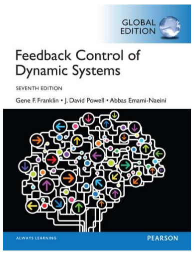
Feedback Control of System   Prof. Cheng-Hsien Liu



國立清華大學
National Tsing Hua University

PME320702
Feedback Control of Systems

Chapter 3

Instructor : Cheng-Hsien Liu 劉 承 賢

liuch@pme.nthu.edu.tw

http://mx.nthu.edu.tw/~chhsliu

Lecture 3 1

1

Feedback Control of System   Prof. Cheng-Hsien Liu

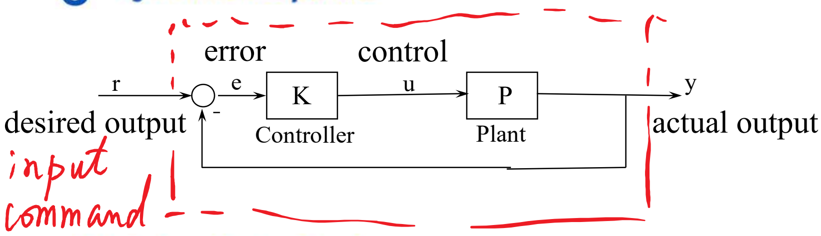
Lecture 3 2

2

Chapter 3

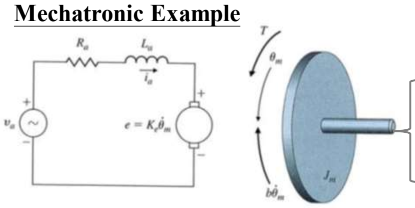
Dynamic Response

3 Dynamic Response



Linear Time Invariant Dynamic System

Mechatronic Example



2^{nd} order linear dynamic system

$$J_m\ddot{\theta}_m + b\dot{\theta}_m = K_t i_a. \qquad (2.52)$$

$$L_a\frac{di_a}{dt} + R_a i_a = v_a - K_e\dot{\theta}_m. \qquad (2.53)$$

1^{st} order linear dynamic system

n阶　　线性

For a general nth order linear dynamic system, represented by

parameter
variable

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + a_{n-2}\frac{d^{n-2}y(t)}{dt^{n-2}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t)$$
$$= b_m\frac{d^m f(t)}{dt^m} + b_{m-1}\frac{d^{m-1}f(t)}{dt^{m-1}} + \cdots + b_1\frac{df(t)}{dt} + b_0 f(t)$$

the coefficients $a_i, i = 0, 1, 2, \ldots, n-1$, and $b_i, i = 0, 1, 2, \ldots, m$, are assumed to be constant, which indicates that the given *system is time invariant*.

Linear Dynamic System

For a general nth order linear dynamic system, represented by

parameter
variable

y: 因变数
t: 自变数

$$\frac{d^n y(t)}{dt^n} + a_{n-1}\frac{d^{n-1}y(t)}{dt^{n-1}} + a_{n-2}\frac{d^{n-2}y(t)}{dt^{n-2}} + \cdots + a_1\frac{dy(t)}{dt} + a_0 y(t)$$
$$= b_m\frac{d^m f(t)}{dt^m} + b_{m-1}\frac{d^{m-1}f(t)}{dt^{m-1}} + \cdots + b_1\frac{df(t)}{dt} + b_0 f(t)$$

⇒ 线性方程式

⇒ 因变数不自乘 (1)
不相乘 (2)
不在 sin cos log | | 内 (3)

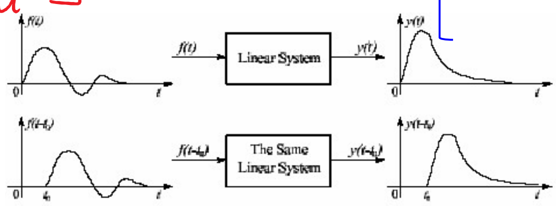
Linear Time variant Dynamic System

$a_i(t)$

LTI Dynamic System

Linear Time Invariant Dynamic System

$a_i = \text{constant}$



Graphical representation of system time invariance

Transfer Functions 转移函数

LTI Dynamic System

$$\Rightarrow T.F. \quad G(s) = \frac{Y(s)}{F(s)} = \frac{b_m s^m + b_{m-1}s^{m-1} + \cdots + b_1 s + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0}$$

$$= K\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

Def.:

1. G(s) is said to be proper if G(∞) is a finite constant -> $n \geq m$
2. G(s) is said to be strictly proper if G(∞) = 0 -> $n > m$
3. Relative order : n-m
4. The order of system: n
5. Poles: $-p_1, -p_2, \ldots, -p_n$ 极点
6. Zeros: $-z_1, -z_2, \ldots, -z_n$ 零点

Note:
- T.F. is to describe the relationship between input and output
- T.F. is independent to the input and initial condition

The $\mathcal{L}_-$ Laplace Transform

Inverse Laplace Transform

Laplace transform of $f(t)$, denoted by $\mathcal{L}_-\{f(t)\} = F(s)$ $\quad s = \sigma + j\omega,$

$$F(s) \triangleq \int_{0^-}^{\infty} f(t)e^{-st}\,dt. \quad (3.24)$$ Laplace

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_c - j\infty}^{\sigma_c + j\infty} F(s)e^{st}\,ds, \quad (3.25)$$ inverse Laplace

The $\mathcal{L}_-$ Laplace Transform

Inverse Laplace Transform

TABLE A.1 Properties of Laplace Transforms

Number	Laplace Transform	Time Function	Comment
—	$F(s)$	$f(t)$	Transform pair
1	$\alpha F_1(s) + \beta F_2(s)$	$\alpha f_1(t) + \beta f_2(t)$	Superposition
2	$F(s)e^{-s\lambda}$	$f(t-\lambda)$	Time delay ($\lambda \geq 0$)
3	$\frac{1}{\lvert a \rvert} F\left(\frac{s}{a}\right)$	$f(at)$	Time scaling
4	$F(s+a)$	$e^{-at}f(t)$	Shift in frequency
5	$s^m F(s) - s^{m-1}f(0) - s^{m-2}\dot{f}(0) - \cdots - f^{(m-1)}(0)$	$f^{(m)}(t)$	Differentiation
6	$\frac{1}{s}F(s)$	$\int_0^t f(\zeta)\, d\zeta$	Integration
7	$F_1(s)F_2(s)$	$f_1(t) * f_2(t)$	Convolution
8	$\lim_{s\to\infty} sF(s)$	$f(0^+)$	Initial Value Theorem
9	$\lim_{s\to 0} sF(s)$	$\lim_{t\to\infty} f(t)$	Final Value Theorem
10	$\frac{1}{2\pi j}\int_{\sigma_c - j\infty}^{\sigma_c + j\infty} F_1(\zeta)F_2(s-\zeta)d\zeta$	$f_1(t)f_2(t)$	Time product
11	$\frac{1}{2\pi}\int_{-j\infty}^{+j\infty} Y(-j\omega)U(j\omega)\, d\omega$	$\int_0^{\infty} y(t)u(t)\, dt$	Parseval's Theorem
12	$-\frac{d}{ds}F(s)$	$tf(t)$	Multiplication by time

TABLE A.2 Table of Laplace Transforms

Number	$F(s)$	$f(t), t \geq 0$
1	1	$\delta(t)$
2	$1/s$	$1(t)$
3	$1/s^2$	t
4	$2!/s^3$	t^2
5	$3!/s^4$	t^3
6	$m!/s^{m+1}$	t^m
7	$\frac{1}{s+a}$	e^{-at}
8	$\frac{1}{(s+a)^2}$	te^{-at}
9	$\frac{1}{(s+a)^3}$	$\frac{1}{2!}t^2e^{-at}$
10	$\frac{1}{(s+a)^m}$	$\frac{1}{(m-1)!}t^{m-1}e^{-at}$
11	$\frac{a}{s(s+a)}$	$1-e^{-at}$
12	$\frac{a}{s^2(s+a)}$	$\frac{1}{a}(at-1+e^{-at})$
13	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$
14	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$
15	$\frac{a^2}{s(s+a)^2}$	$1-e^{-at}(1+at)$
16	$\frac{(b-a)s}{(s+a)(s+b)}$	$be^{-bt}-ae^{-at}$
17	$\frac{a}{s^2+a^2}$	$\sin at$
18	$\frac{s}{s^2+a^2}$	$\cos at$
19	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
20	$\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$
21	$\frac{a^2+b^2}{s[(s+a)^2+b^2]}$	$1-e^{-at}\left(\cos bt + \frac{a}{b}\sin bt\right)$

