Lecture 4
Gauss-Seidel Method (GS)

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We consider again Example 3.1

$$\begin{bmatrix} 3 & 2 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{4.1}$$

Jacobi's Method:

$$\begin{cases} x_1^{(1)} = (b_1 - a_{12}x_2^{(0)})/a_{11} \\ x_2^{(1)} = (b_2 - a_{21}x_1^{(0)})/a_{22} \end{cases} \tag{4.2}$$

For the GS method, we replace $x_1^{(0)} = 0$ by $x_1^{(1)}$, the most recent value for x_1,

$$\therefore \quad x_2^{(1)} = (b_2 - a_{21}x_1^{(1)})/a_{22} \tag{4.3}$$

$$\Rightarrow \begin{cases} x_1^{(1)} = (b_1 - a_{12}x_2^{(0)})/a_{11} = (5 - 2 \cdot 0)/3 = \frac{5}{3} \\ x_2^{(2)} = (b_1 - a_{21}x_1^{(1)})/a_{21} = (6 - 1 \cdot \frac{5}{3})/5 = \frac{13}{15} \end{cases} \tag{4.4}$$

In general, if we proceed as with the Jacobi method, but now assume that the equation are examined one at a time in sequence, and that previously computed results are used as soon as they are available, we obtain the Gauss-Seidel method

$$x_i^{(k)} = (b_i - \sum_{j\,<\,i} a_{ij}x_j^{(k)} - \sum_{j\,>\,i} a_{ij}x_j^{(k-1)})/a_{ii} \tag{4.5}$$

Table 4.1. GS Iteration

k	0	1	2	3	$\cdots$	∞
$x_1^{(k)}$	0	$\frac{5}{3}$	$\frac{49}{45}$	$\cdots$	$\cdots$	1
$x_2^{(k)}$	0	$\frac{13}{15}$	$\frac{221}{225}$	$\cdots$	$\cdots$	1

Considering the matrix form

$$A\overrightarrow{x} = \begin{bmatrix} a_{11} & a_{12} & \cdots & \cdots & a_{1N} \\ a_{21} & a_{22} & \cdots & \cdots & a_{2N} \\ \vdots & & \ddots & & \vdots \\ \vdots & & & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & \cdots & a_{NN} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ \vdots \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ \vdots \\ b_N \end{bmatrix} \tag{4.6}$$

for the case of $i = 3$, (4.5) can be written as

$$\begin{bmatrix} \times & 0 & \cdots & \cdots & 0 \\ 0 & \times & 0 & \cdots & \vdots \\ \vdots & 0 & a_{33} & 0 & \vdots \\ \vdots & \vdots & 0 & \times & 0 \\ 0 & \cdots & \cdots & 0 & \times \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ \vdots \\ x_3^{(k)} \\ x_4^{(k-1)} \\ \vdots \\ x_N^{(k-1)} \end{bmatrix} + \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \times & 0 & 0 & \cdots & \vdots \\ a_{31} & a_{32} & 0 & 0 & \vdots \\ \vdots & \vdots & \times & 0 & 0 \\ \times & \cdots & \cdots & \times & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ \vdots \\ x_3^{(k)} \\ x_4^{(k-1)} \\ \vdots \\ x_N^{(k-1)} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \times & \cdots & \cdots & \cdots & \times \\ 0 & 0 & \times & \cdots & \cdots & \cdots \\ \vdots & 0 & 0 & a_{34} & a_{35} & \cdots \\ \vdots & \cdots & 0 & 0 & \times & \cdots \\ \vdots & \cdots & \cdots & 0 & 0 & \times \\ 0 & \cdots & \cdots & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ \vdots \\ x_3^{(k)} \\ x_4^{(k-1)} \\ \vdots \\ x_N^{(k-1)} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_3 \\ \vdots \\ \vdots \\ b_N \end{bmatrix} \tag{4.7}$$

