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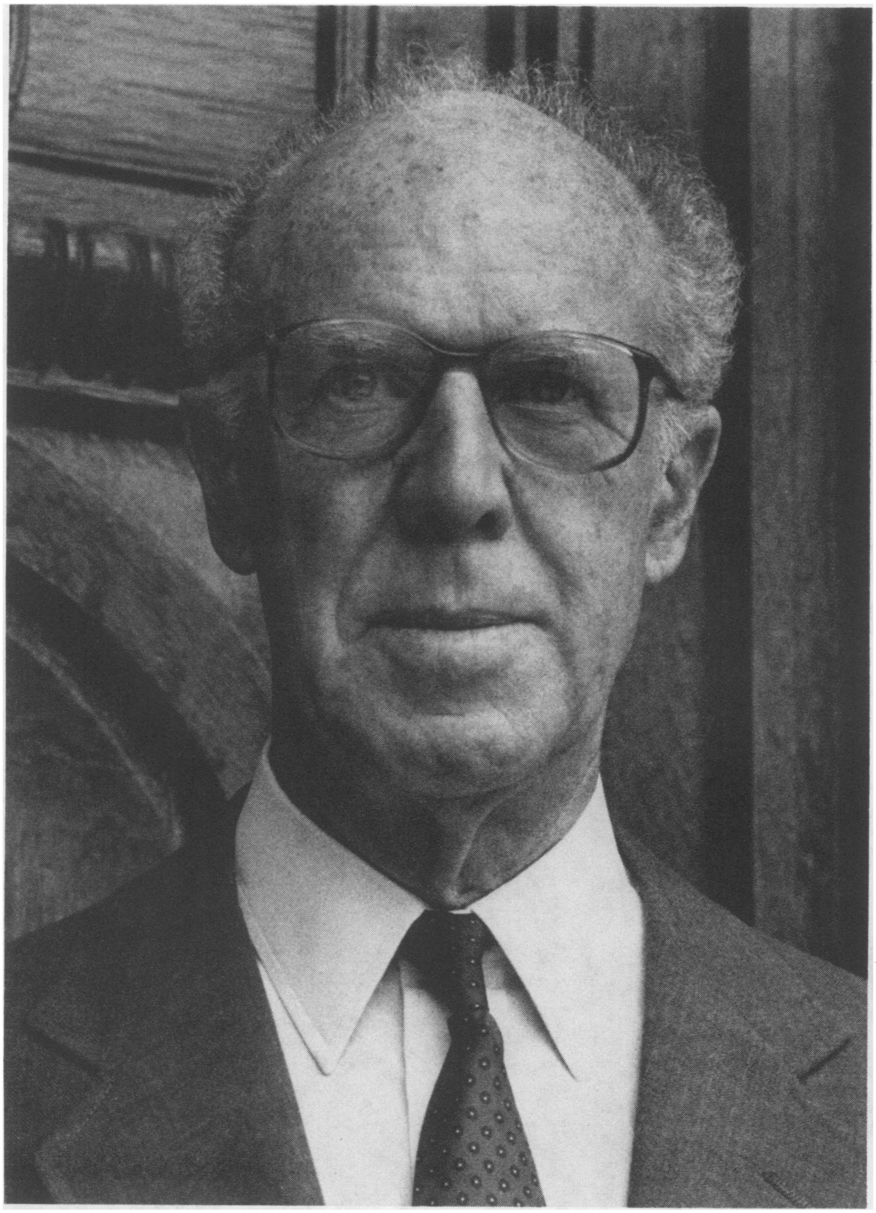

Gerard Debreu

The Mathematization of Economic Theory†

By GERARD DEBREU*

I

As the Second World War was drawing near its resolution, economic theory entered a phase of intensive mathematization that profoundly transformed our profession. In several of its main features that phase had no precedent, and it will have no successor. Assessing it requires a multidimensional analysis acknowledging the contributions to economics that were made, as well as the tensions among economists that were heightened.