The $\mathcal{L}_-$ Laplace Transform

Inverse Laplace Transform

Laplace transform of $f(t)$, denoted by $\mathcal{L}_-\{f(t)\} = F(s)$ $\quad s = \sigma + j\omega,$

$$F(s) \triangleq \int_{0^-}^{\infty} f(t)e^{-st}\,dt. \tag{3.24}$$

Laplace

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_c - j\infty}^{\sigma_c + j\infty} F(s)e^{st}\,ds, \tag{3.25}$$

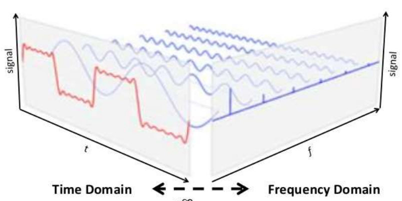
inverse Laplace

Joseph-Louis Lagrange
Italian 1736-1813
Pierre-Simon Laplace
French 1749-1827

Any periodic function can be rewritten as a weighted sum of Sines and Cosines of different frequencies

任何函數都可以展開為三角級數

Fourier Transform - Review



$$\hat{f}(\xi) = \int_{-\infty}^{\infty} f(x)\,e^{-2\pi i \xi x}\,dx$$

Fourier transform for functions f

$$f(x) = \sum_{n=-\infty}^{\infty} c_n\, e^{2\pi i (n/T) x} = \sum_{n=-\infty}^{\infty} \hat{f}(\xi_n)\, e^{2\pi i \xi_n x}\,\Delta\xi,$$

$$\begin{cases} c_n = \dfrac{1}{T}\displaystyle\int_{-T/2}^{T/2} f(x)\, e^{-2\pi i (n/T) x}\,dx. \\ c_n = (1/T)\hat{f}(n/T) \end{cases}$$

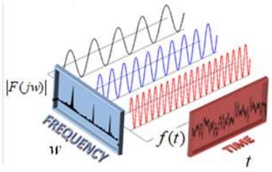
$$F(s) \triangleq \int_{0^-}^{\infty} f(t)e^{-st}\,dt.$$ Laplace

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_c - j\infty}^{\sigma_c + j\infty} F(s)e^{st}\,ds,$$ inverse Laplace

$s = j2\pi\xi$

Fourier Transform

$$\begin{cases} F(\xi) = \int_{-\infty}^{\infty} f(x)\, e^{-j2\pi\xi x}\, dx \\ f(x) = \int_{-\infty}^{\infty} F(\xi)\, e^{j2\pi\xi x}\, d\xi \end{cases}$$



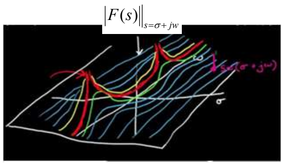
Fourier Transform

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \boxed{e^{-iwx}}\, dx.$$

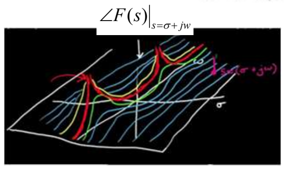
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) \boxed{e^{iwx}}\, dw$$

Laplace transform of $f(t)$, denoted by $\mathcal{L}_{-}\{f(t)\} = F(s)$ $s = \sigma + j\omega$,

$|F(s)|_{s=\sigma+jw}$



$\angle F(s)|_{s=\sigma+jw}$



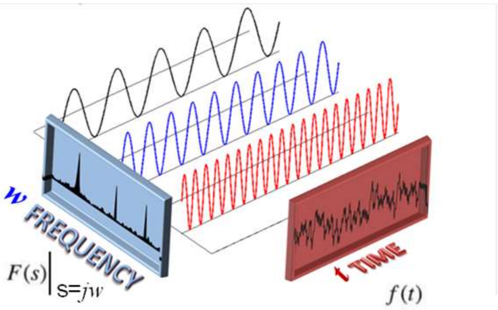
The $\mathcal{L}_{-}$ Laplace Transform

Inverse Laplace Transform

Laplace transform of $f(t)$, denoted by $\mathcal{L}_{-}\{f(t)\} = F(s)$ $\quad s = \sigma + j\omega$,

$$F(s) \triangleq \int_{0^-}^{\infty} f(t)e^{-st}\,dt. \tag{3.24}$$ Laplace

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_c - j\infty}^{\sigma_c + j\infty} F(s)e^{st}\,ds, \tag{3.25}$$ inverse Laplace

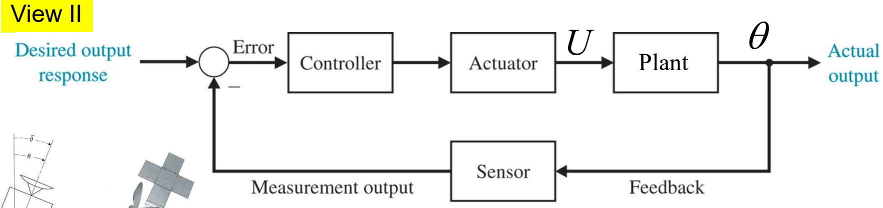


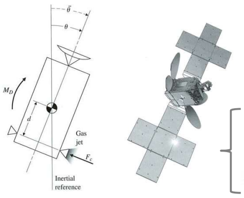
View I

$$\begin{cases} M\ddot{x} + b\dot{x} = 0.63i, \\ L\dfrac{di}{dt} + Ri = v_a - 0.63\dot{x} \end{cases} \Rightarrow \begin{cases} 2\ddot{x} + 3\dot{x} = 0.63i \\ 4\dot{i} + 5i = 6\sin(2t) - 0.63\dot{x} \end{cases}$$

$$\Rightarrow \begin{cases} 2s^2X + 3sX = 0.63I \\ 4sI + 5I = 6\dfrac{2}{s^2+4} - 0.63X \end{cases} \Rightarrow$$

View II



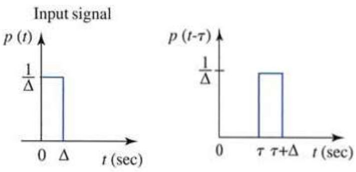


$$\begin{cases} F_c d + M_D = I\ddot{\theta}. \\ U = F_c d + M_D. \end{cases} \Rightarrow I\ddot{\theta} = U \Rightarrow \frac{\Theta(s)}{U(s)} = \frac{1}{I}\frac{1}{s^2}$$

Def : I

EXAMPLE 3.7 *Impulse Function Transform*

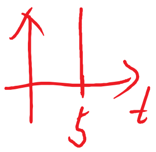
$$p_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t \le \Delta \\ 0, & \text{elsewhere} \end{cases} \qquad (3.3)$$



$$\lim_{\Delta\to 0} p_\Delta(t) = \delta(t), \quad \textbf{impulse}. \qquad (3.5)$$

$$\delta(t) = 0 \quad t \ne 0, \qquad (3.8)$$

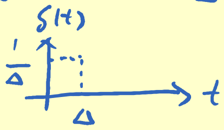
$\delta(t-5)$



$$\Rightarrow \int_{-\infty}^{\infty} \delta(t)dt = 1. \qquad (3.9)$$

$$\Rightarrow \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\,d\tau = f(t). \qquad (3.10)$$

$$\int_{-\infty}^{\infty} \delta(t)\,dt = \lim_{\Delta \to 0} \int_{0}^{\Delta} \frac{1}{\Delta}\,dt = \lim_{\Delta \to 0} 1 = 1$$

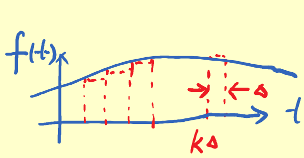


$$\int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\,d\tau = f(t)$$

$$\delta(t) = \begin{cases} \frac{1}{\Delta} & 0 < t < \Delta \\ 0 & \text{elsewhere} \end{cases}$$

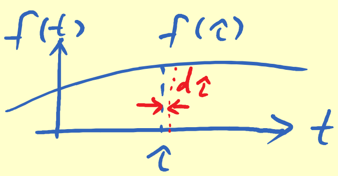
$$\delta(t-k\Delta) = \begin{cases} \frac{1}{\Delta} & k\Delta < t < (k+1)\Delta \\ 0 & \text{elsewhere} \end{cases}$$

Proof:



$$\hat{f}(t) = \sum_{k=-\infty}^{\infty} f(k\Delta)\,\underline{\delta(t-k\Delta)\cdot\Delta}$$

$$(\ \delta(t-k\Delta)\cdot\Delta = 1\ )$$



$$f(t) = \int_{-\infty}^{\infty} f(\tau)\,\delta(t-\tau)\,d\tau$$

$\delta(t) \Rightarrow$ ① $\displaystyle\int_{-\infty}^{\infty} \delta(t-a)\, dt = 1$