Now using the decomposition $A = D+L+U$, this equation can be viewed as a particular case ($i = 3$) of the following general form

$$D \begin{bmatrix} x_1^{(k)} \\ \vdots \\ x_3^{(k)} \\ x_4^{(k)} \\ \vdots \\ x_N^{(k)} \end{bmatrix} + L \begin{bmatrix} x_1^{(k)} \\ \vdots \\ x_3^{(k)} \\ x_4^{(k)} \\ \vdots \\ x_N^{(k)} \end{bmatrix} + U \begin{bmatrix} x_1^{(k-1)} \\ \vdots \\ x_3^{(k-1)} \\ x_4^{(k-1)} \\ \vdots \\ x_N^{(k-1)} \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_3 \\ \vdots \\ \vdots \\ b_N \end{bmatrix} \tag{4.8}$$

or in more compact form

$$D\overrightarrow{x}^{(k)} + L\overrightarrow{x}^{(k)} + U\overrightarrow{x}^{(k-1)} = \overrightarrow{b} \tag{4.9}$$

which is equivalent to

$$\overrightarrow{x}^{(k)} = (D+L)^{-1}(\overrightarrow{b} - U\overrightarrow{x}^{(k-1)}) \tag{4.10}$$

Recall that any iterative method for (4.6) can be expressed by

$$\overrightarrow{x}^{(k)} = B\overrightarrow{x}^{(k-1)} + \overrightarrow{c} \tag{4.11}$$

In summary, the stationary iterative methods so far are

$$\begin{array}{lll} \text{JM:} & B = -D^{-1}(L+U), & \overrightarrow{c} = D^{-1}\overrightarrow{b} \\ \text{GS:} & B = -(D+L)^{-1}U, & \overrightarrow{c} = (D+L)^{-1}\overrightarrow{b} \end{array} \tag{4.12}$$

Algorithm GS: Gauss-Seidel Method Solve $A\vec{x} = \vec{b}$.

Input: N: Number of unknowns and equations; a_{ij}: Entries of A, $i, j = 1 \cdots N$; b_i: Entries of $\vec{b}$, $i = 1 \cdots N$; TOL: Error Tolerance.

Output: $x_i^{(k)}$: Entries of $\vec{x}^{(k)}$(approximate solution) or Error Message.

Step 1. Choose an initial guess $\vec{x}^{(0)}$to the solution $\vec{x}$.

Step 2. For $k = 1, 2, 3 \cdots, k_{\max}$

Step 3. For $i = 1, 2, \cdots, N$

Step 4. $\sigma = 0$

Step 5. For $j = 1, 2, \cdots, i-1$

Step 6. $\sigma = \sigma + a_{ij}x_j^{(k)}$

Step 7. End j loop

Step 8. For $j = i+1, ..., N$

Step 9. $\sigma = \sigma + a_{ij}x_j^{(k-1)}$

Step 10. End j loop

Step 11. $x_i^{(k)} = (b_i - \sigma)/a_{ii}$

Step 12. End i loop

Step 13. If $||\vec{r}^{(k)}||_\infty < \text{TOL} = 10^{-6}$ then Stop otherwise Set $\vec{x}^{(k-1)} = \vec{x}^{(k)}$ and Go To Step 2.

Step 14. End k loop

Step 15. Error: Not convergent with the max number of iterations $k_{\max}$ and TOL.

$\vec{x}^{(k)} \approx \vec{x}$: An approximate solution.
$\vec{r}^{(k)} = A\vec{x}^{(k)} - \vec{b}$: Residual vector.
$E^{\vec{x}} = ||\vec{r}^{(k)}||_\infty := \max_{1 \le i \le N} |r_i^{(k)}|$: Residual error in maximum norm.

Project 4.1. Consider the 1D Poisson Problem (1.1) (with $f(x) = 2$, $g_D = 0$, and $g_N = 0$) and implement the methods FDM and GS.

Input: N, A, $\vec{b}$, $k_{\max}$, TOL (write the input in the program).

Output:

N	k	$E^{\vec{x}}$	E^u	α
5				
9				
17				
33				
65				
129				