# Schrödinger's Equation

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2/21/2001, 10/15/2006, 1/10/2010

Max Planck Formula (1900)：A natural fact.

$$E = h\upsilon = \hbar\omega$$

$E$ : energy
$\upsilon$ : frequency
$\omega$ : $2\pi\upsilon$
$h$ : Planck's constant (6.63 × $10^{-34}$ Js)
$\hbar = h/2\pi$ (1.05 × $10^{-34}$ Js)

Particle–Wave Duality Relation:

$$p\lambda = h, \quad p = \hbar k$$

$p$: momentum
$\lambda$: wavelength
$k$: wave number

Collapse of Determinism (Probability)

Polarized Wave : Electric or magnetic wave.

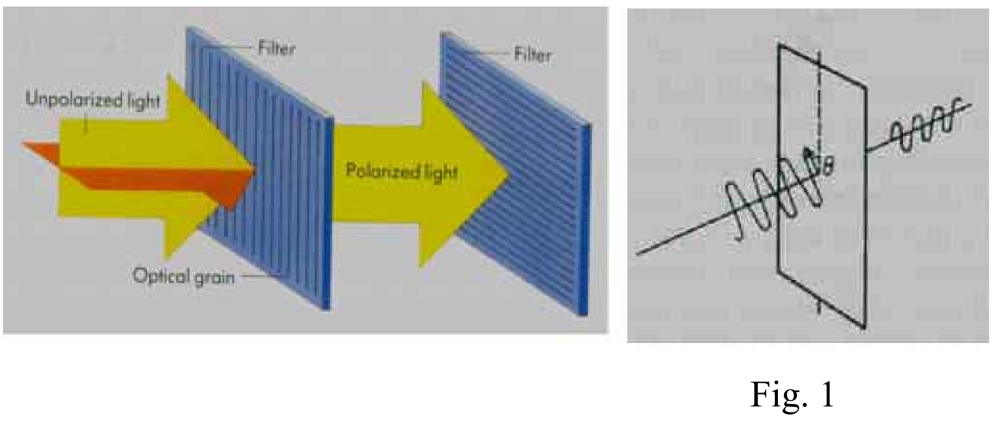


Fig. 1

Fig.1: A plane-polarized light encounters an obliquely oriented polarizer only a fraction $\cos^2\theta$ of the intensity is transmitted.

$\theta = 0^\circ$: All the light is transmitted
$\theta = 45^\circ$: Half gets through
$\theta = 90^\circ$: No transmission

Weird Feature: Suppose the intensity of the light is reduced so that only one photon at a time arrives at the polarizer.
Transmission Probability = $\cos^2\theta$

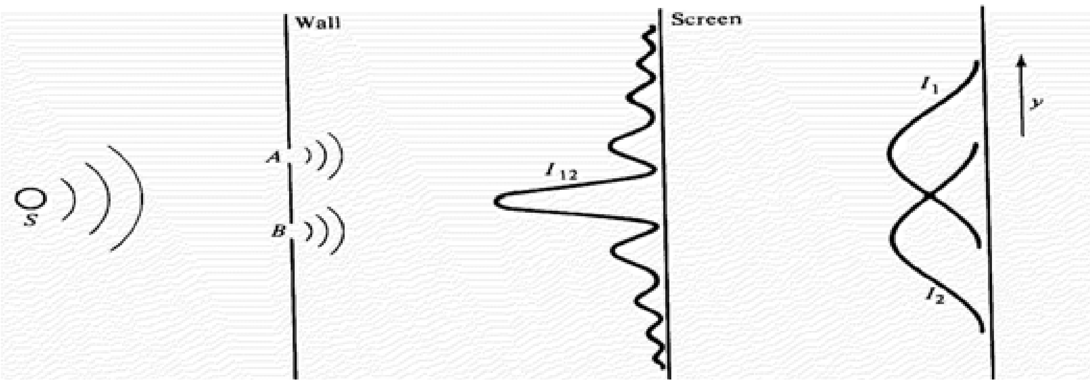


Fig. 2: Wave double-slit experiment. Amplitudes add.

$\psi = \psi(\vec{r}, t)$ : wave function

$\psi = \psi(\vec{r}, t) = |\psi| e^{i\alpha}$  α : phase

$I = |\psi|^2$

Superposition:

$$\psi = \psi_1 + \psi_2$$

$$I = |\psi|^2 = |\psi_1 + \psi_2|^2 = \overline{(\psi_1 + \psi_2)}(\psi_1 + \psi_2)$$

$$= |\psi_1|^2 + |\psi_2|^2 + |\psi_1||\psi_2|[e^{i(\alpha_1 - \alpha_2)} + e^{-i(\alpha_1 - \alpha_2)}]$$

$$= I_1 + I_2 + \underline{2\sqrt{I_1 + I_2}\,Cos(\alpha_1 - \alpha_2)}$$

⇩

Interference

The wave of each individual particle passes through both slits but the particle passes through only one.
Louis de Broglie (1924): Matter waves.

Matter : protons, neutrons, mesons, atoms, molecules $\cdots$

Peter Debye (1926): If matter is a wave, there should be a wave equation to describe a matter wave.

A Traveling Sine Wave: $\psi(x,t) = A\sin\frac{2\pi}{\lambda}(x - vt)$

A Matter Wave:

$$\psi(x,t) = A\exp\left(\frac{2\pi i}{\lambda}(x - ct)\right) = A\exp\left(\frac{2\pi i}{\lambda}(x - \lambda\upsilon t)\right)$$

$$= A\exp\left(2\pi i(\frac{p}{h}x - \frac{E}{h}t)\right) = A\exp\left(\frac{i}{\hbar}(px - Et)\right)$$

$$\frac{\partial\psi}{\partial x} = \frac{ip}{\hbar}\psi \qquad \Rightarrow \qquad p\psi = -i\hbar\frac{\partial\psi}{\partial x}$$

$$\frac{\partial^2\psi}{\partial x^2} = -\frac{p^2}{\hbar^2}\psi \qquad \Rightarrow \qquad p^2\psi = -\hbar\frac{\partial^2\psi}{\partial x^2}$$

$$\frac{\partial\psi}{\partial t} = -\frac{iE}{\hbar}\psi \qquad \Rightarrow \qquad E\psi = -i\hbar\frac{\partial\psi}{\partial t}$$

$$E = \frac{1}{2}mv^2 + V = \frac{p^2}{2m} + V = K + V$$

= Kinetic Energy + Potential Energy

Erwin Schrödinger (1926): Schrödinger's Equation

1D: $$i\hbar\frac{\partial\psi(x,t)}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} + V(x,t)\psi(x,t)$$

2D, 3D: $$\boxed{i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi}$$

- Key equation of the quantum theory.
- Must be accepted as a fundamental postulate.

$|\psi(\vec{r},t)|^2 = \overline{\psi}\psi$ is the probability density for a particle to be located at point $\vec{r}$ at time *t*.

$|\psi|^2 d\vec{r}$ is the probability it will be in the infinitesimal volume $d\vec{r}$ at time *t*.

$\psi$ is not an observable quantity $\Rightarrow$ the phase of $\psi$ is arbitrary (changing) without changing the observable quantity $|\psi|^2$.

Normalization Condition: $\int_{\Re^3} |\psi(\vec{r},t)|^2 d\vec{r} = 1$