$\Rightarrow$ ② $\displaystyle\int_{b}^{c} \delta(t-a)\, dt = \begin{cases} 1 & a \in [b,c] \\ 0 & a \notin [b,c] \end{cases}$

$\Rightarrow$ ③ $\displaystyle\int_{-\infty}^{\infty} \delta(t-a)\, f(t)\, dt = f(a)$

$\Rightarrow$ ④ $\displaystyle\int_{b}^{c} \delta(t-a)\, f(t)\, dt = \begin{cases} f(a) & a \in [b,c] \\ 0 & a \notin [b,c] \end{cases}$

$$\mathcal{L}[\delta(t)] = \int_0^{\infty} \delta(t) e^{-st} dt$$

$\Rightarrow \displaystyle\int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\, d\tau = f(t).$

$\parallel$

$$\int_{-\infty}^{\infty} f(\tau)\, \delta(\tau - t)\, d\tau = f(t)$$

$$= \int_0^{\infty} e^{-s\tau} \delta(\tau - 0)\, d\tau$$

$$= e^{-s0} = 1$$

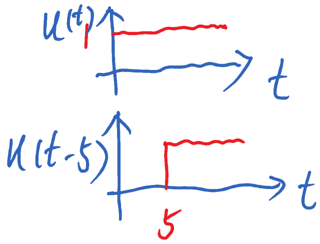
$\Rightarrow \mathcal{L}[\delta(t)] = 1$

Def 2　unit step function $u(t) \leftrightarrow 1(t)$

$$u_s(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

$$\Rightarrow u_s(t-a) = \begin{cases} 1 & t \geq a \\ 0 & t < a \end{cases}$$

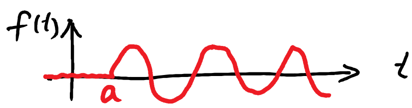
$$\Rightarrow \delta(t) = \frac{d}{dt} u_s(t)$$



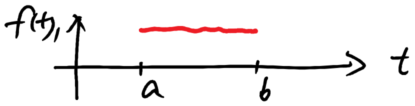
⊙ $f(t) = u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$

$$\mathcal{L}[u(t)] = \mathcal{L}[1] = \int_0^\infty e^{-st} dt = -\frac{e^{-st}}{s}\Big|_0^\infty = \frac{1}{s} \Leftarrow$$

⊙ Unit function 具選段功能



$\Rightarrow f(t) = \sin(t-a) \times u(t-a)$



$\Rightarrow f(t) = u(t-a) - u(t-b)$

$$u'(t) = \delta(t)$$

$$\Rightarrow \begin{cases} \mathcal{L}[u(t)] = \dfrac{1}{s} \\ \mathcal{L}[\delta(t)] = \int_0^\infty e^{-st}\delta(t)\,dt = e^{-st}\Big|_0 = 1 \ \# \end{cases}$$

$$\Rightarrow \begin{cases} \mathcal{L}^{-1}[1] = \delta(t) \\ \mathcal{L}^{-1}\left[\dfrac{1}{s}\right] = u(t) \end{cases} \ \text{推} \longleftrightarrow \begin{cases} \mathcal{L}^{-1}[s] = \delta'(t) \\ \mathcal{L}^{-1}[s^n] = \delta^n(t) \end{cases}$$

$$\Rightarrow \mathcal{L}[\delta(t-a)] = e^{-sa}$$

$$\int_{-\infty}^{\infty} \delta'(t) f(t)\,dt = \cancel{\delta(t) f(t)\Big|_{-\infty}^{\infty}} - \int_{-\infty}^{\infty} \delta(t) f'(t)\,dt$$

$$= (-1) f'(t)\Big|_{t=0}$$

$$\Rightarrow \int_{-\infty}^{\infty} \delta^{(n)}(t-a) f(t)\,dt = (-1)^n f^n(t)\Big|_{t=a}$$

$$\mathcal{L}[t^a] = \int_0^\infty e^{-st}\, t^a\, dt$$

令 $st = y \Rightarrow dt = dy/s$

$$= \int_0^\infty \frac{y^a}{s^a} e^{-y} \frac{dy}{s} = \frac{1}{s^{a+1}} \int_0^\infty y^a e^{-y} dy$$

$$= \frac{1}{s^{a+1}} \left[-y^a e^{-y} \Big|_0^\infty + (a) \int_0^\infty y^{a-1} e^{-y} dy \right]$$

$$= \frac{a}{s^{a+1}} \int_0^\infty y^{a-1} e^{-y} dy$$

$$= \cdots = \frac{a!}{s^{a+1}} \int_0^\infty e^{-y} dy$$

$$= \frac{a!}{s^{a+1}} \left[-e^{-y} \Big|_0^\infty \right]$$

$$= \frac{a!}{s^{a+1}} [-0 + 1] = \frac{a!}{s^{a+1}}$$ 必背

$$\begin{cases}\mathcal{L}[\sin wt] \\ \mathcal{L}[\cos wt]\end{cases} \quad e^{iwt} = \cos wt + i \sin wt$$

$$\Rightarrow \mathcal{L}[e^{iwt}] = \int_0^\infty [e^{iwt} e^{-st}]\, dt$$

$$= \int_0^\infty e^{(iw-s)t}\, dt = \frac{1}{iw-s} e^{(iw-s)t} \Big|_0^\infty$$

$$= \cdots = \frac{s+iw}{s^2+w^2}$$

$$\Rightarrow \begin{cases}\mathcal{L}[\sin wt] = \dfrac{w}{s^2+w^2} & 背 \\ \mathcal{L}[\cos wt] = \dfrac{s}{s^2+w^2} & 背\end{cases}$$

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{at} e^{-st}\, dt = \frac{-e^{-(s-a)t}}{s-a}\Big|_0^\infty$$

$$= \frac{1}{s-a} \quad \# \quad 背$$

TABLE A.2 Table of Laplace Transforms

Mentioned So far!

Number	$F(s)$	$f(t), t \geq 0$	
1	1	$\delta(t)$	p. 20
2	$1/s$	$1(t)$	p. 22/ p. 28
3	$1/s^2$	t	p. 25-26
4	$2!/s^3$	t^2	p. 25-26
5	$3!/s^4$	t^3	p. 25-26
6	$m!/s^{m+1}$	t^m	p. 25-26
7	$\frac{1}{s+a}$	e^{-at}	p. 28
8	$\frac{1}{(s+a)^2}$	te^{-at}	
9	$\frac{1}{(s+a)^3}$	$\frac{1}{2!}t^2e^{-at}$	
10	$\frac{1}{(s+a)^m}$	$\frac{1}{(m-1)!}t^{m-1}e^{-at}$	
11	$\frac{a}{s(s+a)}$	$1-e^{-at}$	
12	$\frac{a}{s^2(s+a)}$	$\frac{1}{a}(at-1+e^{-at})$	
13	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	
14	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	
15	$\frac{a^2}{s(s+a)^2}$	$1-e^{-at}(1+at)$	
16	$\frac{(b-a)s}{(s+a)(s+b)}$	$be^{-bt}-ae^{-at}$	
17	$\frac{a}{s^2+a^2}$	$\sin at$	p. 27
18	$\frac{s}{s^2+a^2}$	$\cos at$	p. 27
19	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$	
20	$\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$	
21	$\frac{a^2+b^2}{s[(s+a)^2+b^2]}$	$1-e^{-at}\left(\cos bt+\frac{a}{b}\sin bt\right)$	

Appendix A Laplace Transforms

4. Shift in Frequency

Multiplication (modulation) of $f(t)$ by an exponential expression in the time domain corresponds to a shift in frequency:

$$F_1(s) = \int_0^\infty e^{-at}f(t)e^{-st}\,dt = \int_0^\infty f(t)e^{-(s+a)t}\,dt = F(s+a). \quad \text{(A.9)}$$

TABLE A.2 Table of Laplace Transforms

Mentioned So far!