The development of mathematical economics during the past half-century can be read in the total number of pages published each year by the leading periodicals in the field, an index that I will follow at first. From 1933, the date when they both started publication, to 1959, those periodicals were *Econometrica* and the *Review of Economic Studies*, and the index tells of the decline from a high point, above 700 pages in 1935 to the lowest point, below 400 pages in 1943–1944. But 1944 marked the beginning of a period of explosive growth in which *Econometrica* and the *Review of Economic Studies* were joined in 1960 by the *International Economic Review*, in 1969 by the *Journal of Economic Theory*, and in 1974 by the *Journal of Mathematical Economics*. In 1977, these five periodicals together published over 5,000 pages. During the period 1944–1977, the index more than doubled every nine years. By that measure, 1944 was a sharp turning point in the history of mathematical economics. It was also the year in which John von Neumann and Oskar Morgenstern published the *Theory of Games and Economic Behavior*.

While the professional journals in the field of mathematical economics grew at an unsustainably rapid rate, the *American Economic Review* underwent a radical change in identity. In 1940, less than 3 percent of the refereed pages of its 30th volume ventured to include rudimentary mathematical expressions. Fifty years later, nearly 40 percent of the refereed pages of the 80th volume display mathematics of a more elaborate type.

At the same time, the mathematization of economists proceeded at an even faster pace in the 13 American departments of economics labeled by a recent assessment of research-doctorate programs in the United States (Lyle V. Jones et al., 1982) as "distinguished" or "strong" according to the scholarly quality of their faculties. Every year the Fellows of the Econometric Society (ES) certify new members by election into their international guild, which increased in size from 46 in 1940 to 422 in 1990. For those 13 departments together, the proportion of ES Fellows among professors was less than 1 percent in 1940; it is now close to 50 percent. It equals or exceeds 50 percent for six of them, which were among those assessed as the eight strongest. So mathematized a faculty expects its students to have what it considers to be minimal mathematical proficiency, and knowledge of calculus and linear algebra is required, or forcefully recommended, for admission to all 13 graduate programs.

Several scholarly recognitions lay additional emphasis on the role that mathematical culture is now playing in our profession. Of the 152 members of the economics section of the American Academy of Arts and Sciences, 87 are Fellows of the Econometric Society; and of the 40 members of the economics section of the National Academy of

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Sciences of the United States, 34 are ES Fellows. From 1969 to 1990, 30 economics Nobel awards were made, and 25 of the laureates are, or were, ES Fellows. Since it was first presented to Paul Samuelson in 1947, the John Bates Clark medal of the American Economic Association has been given to 21 economists, of whom 20 are ES Fellows; and of the 26 living past presidents of our Association, 13 are ES Fellows.

One may wish that those counts had not been made. One may argue about points of their interpretation. But they belong in our common knowledge, and their thrust is unequivocal. They indicate how extensive the mathematization of economics and how deep the accompanying change of our field were over the past five decades.

The perception of the depth of that change is reinforced by a comparison of the levels of mathematics required in 1940 and in 1990 to follow the development of economic theory in every direction it was taking. Fifty years ago, basic undergraduate preparation in mathematics was almost always sufficient. Today, graduate training in mathematics is necessary. If, instead of being a follower, one wishes to be an active participant in that development along its most technical avenues, a high degree of mathematical professionalism is called for. Several faculty members of the 13 departments of economics mentioned previously were actually identified as mathematicians by their doctorates; four of them served as chairmen of those departments during the past 25 years. If still sharper focus brings out the intellectual leaders of that development, prominent among them is John von Neumann, one of the foremost mathematicians of his generation.

In that development process, mathematical economics was continuously redefined as new territories were included within its outward-moving frontier and as topics that were once at that frontier became standard parts of the graduate, if not of the undergraduate, economic-theory curriculum.

II

Before the contemporary period of the past five decades, theoretical physics had been an inaccessible ideal toward which economic theory sometimes strove. During that period, this striving became a powerful stimulus in the mathematization of economic theory.

The great theories of physics cover an immense range of phenomena with a supreme economy of expression. Of this, James Clerk Maxwell (1865) had given a notable example, as he described the electromagnetic field by means of eight equations at the time when mathematical economics was born and came of age in the middle of the 19th century. This extreme conciseness is made possible by the privileged relationship that developed over several centuries between physics and mathematics. In turn, the former presented the latter with open problems, or found to questions raised by physical theory ready-made answers discovered by mathematicians in their abstract universe. Sometimes the causal linkage of research done in each one of the two fields could not easily be unraveled; and, on occasion, the same scientist made inextricably intertwined contributions to both disciplines.

