
<td>7.0</td>
<td>298.5</td>
</tr>
<tr>
<td>4093</td>
<td>3.1</td>
<td>143.9</td>
<td>2.2</td>
<td>140.1</td>
</tr>
<tr>
<td>8191</td>
<td>4.2</td>
<td>347.8</td>
<td>4.9</td>
<td>286.0</td>
</tr>
<tr>
<td>16381</td>
<td>3.1</td>
<td>227.9</td>
<td>7.8</td>
<td>94.8</td>
</tr>
<tr>
<td>32749</td>
<td>6.4</td>
<td>728.7</td>
<td>15.1</td>
<td>741.6</td>
</tr>
</tbody>
</table>

Table 3: Comparison of time (in seconds) used in the simulation of the above European option.
Table 3: Comparison of time (in seconds) used in simulation of the above European option (continued).

<table>
<thead>
<tr>
<th>$N$</th>
<th>MC</th>
<th>MC+CV</th>
<th>Sobol'</th>
<th>Sobol+CV</th>
<th>Sobol+BB</th>
<th>Sobol+CV+BB</th>
</tr>
</thead>
<tbody>
<tr>
<td>1024</td>
<td>7</td>
<td>10</td>
<td>7</td>
<td>9</td>
<td>7</td>
<td>12</td>
</tr>
<tr>
<td>2048</td>
<td>13</td>
<td>19</td>
<td>13</td>
<td>17</td>
<td>13</td>
<td>18</td>
</tr>
<tr>
<td>4096</td>
<td>26</td>
<td>40</td>
<td>26</td>
<td>35</td>
<td>28</td>
<td>39</td>
</tr>
<tr>
<td>8192</td>
<td>54</td>
<td>82</td>
<td>56</td>
<td>70</td>
<td>56</td>
<td>78</td>
</tr>
<tr>
<td>16384</td>
<td>109</td>
<td>167</td>
<td>107</td>
<td>139</td>
<td>107</td>
<td>157</td>
</tr>
<tr>
<td>32768</td>
<td>225</td>
<td>316</td>
<td>222</td>
<td>301</td>
<td>218</td>
<td>318</td>
</tr>
</tbody>
</table>

From Table 2, we observed the following facts. Using the control variate technique, the variance reduction ratios are around 48 for pseudo-random sequences. Without control variate, both Sobol’ sequence and L’Ecuyer type lattice rule points, even combined with Brownian bridge sampling technique, the variance reduction ratios are only a few times better than the MC sampling at most. However, when combined with control variate, the variance reduction ratios for the Sobol’ sequence vary from about 124 to 339 for Sobol’+CV and from 115 to 480 for Sobol’+CV+BB; and the variance reduction ratios for the L’Ecuyer type lattice rule points range from about 75 to 729 for LTLRP+CV and from 94 to 742 for LTLRP+CV+BB. This implicitly indicates that the new controlled payoff $e^{-rT}(S_T - K)^+ - M_0(P_BS; T)$ is smoother than the original call payoff $e^{-rT}(S_T - K)^+$ so that QMC methods become effective. It can be easily seen that under the Black-Scholes model with the constant volatility $\sigma$, the controlled payoff is exactly equal to the Black-Scholes option price $P_{BS}(0, S_0; \sigma)$, which is a constant so as a smooth function; while the original call payoff function is only continuous and even not differentiable.

Another interesting observation is that the variance reduction ratios do not always increase when the two low-discrepancy sequences are combined with control variate and Brownian bridge sampling, compared with when they are combined with control variate without Brownian bridge sampling.

Regarding time used in simulations, from Table 3 we observed that the time differences among methods without control variates are not significant, but the time differences between methods with and without control variates are not ignorable. Similar conclusions are true regarding time used in simulations for other cases. In our martingale control variate method, it is nor surprising to attribute extra computing time to the calculation of approximate deltas along every simulated path. Technically speaking we can use efficient schemes such as numerical partial differential methods to calculate these deltas in order to reduce the computational demand. However in this paper we will not pursue this direction but focusing on the effect of the martingale control.