Number	$F(s)$	$f(t), t \geq 0$	
1	1	$\delta(t)$	p. 20
2	$1/s$	$1(t)$	p. 22/ p. 28
3	$1/s^2$	t	p. 25-26
4	$2!/s^3$	t^2	p. 25-26
5	$3!/s^4$	t^3	p. 25-26
6	$m!/s^{m+1}$	t^m	p. 25-26
7	$\frac{1}{s+a}$	e^{-at}	p. 28
8	$\frac{1}{(s+a)^2}$	te^{-at}	p. 28 + p. 25-26
9	$\frac{1}{(s+a)^3}$	$\frac{1}{2!}t^2e^{-at}$	p. 28 + p. 25-26
10	$\frac{1}{(s+a)^m}$	$\frac{1}{(m-1)!}t^{m-1}e^{-at}$	p. 28 + p. 25-26
11	$\frac{a}{s(s+a)}$	$1-e^{-at}$	
12	$\frac{a}{s^2(s+a)}$	$\frac{1}{a}(at-1+e^{-at})$	
13	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$	
14	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$	
15	$\frac{a^2}{s(s+a)^2}$	$1-e^{-at}(1+at)$	
16	$\frac{(b-a)s}{(s+a)(s+b)}$	$be^{-bt}-ae^{-at}$	
17	$\frac{a}{s^2+a^2}$	$\sin at$	p. 27
18	$\frac{s}{s^2+a^2}$	$\cos at$	p. 27
19	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$	p. 28 + p.27
20	$\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$	p. 28 + p.27
21	$\frac{a^2+b^2}{s[(s+a)^2+b^2]}$	$1-e^{-at}\left(\cos bt+\frac{a}{b}\sin bt\right)$	

$$\int_0^\infty e^{-at}f(t)e^{-st}\,dt = \int_0^\infty f(t)e^{-(s+a)t}\,dt = F(s+a)$$

$$\int_0^\infty e^{-at}f(t)e^{-st}\,dt = \int_0^\infty f(t)e^{-(s+a)t}\,dt = F(s+a)$$

The $\mathcal{L}_-$ Laplace Transform Laplace transform of $f(t)$, denoted by $\mathcal{L}_-\{f(t)\} = F(s)$

$$F(s) \triangleq \int_{0^-}^{\infty} f(t)e^{-st}\,dt$$

Inverse Laplace Transform

$$f(t) = \frac{1}{2\pi j}\int_{\sigma_c-j\infty}^{\sigma_c+j\infty} F(s)e^{st}\,ds$$

① $\mathcal{L}[y'(t)] = \int_0^\infty y'(t)e^{-st}dt = [y(t)e^{-st}]_0^\infty - \int_0^\infty y(t)e^{-st}dt$

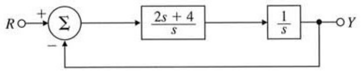
$= 0 - y(0) + s\int_0^\infty y(t)e^{-st}dt$

$= sY(s) - y(0)$

$$\mathcal{L}\left\{\frac{df}{dt}\right\} = \int_{0^-}^{\infty}\left(\frac{df}{dt}\right)e^{-st}\,dt = -f(0^-) + sF(s)$$

$$\mathcal{L}\{\ddot{f}\} = s^2F(s) - sf(0^-) - \dot{f}(0^-)$$

$$\mathcal{L}\{f^m(t)\} = s^mF(s) - s^{m-1}f(0^-) - s^{m-2}\dot{f}(0^-) - \cdots - f^{(m-1)}(0^-)$$



Convolution

$$\mathcal{L}\{f_1(t)\} = F_1(s) \text{ and } \mathcal{L}\{f_2(t)\} = F_2(s)$$

$$\Rightarrow \mathcal{L}\{f_1(t) * f_2(t)\} = \int_0^\infty f_1(t) * f_2(t) e^{-st}\, dt = \int_0^\infty \left[\int_0^t f_1(\tau) f_2(t-\tau)\, d\tau\right] e^{-st}\, dt$$

$$= \int_0^\infty \int_\tau^\infty f_1(\tau) f_2(t-\tau) e^{-st}\, dt\, d\tau$$

Multiplying by $e^{-s\tau}e^{s\tau}$

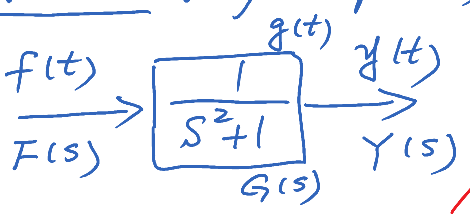
$$= \int_0^\infty f_1(\tau) e^{-s\tau} \left[\int_\tau^\infty f_2(t-\tau) e^{-s(t-\tau)}\, dt\right] d\tau$$

change variables $t' \triangleq t - \tau$

$$= \int_0^\infty f_1(\tau) e^{-st}\, d\tau \int_0^\infty f_2(t') e^{-st'}\, dt' \quad = F_1(s)F_2(s)$$

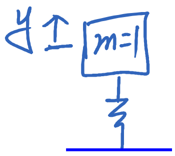
$$\Rightarrow \mathcal{L}^{-1}\{F_1(s)F_2(s)\} = f_1(t) * f_2(t)$$

Convolution (My viewpoint)



$S^2Y + Y = F$

物理系統



方程式 $\begin{cases} \ddot{y} + y = f(t) \\ F \cdot \frac{1}{S^2+1} = Y \end{cases}$

Initial Condition

$\begin{cases} y(t=0) = 0 \\ y'(t=0) = 0 \end{cases}$

$\Rightarrow f(t) * g(t) = y(t)$

convolution

$$\ddot{y}+y=\delta(t)$$

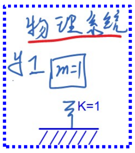
Laplace

$$s^2Y+Y=1$$

$$Y=\frac{1}{s^2+1}$$

$\mathcal{L}^{-1}$ Laplace

$$y=\sin t \quad \#$$



$$\mathcal{L}[\delta(t)]=\int_0^\infty \delta(t)e^{-st}dt$$

$$\Rightarrow \int_{-\infty}^{\infty} f(\tau)\delta(t-\tau)\,d\tau = f(t).$$

$$\int_{-\infty}^{\infty} f(\tau)\,\delta(\tau-t)\,d\tau = f(t)$$

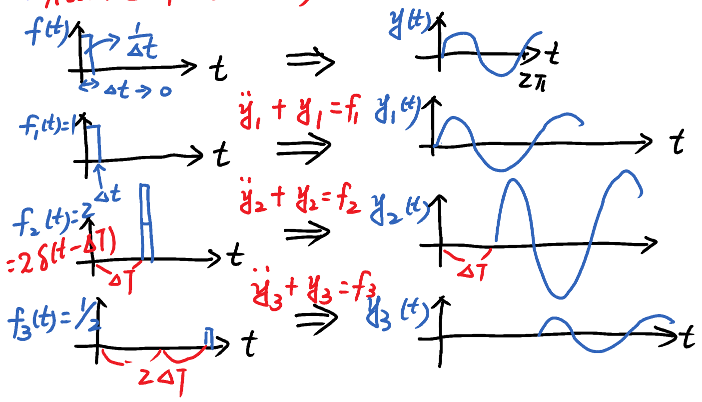
$$=\int_0^\infty e^{-s\tau}\,\delta(\tau-0)\,d\tau$$

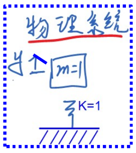
$$=e^{-s0}=1$$

$$\Rightarrow \mathcal{L}[\delta(t)]=1$$

$$\ddot{y}+y=f(t)$$

$\Rightarrow$ Assume $f(t)=\delta(t) \longleftrightarrow y(t)=g(t)=\sin t$





$$\ddot{y}+y=\delta$$

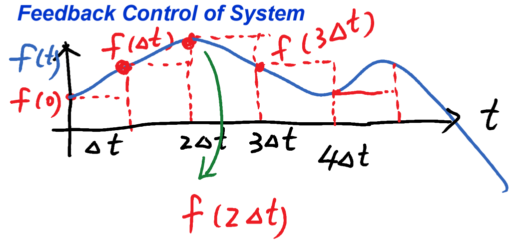
Laplace

$$s^2Y+Y=1$$

$$Y=\frac{1}{s^2+1}$$

$\mathcal{L}^{-1}$ Laplace

$$y=\sin t \quad \#$$



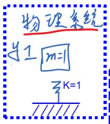
Assume $\Delta t \to 0$

$$\Rightarrow f(t) = \sum_{n=0}^{(t/\Delta t - 1)} f(n\Delta t)$$

$$\Rightarrow y(t) = f(0)\sin t + f(\Delta t)\sin(t-\Delta t) + f(2\Delta t)\sin(t-2\Delta t) + \cdots$$