The benefits of that special relationship were large for both fields; but physics did not completely surrender to the embrace of mathematics and to its inherent compulsion toward logical rigor. The experimental results and the factual observations that are at the basis of physics, and which provide a constant check on its theoretical constructions, occasionally led its bold reasonings to violate knowingly the canons of mathematical deduction.

In these directions, economic theory could not follow the role model offered by physical theory. Next to the most sumptuous scientific tool of physics, the Superconducting Super Collider whose construction cost is estimated to be on the order of $\$10^{10}$ (David P. Hamilton, 1990; see also *Science*, 5 October 1990), the experiments of economics look excessively frugal. Being denied a sufficiently secure experimental base, economic theory has to adhere to the rules of logical discourse and must renounce the facility of internal inconsistency. A deductive structure that tolerates a contradiction does so under the penalty of being useless, since any statement can be derived flawlessly and immediately from that contradiction.

In its mathematical form, economic theory is open to an efficient scrutiny for logical errors. The rigor that has been reached as a consequence is in sharp contrast to the standards of reasoning that were accepted in the late 1930's. Few of the articles published then by *Econometrica* or by the *Review of Economic Studies* would pass the acid test of removing all their economic interpretations and letting their mathematical infrastructure stand on its own. The greater logical solidity of more recent analyses has contributed to the rapid contemporary construction of economic theory. It has enabled researchers to build on the work of their predecessors and to accelerate the cumulative process in which they are participating.

But a Grand Unified Theory will remain out of the reach of economics, which will keep appealing to a large collection of individual theories. Each one of them deals with a certain range of phenomena that it attempts to understand and to explain. When it acquires an axiomatic form, its explicit assumptions delimit its domain of applicability and make illegitimate overstepping of its boundary flagrant. Some of those theories take a comprehensive view of an economic system and bring insights into the solutions of several global problems. For instance, prices contribute to achieving an efficient use of resources, to equalizing supply and demand for commodities, and to preventing the formation of destabilizing coalitions. In every case, a theoretical explanation must be provided. The assumptions, which cannot be satisfied by all economic observations, are the present outcome of a continuing weakening process.

A global view of an economy that wants to take into account the large number of its commodities, the equally large number of its prices, the multitude of its agents, and their interactions requires a mathematical model. Economists have successfully constructed such a model because the central concept of the quantity of a commodity has a natural linear structure. The action of an agent can then be described by listing the quantity of its input or output for each commodity (opposite signs differentiating inputs from outputs). That list can be treated as the list of the coordinates of a point in the linear commodity space. Similarly, the price system of an economy can be treated as a point in the linear price space, dual of the commodity space, whose dimension is also the number of commodities.

In those two linear spaces, the stage was set for sometimes dazzling mathematical developments that began with the elements of differential calculus and linear algebra and that gradually called on an ever broader array of powerful techniques and fundamental results offered by mathematics. Thus, the three roles of prices given earlier as instances were illuminated by basic mathematical theorems: the first, the achievement of an efficient use of resources, by results of convex analysis; the second, the equalization of supply and demand for commodities, by results of fixed point theory; the third, the prevention of the formation of destabilizing coalitions, by results of the theory of integration and of nonstandard analysis. In those three cases, the lag between the date of a mathematical discovery and the date of its application to economic theory decreased over time. It was notably short for nonstandard analysis, founded at the beginning of the 1960's by Abraham Robinson[1] and applied to economics by Donald Brown and Abraham Robinson (1972).

