個体經濟學
實習課
補充作業解答

1 $y = a_0 + a_1 X + a_2 X^2 + a_3 X^3$

(1) $\frac{dy}{dx} = a_1 + 2a_2 X + 3a_3 X^2$

$\frac{d^2y}{dx^2} = 2a_2 + 6a_3 X$

二階導數值受 a_2 和 a_3 的符号及大小影响.
故隨着 X 的不同而可能為正值或負值. 因此
無法判斷是否為凹函数或凸函数.
(参個体經濟學, 蔡攀龍. 張宏培 著 (*), 例 2.5 (Ⅲ), P. 57)

(2) $f(x, y) = x^{\frac{1}{2}} \cdot y^{\frac{1}{3}}$, $x>0$. $y>0$

$f_x = \frac{\partial f}{\partial x} = \frac{1}{2} x^{-\frac{1}{2}} y^{\frac{1}{3}}$, $f_{xx} = -\frac{1}{4} x^{-\frac{3}{2}} y^{\frac{1}{3}}$

$f_y = \frac{\partial f}{\partial y} = \frac{1}{3} x^{\frac{1}{2}} y^{-\frac{2}{3}}$, $f_{yy} = -\frac{2}{9} x^{\frac{1}{2}} y^{-\frac{5}{3}}$

$f_{xy} = f_{yx} = \frac{1}{6} x^{-\frac{1}{2}} y^{-\frac{2}{3}}$

$$|H| = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} -\frac{1}{4} x^{-\frac{3}{2}} y^{\frac{1}{3}} & \frac{1}{6} x^{-\frac{1}{2}} y^{-\frac{2}{3}} \\ \frac{1}{6} x^{-\frac{1}{2}} y^{-\frac{2}{3}} & -\frac{2}{9} x^{\frac{1}{2}} y^{-\frac{5}{3}} \end{vmatrix}$$

$$= \frac{1}{18} x^{-1} y^{-\frac{4}{3}} - \frac{1}{36} x^{-1} y^{-\frac{4}{3}} = \frac{1}{36} x^{-1} y^{-\frac{4}{3}}$$

$\because x>0$. $y>0$ $\therefore |H| > 0$

又 $f_{xx} < 0$

$\therefore f(x, y)$ 為一凹函数

(参 (*), 例 2.7 (Ⅱ), P. 69)

2. $F(x,y,z) = 3xy + 2yz^2 + x^2yz - 10 = 0$

$$F_x = \frac{\partial F}{\partial x} = 3y + 2xyz$$

$$F_y = 3x + 2z^2 + x^2z$$

$$F_z = 4yz + x^2y$$

偏導數

全微分：

$$dF(x,y,z) = F_x\,dx + F_y\,dy + F_z\,dz$$

$$= (3y+2xyz)\,dx + (3x+2z^2+x^2z)\,dy + (4yz+x^2y)\,dz = 0$$

其他偏導數

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3y+2xyz}{4yz+x^2y} = -\frac{3+2xz}{4z+x^2}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{3x+2z^2+x^2z}{4yz+x^2y}$$

$$\frac{\partial y}{\partial x} = -\frac{F_x}{F_y} = -\frac{3y+2xyz}{3x+2z^2+x^2z}$$

(參(*), 例 2.6 (Ⅲ), P.64)

3. $y = \ln x^2 - 4x,\ x > 0$

$$f'(x) = \frac{\partial y}{\partial x} = \frac{2x}{x^2} - 4 = \frac{2}{x} - 4$$

F.O.C (一階條件)

$f'(x^*) = 0 \quad \frac{2}{x^*} - 4 = 0 \quad \therefore x^* = \frac{1}{2}$

S.O.C (二階條件) $f''(x) = -\frac{2}{x^2}$

$$f''(x^*) = -\frac{2}{(\frac{1}{2})^2} = -8 < 0$$

$\therefore$ 由二階條件可知 $x^* = \frac{1}{2}$ 為極大值 且函數值

$y^* = \ln(\frac{1}{2})^2 - 4\cdot\frac{1}{2} = -2\ln 2 - 2 = -2(\ln 2 - 1)$ ☆ (參(*), 例 2.8 (Ⅲ). P.84)