$$= \sum_{n=0}^{N-1} f(n\Delta t)\sin(t-n\Delta t)$$

$$= \sum_{n=0}^{N-1} f(n\Delta t)\cdot g(t-n\Delta t)$$



$\ddot{y} + y = \delta$

Laplace

$s^2 Y + Y = 1$

$Y = \frac{1}{s^2+1}$

$\mathcal{L}^{-1}$ Laplace

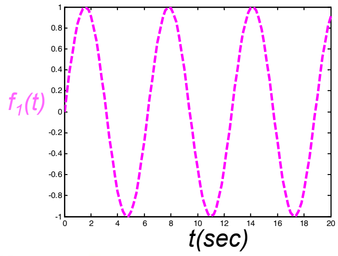
$y = \sin t$

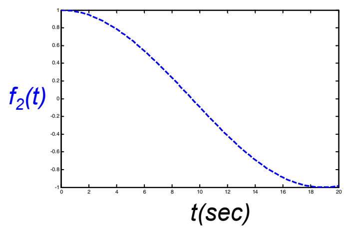
as $\Delta t \to 0$

$$\Rightarrow y(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau$$

$$= f(t) * g(t)$$

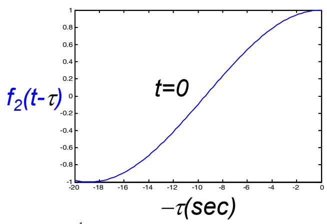
convolution

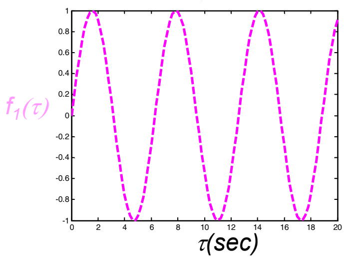


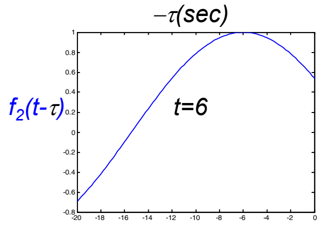


Convolution

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau)\, d\tau$$

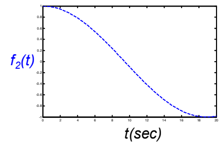


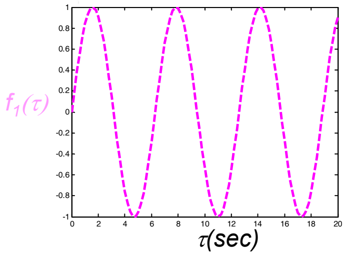


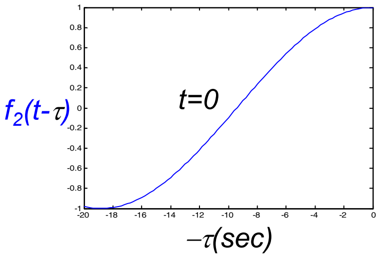


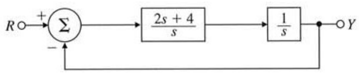
Convolution

$$f_1(t) * f_2(t) = \int_0^t f_1(\tau) f_2(t-\tau)\, d\tau$$









Convolution

$$\mathcal{L}\{f_1(t)\} = F_1(s) \text{ and } \mathcal{L}\{f_2(t)\} = F_2(s)$$

$$\mathcal{L}\{f_1(t) * f_2(t)\} = \int_0^\infty f_1(t) * f_2(t) e^{-st}\, dt = \int_0^\infty \left[\int_0^t f_1(\tau) f_2(t-\tau)\, d\tau\right] e^{-st}\, dt$$

$$= \int_0^\infty \int_\tau^\infty f_1(\tau) f_2(t-\tau) e^{-st}\, dt\, d\tau$$

Multiplying by $e^{-s\tau} e^{s\tau}$

$$= \int_0^\infty f_1(\tau) e^{-s\tau} \left[\int_\tau^\infty f_2(t-\tau) e^{-s(t-\tau)}\, dt\right] d\tau$$

change variables $t' \triangleq t - \tau$

$$= \int_0^\infty f_1(\tau) e^{-st}\, d\tau \int_0^\infty f_2(t') e^{-st'}\, dt' \quad = F_1(s) F_2(s)$$

$$\mathcal{L}^{-1}\{F_1(s) F_2(s)\} = f_1(t) * f_2(t)$$

Inverse Laplace Transform by Partial-Fraction Expansion

$m < n$

$$F(s) = \frac{b_1 s^m + b_2 s^{m-1} + \cdots + b_{m+1}}{s^n + a_1 s^{n-1} + \cdots + a_n} = \frac{C_1}{s-p_1} + \frac{C_2}{s-p_2} + \cdots + \frac{C_n}{s-p_n}. \quad (3.43)$$

$$(s-p_1)F(s) = C_1 + \frac{s-p_1}{s-p_2}C_2 + \cdots + \frac{(s-p_1)C_n}{s-p_n}. \quad (3.44)$$

$$C_1 = (s-p_1)F(s)|_{s=p_1} \qquad C_i = (s-p_i)F(s)|_{s=p_i}$$

EXAMPLE 3.9 $Y(s) = \dfrac{(s+2)(s+4)}{s(s+1)(s+3)}$ Find $y(t)$

$$Y(s) = \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3}{s+3}$$

$$C_1 = \left.\frac{(s+2)(s+4)}{(s+1)(s+3)}\right|_{s=0} = \frac{8}{3}$$

$$C_2 = \left.\frac{(s+2)(s+4)}{s(s+3)}\right|_{s=-1} = -\frac{3}{2}$$

$$C_3 = \left.\frac{(s+2)(s+4)}{s(s+1)}\right|_{s=-3} = -\frac{1}{6}$$

$$\Rightarrow Y = \frac{8/3}{s} + \frac{(-\frac{3}{2})}{s+1} + \frac{(-\frac{1}{6})}{s+3}$$

$$\Rightarrow y(t) = \frac{8}{3}1(t) - \frac{3}{2}e^{-t}1(t) - \frac{1}{6}e^{-3t}1(t)$$

Ex

$$F(s) = \frac{s^5+1}{(s+1)(s+2)(s+3)^3}$$

$$= 1 + \left(\frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+3)^3} + \frac{D}{(s+3)^2} + \frac{E}{(s+3)}\right)$$

$\Rightarrow$ 解 A, B, C, D, E.

$$A = \lim_{s \to (-1)} F \cdot (s+1)$$

$$B = \lim_{s \to (-2)} F \cdot (s+2)$$

$$C = \lim_{s \to (-3)} F \cdot (s+3)^3 = \lim_{s \to (-3)} \left[(s+3)^3 + A\frac{(s+3)^3}{s+1} + B\frac{(s+3)^3}{s+2} + C + D(s+3) + E(s+3)^2\right]$$

$$\begin{cases} D = \lim\limits_{S \to (-3)} \frac{1}{1!} \frac{d}{ds} [F \cdot (S+3)^3] \\ E = \lim\limits_{S \to (-3)} \frac{1}{2!} \frac{d}{ds^2} [F \cdot (S+3)^3] \end{cases}$$

for E only $\Rightarrow \lim\limits_{S \to \infty} S \cdot [F(S) - 1]$

$$= \lim_{S \to \infty} S \cdot \frac{s^5 + 1 - (s^5 + 12s^4 + 56s^3 + 126s^2 + 135s + 54)}{(S+1)(S+2)(S+3)^3}$$

$$= -12 = A + B + E$$

Summary

The $\mathcal{L}_-$ Laplace Transform **Inverse Laplace Transform**

Laplace transform of $f(t)$, denoted by $\mathcal{L}_-\{f(t)\} = F(s)$ $s = \sigma + j\omega$,

省

$$F(s) \triangleq \int_{0^-}^{\infty} f(t) e^{-st}\, dt. \quad (3.24)$$

$$f(t) = \frac{1}{2\pi j} \int_{\sigma_c - j\infty}^{\sigma_c + j\infty} F(s) e^{st}\, ds, \quad (3.25)$$

省 Basic properties of impulse (δ) function and step (u) function!