The last, and most recently developed, of those three instances can be chosen, as can either of the other two, for a more detailed illustration. Competition is perfect when every agent's influence on the outcome of economic activity is insignificant. The influence of their totality on that outcome is, however, significant. It is to solve the problem of aggregating negligible quantities so as to obtain a nonnegligible sum that integration was invented. In this perspective, the application of integration theory to the study of economic competition is entirely natural. That application requires the set of agents to be large—larger than the set of integers. Treating the set of the agents of an economy as the rich collection of the points

[1]See the preface in Robinson (1966).

of an interval of real numbers has long been familiar in descriptions of economic data. It became familiar in economic theory as well after Robert J. Aumann (1964) showed that, in a pure exchange economy composed of insignificant agents, the formation of destabilizing coalitions is prevented if and only if all those agents base their decisions on a price system.

The concept of a convex set (i.e., a set containing the segment connecting any two of its points) had repeatedly been placed at the center of economic theory before 1964. It appeared in a new light with the introduction of integration theory in the study of economic competition: if one associates with every agent of an economy an arbitrary set in the commodity space and if one averages those individual sets over a collection of insignificant agents, then the resulting set is necessarily convex.[2] But explanations of the three functions of prices taken as examples can be made to rest on the convexity of sets derived by that averaging process. Convexity in the commodity space obtained by aggregation over a collection of insignificant agents is an insight that economic theory owes in its revealing clarity to integration theory.

An economist who experiences such an insight belongs to the group of applied mathematicians, whose values he espouses. Mathematics provides him with a language and a method that permit an effective study of economic systems of forbidding complexity; but it is a demanding master. It ceaselessly asks for weaker assumptions, for stronger conclusions, for greater generality. In taking a mathematical form, economic theory is driven to submit to those demands. The gains in generality that it has achieved as a result, in little more than a century, stand out when the first formulations of the theories of general equilibrium (Léon Walras, 1874–1877) and of the core of an economy (Francis Y. Edgeworth, 1881 pp. 34–8) are placed side by side with the recent treatments of those subjects to which *The New Palgrave* is an introduction and a bibliographical key (John Eatwell et al., 1987–1989). Walras's consumers and producers have been freed from many of their constraining characteristics; Edgeworth's universe of two consumers and two commodities has been vastly expanded.

Mathematics also dictates the imperative of simplicity. It relentlessly searches for short transparent proofs and for the theoretical frameworks in which they will be inserted. Participating in that pursuit, economic theory was sometimes drawn by drives toward greater generality and toward greater simplicity in the same direction, rather than in opposite directions. Cohort after cohort, students of consumer theory have learned about the concept of decreasing marginal rate of substitution for two commodities on an indifference curve and about its extension to the multicommodity case. Notably more general, and notably simpler, is the concept of convexity of the set of points preferred to a given point in the commodity space. Welfare economics presents another instance. One of its main theorems formulates precisely the principle enunciated by Adam Smith (1776). If all the agents of an economy are in equilibrium relative to a price system, then they utilize their collective resources optimally. The proof of that theorem (Kenneth J. Arrow, 1951) has become so simple that it can be given without mathematical symbols. It is, at the same time, of utmost generality; in relating two basic concepts of economic theory to each other, it uses no assumption.

In its attempts to attain its many objectives, economic theory was helped by greater abstraction. Preference theory supplies an example again. Significant research efforts were expended on solutions of the integrability problem. That problem can be bypassed altogether, and greater simplicity can be achieved by moving from the commodity space to the more abstract space of the pairs of its points. In this space, whose dimension is twice the number of commodities, the pairs of commodity points indifferent to each other are now assumed to form a smooth (hyper)surface. As another instance of the generality permitted by abstraction, consider the notion of a commod-

[2] On this direct consequence of a theorem of A. A. Lyapunov, see Karl Vind (1964).

ity, which can be treated as a primitive concept, with an unspecified interpretation, in an axiomatic economic theory. A newly discovered interpretation can then increase considerably the range of applicability of the theory without requiring any change in its structure. Thus, by making the transfer of a good or service between two agents contingent on the state of the world that will obtain, Arrow (1953) made possible the immediate extension of the economic theory of certainty to an economic theory of uncertainty by a simple reinterpretation of the concept of a commodity. The theory of financial markets has been influenced by that view of uncertainty, and their practice has not been unaffected. Finally, take the problem of existence of a general equilibrium, once considered to be one of the most abstract questions of economic theory. The solutions that were proposed in the early 1950's paved the way for the algorithms for the computation of equilibria of Herbert E. Scarf (1973) and for several of the developments of applied general equilibrium analysis (Scarf and John B. Shoven, 1984). In this case, abstraction in economic theory led to the study of fundamental problems of great generality, but also to a broad range of applications.