In our last numerical examples we present the calculation of a sensitivity of the option price; namely delta ($\Delta$), the first partial derivatives of the option price with respect to its underlying price. A intuitive approach is by some finite difference scheme. For example we use a central difference scheme with a 1% underlying price as the difference. Numerical results are shown in Table 4. As before the notation +CV represents the finite difference of option price estimates obtained from martingale control variate estimators.
Table 4: Comparison of simulated $\Delta$ of the above European option by Finite Difference.

<table>
<thead>
<tr>
<th>$N$</th>
<th>MC</th>
<th>MC+CV</th>
<th>Sobol'</th>
<th>Sobol'+CV</th>
</tr>
</thead>
<tbody>
<tr>
<td>1024</td>
<td>0.8490(0.00507)</td>
<td>15.8</td>
<td>2.6</td>
<td>25.7</td>
</tr>
<tr>
<td>2048</td>
<td>0.8354(0.00357)</td>
<td>14.7</td>
<td>1.8</td>
<td>7.9</td>
</tr>
<tr>
<td>4096</td>
<td>0.8378(0.00253)</td>
<td>14.7</td>
<td>2.8</td>
<td>21.6</td>
</tr>
<tr>
<td>8192</td>
<td>0.8355(0.00179)</td>
<td>14.9</td>
<td>5.5</td>
<td>13.6</td>
</tr>
<tr>
<td>16384</td>
<td>0.8381(0.00126)</td>
<td>14.7</td>
<td>4.8</td>
<td>19.4</td>
</tr>
<tr>
<td>32768</td>
<td>0.8384(0.00090)</td>
<td>14.7</td>
<td>2.8</td>
<td>15.2</td>
</tr>
</tbody>
</table>

Since the call payoff is piecewise continuous it is useful to apply pathwise differentiation [11] or by the derivation Lemma A.1 in [7]. Then we can construct the corresponding martingale control variate estimators shown as +CV. Numerical results are shown in Table 5.

Table 5: Comparison of simulated $\Delta$ of the above European option by pathwise differentiation.

<table>
<thead>
<tr>
<th>$N$</th>
<th>MC</th>
<th>MC+CV</th>
<th>Sobol'</th>
<th>Sobol'+CV</th>
</tr>
</thead>
<tbody>
<tr>
<td>1024</td>
<td>0.8320(0.00513)</td>
<td>13.3</td>
<td>2.4</td>
<td>17.5</td>
</tr>
<tr>
<td>2048</td>
<td>0.8413(0.00358)</td>
<td>13.2</td>
<td>2.2</td>
<td>8.4</td>
</tr>
<tr>
<td>4096</td>
<td>0.8405(0.00254)</td>
<td>13.8</td>
<td>2.9</td>
<td>21.4</td>
</tr>
<tr>
<td>8192</td>
<td>0.8371(0.00180)</td>
<td>13.5</td>
<td>3.8</td>
<td>11.3</td>
</tr>
<tr>
<td>16384</td>
<td>0.8397(0.00127)</td>
<td>13.7</td>
<td>3.0</td>
<td>21.4</td>
</tr>
<tr>
<td>32768</td>
<td>0.8397(0.00090)</td>
<td>13.7</td>
<td>1.3</td>
<td>18.8</td>
</tr>
</tbody>
</table>

From the last two tables, we can still observe the smooth effect of martingale controls but the variance reduction power is not as significant when estimating option prices. This is due to the regularity of the delta is worse than the option price so that the effect of martingale control is also reduced.

5 Conclusion

Using (randomized) QMC methods to estimate high dimensional problems in financial derivatives may not be effective as shown in all examples presented. Based on the delta hedging strategy in trading financial derivatives, the value process of a hedging portfolio is considered as a martingale control in order to reduce the risk (replication error) of traded derivatives. For MC/QMC methods the role of the martingale control is a smoother so that significant variance reduction ratios can are obtained. We give an explanation of the effect of the smoother under a perturbed volatility model.

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References