$$\begin{cases} \mathcal{L}^{-1}[1] = \delta(t) \\ \mathcal{L}^{-1}\left[\frac{1}{s}\right] = u(t) \end{cases} \text{背} \longleftrightarrow \begin{cases} \mathcal{L}^{-1}[s] = \delta'(t) \\ \mathcal{L}^{-1}[s^n] = \delta^{n}(t) \end{cases}$$

$$\mathcal{L}[t^a] = \int_0^\infty e^{-st} t^a \, dt$$

令 $st = y \Rightarrow dt = dy/s$

$$= \int_0^\infty \frac{y^a}{s^a} e^{-y} \frac{dy}{s} = \frac{1}{s^{a+1}} \int_0^\infty y^a e^{-y} dy$$

$$= \frac{1}{s^{a+1}}\left[-y^a e^{-y} \Big|_0^\infty + (a) \int_0^\infty y^{a-1} e^{-y} dy \right]$$

$$= \frac{a}{s^{a+1}} \int_0^\infty y^{a-1} e^{-y} dy$$

$$= \cdots = \frac{a!}{s^{a+1}} \int_0^\infty e^{-y} dy$$

$$= \frac{a!}{s^{a+1}} \left[-e^{-y} \Big|_0^\infty \right]$$

$$= \frac{a!}{s^{a+1}} [-0+1] = \frac{a!}{s^{a+1}}$$ 背

$$\Rightarrow \begin{cases} \mathcal{L}[\sin \omega t] = \frac{\omega}{s^2+\omega^2} & \text{背} \\ \mathcal{L}[\cos \omega t] = \frac{s}{s^2+\omega^2} & \text{背} \end{cases}$$

$$\mathcal{L}[e^{at}] = \int_0^\infty e^{at} e^{-st} dt = \frac{-e^{-(s-a)t}}{s-a}\Big|_0^\infty$$

$$= \frac{1}{s-a}$$ # 背

Feedback Control of System | Textbook pp 123~125 | **Prof. Cheng-Hsien Liu**

背 *Superposition* $\mathcal{L}\{\alpha f(t)\} = \alpha F(s)$

$$\mathcal{L}\{\alpha f_1(t) + \beta f_2(t)\} = \alpha F_1(s) + \beta F_2(s). \quad (3.28)$$

背 *Time Scaling* $$F_1(s) = \int_0^\infty f(at)e^{-st}\,dt = \frac{1}{|a|}F\left(\frac{s}{a}\right). \quad (3.31)$$

背 *Time Delay* $$F_1(s) = \int_0^\infty f(t-\lambda)e^{-st}\,dt = e^{-s\lambda}F(s). \quad (3.30)$$

背 *Shift in Frequency* $$F_1(s) = \int_0^\infty e^{-at}f(t)e^{-st}\,dt = F(s+a). \quad (3.32)$$

背 *Differentiation* $$\mathcal{L}\left\{\frac{df}{dt}\right\} = \int_{0^-}^\infty \left(\frac{df}{dt}\right)e^{-st}\,dt = -f(0^-) + sF(s). \quad (3.33)$$

$$\mathcal{L}\{\ddot{f}\} = s^2F(s) - sf(0^-) - \dot{f}(0^-)$$

$$\mathcal{L}\{f^m(t)\} = s^mF(s) - s^{m-1}f(0^-) - s^{m-2}\dot{f}(0^-) - \cdots - f^{(m-1)}(0^-), \quad (3.35)$$

背 *Integration* $$F_1(s) = \mathcal{L}\left\{\int_0^t f(\xi)\,d\xi\right\} = \frac{1}{s}F(s), \quad (3.36)$$

背 *Multiplication by Time* $$\mathcal{L}\{tf(t)\} = -\frac{d}{ds}F(s). \quad (3.40)$$

Homework

Proof: Textbook pp 827 (Example A.1)~832 (ExampleA.7)

Reference Hint

- $\frac{d}{dt}(e^{-st}) = s e^{-st}$
- $\int_0^\infty u' v\,dt = [uv]_0^\infty - \int_0^\infty u v'\,dt$
- $F(s) = \int_0^\infty f(t) e^{-st}\,dt \Rightarrow \alpha F(s) = \int_0^\infty \alpha f(t) e^{-st}\,dt$
- $\alpha F_1(s) + \beta F_2(s) = \int_0^\infty (\alpha f_1(t) + \beta f_2(t)) e^{-st}\,dt$
- $\mathcal{L}[y'(t)] = \int_0^\infty y'(t) e^{-st}\,dt = [y(t)e^{-st}]_0^\infty - \int_0^\infty y(t) e^{-st}\,dt$

$$= 0 - y(0) + s\int_0^\infty y(t) e^{-st}\,dt$$

$$= sY(s) - y(0)$$

The Final Value Theorem 終值定理

背

If all poles of $sY(s)$ are in the left half of the s-plane, then

$$\Rightarrow \lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s). \tag{3.46}$$

EXAMPLE 3.10

$$Y(s) = \frac{3(s+2)}{s(s^2+2s+10)}$$

$$y(\infty) = sY(s)|_{s=0} = \frac{3\cdot 2}{10} = 0.6.$$

A.1.3 The Initial Value Theorem 起始值定理

背

For any Laplace transform pair,

$$\Rightarrow \lim_{s\to\infty} sF(s) = f(0^+). \tag{A.27}$$

Proof: Textbook pp 836 (A.1.3)~838 (A.1.4)   Homework

Partial Expansion 整理

<1> 先化為真分式

<2> 分解後,一次分母項先解

<3> $\frac{A}{(S+a)^m} + \frac{B}{(S+a)^{m-1}} + \cdots \frac{Z}{(S+a)}$

<i> 消去 $(S+a)^m$ 後代 $(-a)$ 解 A

<ii> 代極限值 $S\,F(S)\big|_{S\to\infty}$ 解 Z

真分式

<iii> $B = \frac{1}{1!}\frac{d}{ds}[\cdots]$

$C = \frac{1}{2!}\frac{d^2}{ds^2}[\quad]$

<4> 分母二次式 $\frac{PS+Q}{(S+a)^2+b^2} = P\frac{(S+a)}{(S+a)^2+b^2} + \frac{m\cdot b}{(S+a)^2+b^2}$

$\mathcal{L}^{-1} \Rightarrow P e^{-at}\cos bt + m e^{-at}\sin bt$ #

Feedback Control of System — *Prof. Cheng-Hsien Liu*

The $\mathcal{L}$_ Laplace Transform — Inverse Laplace Transform

TABLE A.1 Properties of Laplace Transforms

Mentioned so far

Number	Laplace Transform	Time Function	Comment	
—	$F(s)$	$f(t)$	Transform pair	
1	$\alpha F_1(s) + \beta F_2(s)$	$\alpha f_1(t) + \beta f_2(t)$	Superposition	p. 47
2	$F(s)e^{-s\lambda}$	$f(t-\lambda)$	Time delay ($\lambda \geq 0$)	p. 47
3	$\frac{1}{\lvert a\rvert}F\left(\frac{s}{a}\right)$	$f(at)$	Time scaling	p. 47
4	$F(s+a)$	$e^{-at}f(t)$	Shift in frequency	p. 47
5	$s^mF(s) - s^{m-1}f(0) - s^{m-2}\dot{f}(0) - \cdots - f^{(m-1)}(0)$	$f^{(m)}(t)$	Differentiation	p. 47
6	$\frac{1}{s}F(s)$	$\int_0^t f(\zeta)\,d\zeta$	Integration	p. 47
7	$F_1(s)F_2(s)$	$f_1(t) * f_2(t)$	Convolution	p. 33~40
8	$\lim_{s\to\infty} sF(s)$	$f(0^+)$	Initial Value Theorem	p. 49
9	$\lim_{s\to 0} sF(s)$	$\lim_{t\to\infty} f(t)$	Final Value Theorem	p. 49
10	$\frac{1}{2\pi j}\int_{\sigma_c - j\infty}^{\sigma_c + j\infty} F_1(\zeta)F_2(s-\zeta)\,d\zeta$	$f_1(t)f_2(t)$	Time product	
11	$\frac{1}{2\pi}\int_{-j\infty}^{+j\infty} Y(-j\omega)U(j\omega)\,d\omega$	$\int_0^\infty y(t)u(t)\,dt$	Parseval's Theorem	
12	$-\frac{d}{ds}F(s)$	$tf(t)$	Multiplication by time	p. 47