III

The list of advances that the mathematization of economic theory helped or permitted is already long; and in one aspect it may appear lengthy. Ceteris paribus, one cannot prefer less to more rigor, lesser to greater generality, or complexity to simplicity; but other things are not equal, and in the estimate of many members of our Association the cost of that mathematization sometimes outweighs its benefit. Two of its presidential addresses notably confronted that difficult analysis and stressed the price that economics paid for its increased use of mathematics. Wassily Leontief's (1971) observations were factual, and Robert A. Gordon's (1976) comments relevant when they were made in 1970 and in 1975. They still are today, for, in spite of their authorities, enhanced by the platform from which they were speaking, and in spite of the wide diffusion of their critiques, neither Leontief nor Gordon altered the course of the development they were assessing. In the past two decades, economic theory has been carried away further by a seemingly irresistible current that can be explained only partly by the intellectual successes of its mathematization.

Essential to an attempt at a fuller explanation are the values imprinted on an economist by his study of mathematics. When a theorist who has been so typed judges his scholarly work, those values do not play a silent role; they may play a decisive role. The very choice of the questions to which he tries to find answers is influenced by his mathematical background. Thus, the danger is ever present that the part of economics will become secondary, if not marginal, in that judgment.

The reward system of our profession reinforces the effects of that autocriticism. Decisions that shape the career of an economic theorist are made by his peers. Whether they are referees of a journal or of a research organization, members of an appointment or of a promotion committee, when they sit as judges in any capacity, their verdicts will not be independent of their own values. An economist who appears in their court rarely ignores his perception of those values. If he believes that they rate mathematical sophistication highly, and if he can prove that he is one of the sophisticates, the applause that he expects to receive will condition his performance.

The same effects are also amplified by the relentless pressure to publish exerted by his environment. There are indeed instances of extreme restraint in scientific publication, and some of them have become legend. The mathematical papers of Bernhard Riemann (1826–1866) take 506 pages in the volume that collected them (Riemann, 1876). The molecular structure of DNA was announced by James Watson and Francis Crick (1953) in a one-page article. But it is easier to explain those examples away than to follow them. The environment of a scholar demands papers, and the temptation to supply them without restraint may become overpowering to an economic theorist who has developed proficiency in his research style. The precocious development of that proficiency is a comparative advantage that a mathematical approach bestows on him.

The spread of mathematized economic theory was helped even by its esoteric character. Since its messages cannot be deciphered by economists who do not have the proper key, their evaluation is entrusted to those who have access to the code. But acceptance of their technical expertise also implies acceptance of their values. Our profession may take pride in its exceptional intellectual diversity, one of whose clearest symbols is an Ely lecture given by an economic historian at a session chaired by a mathematical economist. Yet that diversity is strained by the increasing impenetrability to the overwhelming majority of our Association of the work done by its most mathematical members.

IV

The bond that ties economists together in their study of a common subject has not been tested only by differences in methodologies. It has also been tried by differences in ideologies. In their endeavors to make their field into a science, economists must renounce a favorite mode of thinking—wishful thinking; they must be impartial spectators of a play in which they are the actors. While they attempt to keep that inhuman stance, they are pressed to give immediate answers to societal questions of immense complexity and thereby to abandon the exacting slowness of the step-by-step scientific approach. Divisions according to methodologies and ideologies, criticism from outside and from inside, and intellectual fashions that sweep our discipline make each one of its steady developments remarkable. The mathematization of economic theory was one of them for a century and a half. During the past five decades it became one of the prime movers in the transformation of our field. The extent of that mathematization has given rise to discordant assessments of its effects and to attempts to change its heading. The quality of assessments of the phase that economic theory underwent and the effectiveness of attempts to alter the course of its evolution will gain from a detailed analysis of the processes that led to its present state.

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[Footnotes]

[2] **Edgeworth-Allocations in an Exchange Economy with Many Traders**
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