Exam 1 Numerical Analysis Fall 2014

Oct. 16, 2014

(Total of 100 points.)

1. 1D Poisson's problem states that given a function $f(x)$ and two constants g_D and g_N, find the solution $u(x)$ satisfying

$$
\begin{aligned}
-u''(x) &= f(x) \text{ for all } x \in (0,1) && (1)\\
u'(0) &= g_N && (2)\\
u(1) &= g_D && (3)
\end{aligned}
$$

(I) Let $g_N = 0$ and $g_D = 0$. The finite difference method for solving this problem consists of the following steps:

Step 1. (5%) Domain Discretization. Let's use a uniform partition on the domain with the number of subintervals being $N-1$. Sketch the partition with proper notations indicating the grid nodes and the mesh size.

Step 2. (10%) For any grid point x_i, write down the central difference approximation formulas of

$$
\begin{aligned}
u'(x_i) &\approx \\
u''(x_i) &\approx
\end{aligned}
$$

Step 3. (A) (10%) Write down the system of difference equations with the unknown scalars U_i after applying the central difference formulas to (1).
(B) (5%) What is the difference between U_i and $u_i \equiv u(x_i)$?
(C) (5%) How do you write the approximation equations for (2) and (3)?
(D) (10%) Combine (A) and (C) to a linear system $A\overrightarrow{U} = \overrightarrow{b}$ and write clearly the definition of the symbols A, $\overrightarrow{U}$, and $\overrightarrow{b}$.

(II) (A) (10%) Let $f(x) = 2$, $g_N = 1$, and $g_D = 0$. Find the (unique) solution $u(x)$ of this problem.

(B) (5%) Write down the system $A\overrightarrow{U} = \overrightarrow{b}$ for the problem of (A) with only two subintervals, i.e., $N = 3$ (you should write element values of A and $\overrightarrow{b}$).
(C) (5%) Will you get a unique solution if (3) is replaced by $u'(1) = 0$?

2. (10%) Consider the linear system $A\overrightarrow{x} = \overrightarrow{b}$ where $A = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 3 & 0 \\ 1 & 0 & -2 \end{bmatrix}$, $\overrightarrow{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$. Solve the system by Jacobi's Method (JM) to the first 2 iterates and fill your answers in the following table. Use the rational number such as $\frac{1}{3}$ do NOT use 0.333.

Table. JM Iteration

k	0	1	2
$x_1^{(k)}$	0		
$x_2^{(k)}$	0		
$x_3^{(k)}$	0		

3. (15%) Write an algorithm for JM.

4. (10%) For solving the linear system $A\overrightarrow{x} = \overrightarrow{b}$, the JM in matrix form is expressed as

$$\overrightarrow{x}^{(k)} = -D^{-1}(L+U)\overrightarrow{x}^{(k-1)} + D^{-1}\overrightarrow{b}, \qquad k = 1, 2, 3, \cdots \tag{4}$$

where $A = D + L + U$ is an NxN matrix, $\overrightarrow{x}$ is the exact (unknown) vector, and $\overrightarrow{x}^{(k)}$ is an approximate vector of $\overrightarrow{x}$. Here D, L, and U are the diagonal, the strictly lower-triangular, and the strictly upper-triangular parts of A, respectively. The JM written in component form is

$$x_i^{(k)} = (b_i - \sum_{\substack{j=1 \\ j \neq i}}^{N} a_{ij} x_j^{(k-1)})/a_{ii} \tag{5}$$

Prove from (4) to (5).