Lecture 3- Review Laplace 51

51

Feedback Control of System — *Prof. Cheng-Hsien Liu*

TABLE A.2 Table of Laplace Transforms

Mentioned so far

Number	$F(s)$	$f(t), t \geq 0$
1	1	$\delta(t)$
2	$1/s$	$1(t)$
3	$1/s^2$	t
4	$2!/s^3$	t^2
5	$3!/s^4$	t^3
6	$m!/s^{m+1}$	t^m
7	$\frac{1}{s+a}$	e^{-at}
8	$\frac{1}{(s+a)^2}$	te^{-at}
9	$\frac{1}{(s+a)^3}$	$\frac{1}{2!}t^2e^{-at}$
10	$\frac{1}{(s+a)^m}$	$\frac{1}{(m-1)!}t^{m-1}e^{-at}$
11	$\frac{a}{s(s+a)}$	$1-e^{-at}$
12	$\frac{a}{s^2(s+a)}$	$\frac{1}{a}(at-1+e^{-at})$
13	$\frac{b-a}{(s+a)(s+b)}$	$e^{-at}-e^{-bt}$
14	$\frac{s}{(s+a)^2}$	$(1-at)e^{-at}$
15	$\frac{a^2}{s(s+a)^2}$	$1-e^{-at}(1+at)$
16	$\frac{(b-a)s}{(s+a)(s+b)}$	$be^{-bt}-ae^{-at}$
17	$\frac{a}{s^2+a^2}$	$\sin at$
18	$\frac{s}{s^2+a^2}$	$\cos at$
19	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-at}\cos bt$
20	$\frac{b}{(s+a)^2+b^2}$	$e^{-at}\sin bt$
21	$\frac{a^2+b^2}{s[(s+a)^2+b^2]}$	$1-e^{-at}\left(\cos bt + \frac{a}{b}\sin bt\right)$

Lecture 3- Review Laplace 52

52

EXAMPLE 3.14　*Forced Differential Equation Solution*

$\ddot{y}(t)+5\dot{y}(t)+4y(t) = 3$, where $y(0) = \alpha$, $\dot{y}(0) = \beta$

$$s^2Y(s) - s\alpha - \beta + 5[sY(s) - \alpha] + 4Y(s) = \frac{3}{s} \qquad Y(s) = \frac{s(s\alpha + \beta + 5\alpha) + 3}{s(s+1)(s+4)}$$

u(t)

$$Y(s) = \frac{\frac{3}{4}}{s} - \frac{\frac{3-\beta-4\alpha}{3}}{s+1} + \frac{\frac{3-4\alpha-4\beta}{12}}{s+4} \Rightarrow y(t) = \left(\frac{3}{4} + \frac{-3+\beta+4\alpha}{3}e^{-t} + \frac{3-4\alpha-4\beta}{12}e^{-4t}\right)1(t).$$

EXAMPLE 3.15　*Forced Equation Solution with Zero Initial Conditions*

$\ddot{y}(t)+5\dot{y}(t)+4y(t) = u(t), y(0) = 0, \dot{y}(0) = 0, u(t) = 2e^{-2t}1(t)$

u(t)

$$s^2Y(s) + 5sY(s) + 4Y(s) = \frac{2}{s+2} \qquad Y(s) = \frac{2}{(s+2)(s+1)(s+4)}$$

$$Y(s) = -\frac{1}{s+2} + \frac{\frac{2}{3}}{s+1} + \frac{\frac{1}{3}}{s+4} \Rightarrow y(t) = \left(-1e^{-2t} + \frac{2}{3}e^{-t} + \frac{1}{3}e^{-4t}\right)1(t)$$

Homework

$$\boxed{G(s) = \frac{5s+3}{(s+1)(s+2)(s+3)} = \frac{5s+3}{s^3+6s^2+11s+6}}$$

$$G(s) = \frac{K_{-1}}{s+1} + \frac{K_{-2}}{s+2} + \frac{K_{-3}}{s+3}$$

$$K_{-1} = [(s+1)G(s)]\Big|_{s=-1} = \frac{5(-1)+3}{(2-1)(3-1)} = -1$$

$$K_{-2} = [(s+2)G(s)]\Big|_{s=-2} = \frac{5(-2)+3}{(1-2)(3-2)} = 7$$

$$K_{-3} = [(s+3)G(s)]\Big|_{s=-3} = \frac{5(-3)+3}{(1-3)(2-3)} = -6$$

$$G(s) = \frac{-1}{s+1} + \frac{7}{s+2} - \frac{6}{s+3}$$

g(t)=

Homework

$$\boxed{G(s)=\frac{1}{s(s+1)^3(s+2)}=\frac{1}{s^5+5s^4+9s^3+7s^2+2s}}$$

$$G(s)=\frac{K_0}{s}+\frac{K_{-2}}{s+2}+\frac{A_1}{s+1}+\frac{A_2}{(s+1)^2}+\frac{A_3}{(s+1)^3}$$

$$K_0=\left[sG(s)\right]\Big|_{s=0}=\frac{1}{2}$$

$$K_{-2}=\left[(s+2)G(s)\right]\Big|_{s=-2}=\frac{1}{2}$$

$$G(s)=\frac{1}{2s}+\frac{1}{2(s+2)}-\frac{1}{s+1}-\frac{1}{(s+1)^3}$$

g(t)=

$$A_3=\left[(s+1)^3G(s)\right]\Big|_{s=-1}=-1$$

$$A_2=\frac{d}{ds}\left[(s+1)^3G(s)\right]\Big|_{s=-1}=\frac{d}{ds}\left[\frac{1}{s(s+2)}\right]\Big|_{s=-1}=0$$

$$A_1=\frac{1}{2!}\frac{d^2}{ds^2}\left[(s+1)^3G(s)\right]\Big|_{s=-1}=\frac{1}{2}\frac{d^2}{ds^2}\left[\frac{1}{s(s+2)}\right]\Big|_{s=-1}=-1$$

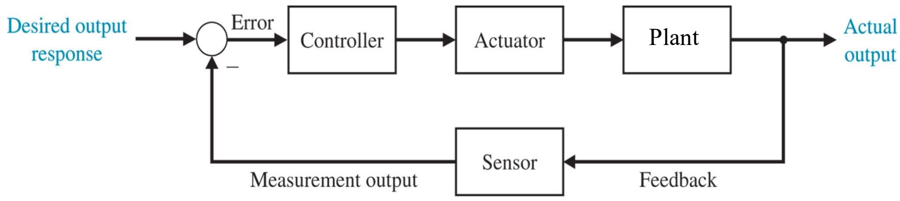
EXAMPLE 3.5 *Frequency Response*

延後

3.1.9 Linear System Analysis Using MATLAB

```
numb=[0 0 100];         % form numerator
denb=[1 10.1 101];      % form denominator
sysb=tf(numb,denb);     % define system by its numerator and denominator
t=0:0.01:5;             % form time vector
y=step(sysb,t)          % compute step response;
plot(t,y)               % plot step response
```

3 Dynamic Response



3.1 Review of Laplace Transforms

3.2.1 The Block Diagram

3.3 Effect of Pole Locations

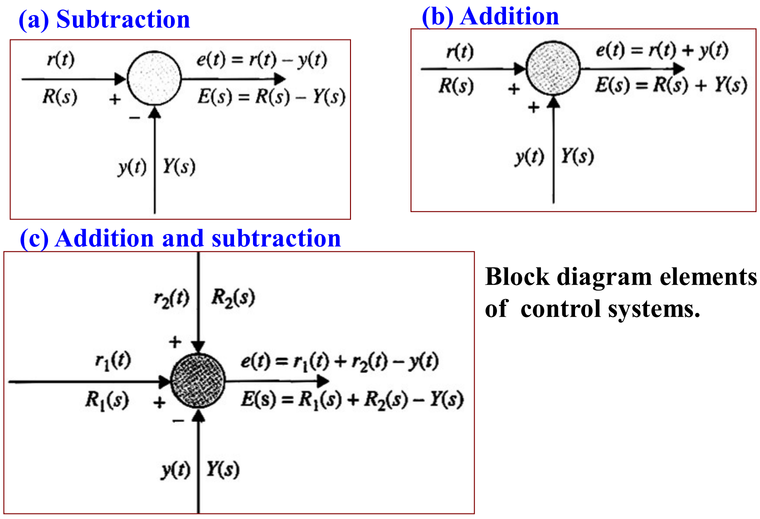
3.4 Time-Domain Specifications

3.5 Effects of Zeros and Additional Poles

3.6.3 Routh's Stability Criterion

1. To study block diagrams, their components, and their underlying mathematics.
2. To obtain transfer function of systems through block diagram manipulation and reduction.

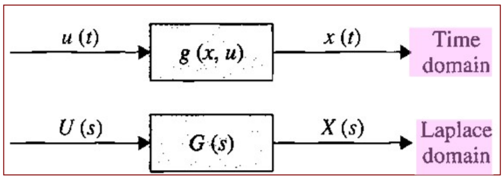
Block-Diagram Elements of Comparators



Block diagram elements of control systems.