4. $f(x, y) = x^4 + x^2 + 3y^2 - 6xy$

F.O.C $\begin{cases} f_x(x^*, y^*) = 4x^{*3} + 2x^* - 6y^* = 0 & —① \\ f_y(x^*, y^*) = 6y^* - 6x^* = 0 & —② \end{cases}$

由②知 $x^* = y^*$ 代入①：

$4x^{*3} + 2x^* - 6x^* = 0 \Rightarrow 4x^{*3} - 4x^* = 0$

$\Rightarrow 4x^*(x^{*2} - 1) = 0 \quad \therefore x^* = 0$ or 1 or -1

$\Rightarrow y^* = 0$ or 1 or -1

臨界点：$(x^*, y^*) = (0, 0)$, or $(1, 1)$ or $(-1, -1)$

又 $f_{xx} = 12x^2 + 2$, $f_{yy} = 6$. $f_{xy} = f_{yx} = -6$

∴ ① 臨界点 $= (0, 0)$ 時：

$f_{xx}(0, 0) = 2 > 0$

$|H| = \begin{vmatrix} 2 & -6 \\ -6 & 6 \end{vmatrix} = 12 - 36 = -24 < 0$

∴ $(0, 0)$ 為一馬鞍点

② 臨界点 $= (1, 1)$ 時：

$f_{xx}(1, 1) = 14 > 0$, $|H| = \begin{vmatrix} 14 & -6 \\ -6 & 6 \end{vmatrix} = 48 > 0$

∴ $(1, 1)$ 為 (局部) 極小点

③ 臨界点為 $(-1, -1)$ 時：

$f_{xx}(-1, -1) = 14 > 0$, $|H| = 48 > 0$

∴ $(-1, -1)$ 為 (局部) 極小点

(參 (*)、例 2.9 Ⅲ, P.88)

5.

(1) $\text{Max } f(x,y) = x^{0.25} y^{0.75}$

$\text{s.t } 100 - 2x - 4y = 0$

$\mathcal{L} = x^{0.25} y^{0.75} + \lambda(100 - 2x - 4y)$

$\frac{\partial \mathcal{L}}{\partial x} = 0 \Rightarrow 0.25 x^{-0.75} y^{0.75} = 2\lambda$ —①

$\frac{\partial \mathcal{L}}{\partial y} = 0 \Rightarrow 0.75 x^{0.25} y^{-0.25} = 4\lambda$ —②

$\frac{\partial \mathcal{L}}{\partial \lambda} = 0 \Rightarrow 100 - 2x - 4y = 0$ —③

$\frac{① \times 2}{②}$: $\frac{0.5}{0.75} \frac{x^{-0.75} y^{0.75}}{x^{0.25} y^{-0.25}} = 1 \Rightarrow \frac{2y}{3x} = 1 \Rightarrow y = \frac{3}{2}x$

代入③ : $100 - 2x - 4 \times \frac{3}{2}x = 0$

$100 - 2x - 6x = 0$ $\therefore x^* = 12.5$, $y^* = 18.75$

(參 hoy 經數 P. 589, Ex. 13.1)

(2) $\text{Min } f(x,y) = x + y$

$\text{s.t } 1 - x^{\frac{1}{2}} - y = 0$

$\mathcal{L} = x + y + \lambda(1 - x^{\frac{1}{2}} - y)$

$\frac{\partial \mathcal{L}}{\partial x} = 0$ $1 = \frac{1}{2}\lambda x^{-\frac{1}{2}}$ —①

$\frac{\partial \mathcal{L}}{\partial y} = 0$ $1 = \lambda$ $\therefore \lambda^* = 1$ —②

$\frac{\partial \mathcal{L}}{\partial \lambda} = 0$ $1 - x^{\frac{1}{2}} - y = 0$ —③

②代入① 可得 $x^* = \frac{1}{4}$ 再代入③ 可得 $y^* = \frac{1}{2}$

(參 hoy 經數, P. 600. Ex. 13.4)