Time & Laplace Domain Block Diagrams

Convolution



$$\begin{cases} X(s) = G(s)\,U(s) \\ G(s) = \dfrac{X(s)}{U(s)} \end{cases}$$

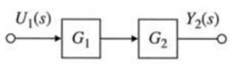
Convolution

$$x(t) = \int_0^\infty u(\tau)g(t-\tau)d\tau$$

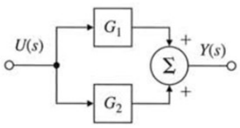
3.2 System Modeling Diagrams

3.2.1 The Block Diagram

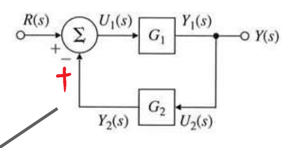
G_1 : gain 增益



$$\frac{Y_2(s)}{U_1(s)} = G_1G_2$$



$$\frac{Y(s)}{U(s)} = G_1 + G_2$$



$$\frac{Y(s)}{R(s)} = \frac{G_1}{1 + G_1G_2}$$

negative feedback

Feedback Control 迴授控制

$$\begin{cases} U_1(s) = R(s) \pm Y_2(s), \\ Y_2(s) = G_2(s)G_1(s)U_1(s) \\ Y_1(s) = G_1(s)U_1(s), \end{cases}$$

$u_1 = R \pm G_2 G_1 u_1$

$R = (1 \mp G_2 G_1) u_1$

$u_1 = \dfrac{1}{1 \pm G_1 G_2} R$

$$\Rightarrow Y_1(s) = \frac{G_1(s)}{1 \pm G_1(s)G_2(s)}R(s).$$

$\Rightarrow$ The gain of a single-loop negative feedback system is given by the forward gain divided by the sum of 1 plus the loop gain.

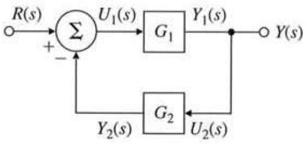
Negative feedback: $Y_1(s) = \dfrac{G_1(s)}{1 + G_1(s)G_2(s)}R(s).$

Positive feedback: $Y_1(s) = \dfrac{G_1(s)}{1 - G_1(s)G_2(s)}R(s).$

unity feedback system $G_2=1$

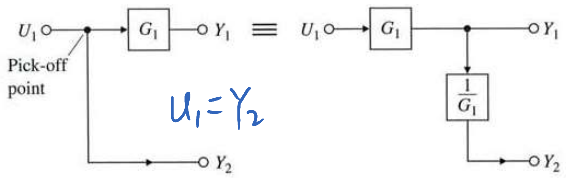
Rules for Block Diagram Manipulation

Rule 1:



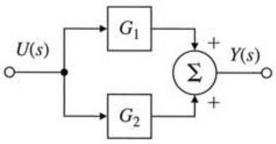
$$\frac{Y(s)}{R(s)} = \frac{G_1}{1 + G_1 G_2}$$

Rule 2:



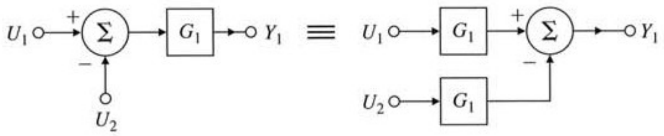
$U_1 = Y_2$

Rule 3:

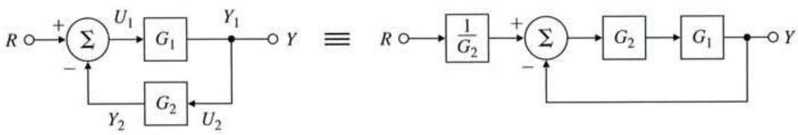


$$\frac{Y(s)}{U(s)} = G_1 + G_2$$

Rule 4:



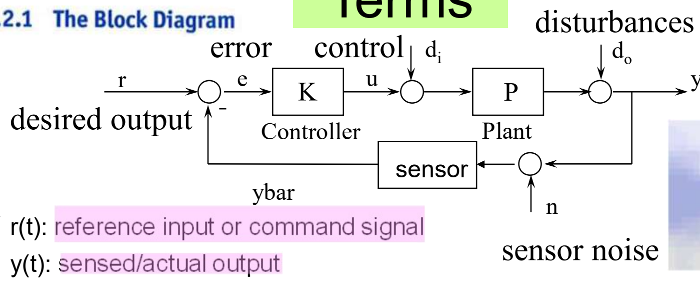
Rule 5:



$$\frac{Y}{R} = \frac{G_1}{1+G_1 G_2} \qquad \frac{Y}{R\frac{1}{G_2}} = \frac{G_1 G_2}{1+G_1 G_2}$$

Terms

3.2.1 The Block Diagram



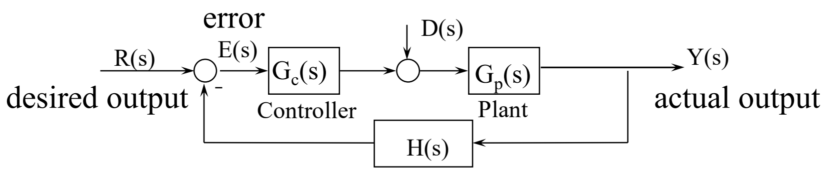
- r(t): reference input or command signal
- y(t): sensed/actual output
- e(t)= r(t) - ybar(t)
- u(t): control input
- Plant P
- ybar: estimate of y

Purpose: find compensator/controller
- error is "small"
- stability is "good"

- If r(t) is a constant -> "regulator" problem
- If r(t) =function of time -> "servo" or "tracking" problem

伺服 追踪

3.2.1 The Block Diagram

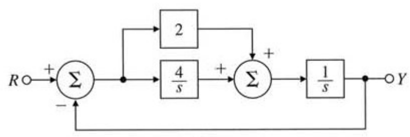


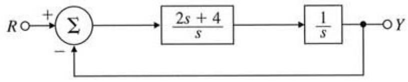
Sensor dynamics

(1) Assume d(t)=0 $\quad \dfrac{Y}{R} = \dfrac{G_c\, G_p}{1+G_c \cdot G_p \cdot H}$

(2) Assume r(t)=0 $\quad \dfrac{Y}{D} = \dfrac{G_p}{1+G_c \cdot G_p \cdot H}$

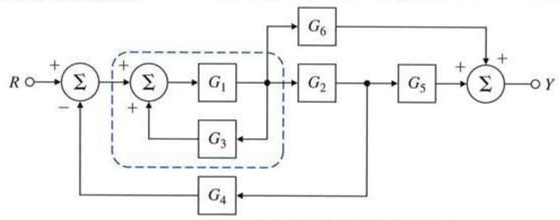
EXAMPLE 3.20 *Transfer Function from a Simple Block Diagram*

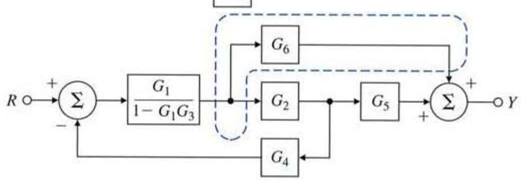


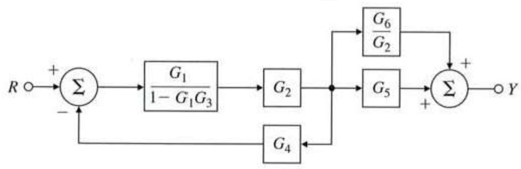


$$T(s) = \frac{Y(s)}{R(s)} = \frac{\frac{2s+4}{s^2}}{1 + \frac{2s+4}{s^2}} = \frac{2s+4}{s^2+2s+4}$$

EXAMPLE 3.21 *Transfer Function from the Block Diagram*



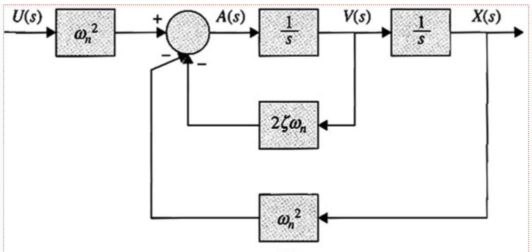




$$T(s) = \frac{Y(s)}{R(s)} = \frac{\frac{G_1G_2}{1-G_1G_3}}{1 + \frac{G_1G_2G_4}{1-G_1G_3}}\left(G_5 + \frac{G_6}{G_2}\right)$$

$$= \frac{G_1G_2G_5 + G_1G_6}{1 - G_1G_3 + G_1G_2G_4}.$$

Homework



(b)

$$\frac{X(s)}{U(s)} = \frac{\omega_n^2}{s^2 + 2\,\zeta\,\omega_n\,s + \omega_n^2}$$

Mason's Rule for Complicated Block Diagram

See Appendix W3.2.3 at www.fpe7e.com

3.2.2 Block Diagram Reduction Using MATLAB