Lecture 8
Finite Element Method (FEM) for 1D Poisson's Equation

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1D Poisson's Problem (**Strong** or **Classical** Problem): Given a function $f(x) \in C^0(\Omega)$ and two constants g_D and g_N, find the solution $u(x) \in$? satisfying

$$
\begin{aligned}
-u'' &= f(x) \quad \forall\, x \in (0,1) = \Omega : \text{Open set} && (8.1a)\\
u(0) &= g_D \quad \text{on } \partial\Omega_D = \{x = 0\} && (8.1b)\\
u'(1) &= g_N \quad \text{on } \partial\Omega_N = \{x = 1\} && (8.1c)
\end{aligned}
$$

$\partial\Omega$ is the boundary of Ω, i.e., $\partial\Omega = \{0, 1\}$, $\overline{\Omega} = \Omega \cup \partial\Omega$

Step1. Weak (or Variational) Formulation

(1D) **Fundamental Theorem of Calculus (Integration by Parts):**

$$\int_a^b (uv)'\, dx = uv|_a^b \Rightarrow \int_a^b u'v dx = uv|_a^b - \int_a^b uv' dx \qquad (8.2)$$

(2D, 3D) **Gauss's Divergence Theorem:** Let B be an open bounded domain in $\mathbf{R}^n$, $n = 2$ or 3 with a piecewise smooth boundary ∂B. Let $\mathbf{u}$ be a differentiable vector function in B. Then

$$
\begin{aligned}
\text{(2D)} \qquad \iint_B \text{div } \mathbf{u} d\mathbf{r} &= \iint_B \nabla \cdot \mathbf{u} d\mathbf{r} = \int_{\partial B} \mathbf{u} \cdot \mathbf{n} dS \qquad (8.3)\\
\text{Area Integral (2D)} &= \text{Line Integral (1D)}\\
\text{Total Mass Change in } B &= \text{Mass Flows across } \partial B,
\end{aligned}
$$

$$
\begin{aligned}
\text{(3D)} \qquad \iiint_B \text{div } \mathbf{u} d\mathbf{r} &= \iiint_B \nabla \cdot \mathbf{u} d\mathbf{r} = \iint_{\partial B} \mathbf{u} \cdot \mathbf{n} dS \qquad (8.4)\\
\text{Volume Integral (3D)} &= \text{Surface Integral (2D)}\\
\text{Total Mass Change in } B &= \text{Mass Flows across } \partial B,
\end{aligned}
$$

where $\mathbf{n}$ is an outward unit normal vector on ∂B. The integral $\iint_{\partial B} \mathbf{u} \cdot \mathbf{n} dS$ is also called the **flux u** across the surface S.

HW 8.1. Consider the domain B as a cube centered at (x, y, z) of sides dx, dy, dz with face S_1 at $x - \frac{dx}{2}$ and face S_2 at $x + \frac{dx}{2}$. Compute the outward fluxes through S_1 and S_2 and the total flux through S_1 and S_2. Repeat this procedure for the remaining four faces and find the total outward flux from cube. And then prove (8.4) with the cube domain.

HW 8.2. Let B be the region defined by $x^2 + y^2 + z^2 < 1$. Use the divergence theorem to evaluate $\iiint_B z^2 d\mathbf{r}$.

Multiply (8.1a) by an arbitrary *test* function

$$v(x) \in H^1(\Omega) := \{v(x) : \int_0^1 (v^2 + (v')^2) dx < \infty\} \tag{8.5}$$

where $H^1(\Omega)$ a **Hilbert** (**Sobolev**) space (i.e., $H^1(\Omega)$ is a function space such that any function of H^1 and it's first derivative are square integrable) and integrate over Ω so that

$$\int_0^1 (-u''v)\, dx = -u'v|_0^1 + \int_0^1 (u'v') dx = \int_0^1 fv \qquad \forall v \in H^1 \tag{8.6}$$

Now choose $v(x)$ such that $v(0) = 0$ and define

$$H_0^1(\Omega) \;:\; = \{v \in H^1(\Omega) : v(0) = 0\} \tag{8.7}$$
$$H_D^1(\Omega) \;:\; = \{v \in H^1(\Omega) : v(0) = g_D\} \tag{8.8}$$
$$H_0^1(\Omega) \subset H^1(\Omega) \tag{8.9}$$

Define the *bilinear form* (*functional*)

$$B(u, v) = \int_0^1 u'v' dx \tag{8.10}$$
$$B(\alpha_1 u_1 + \alpha_2 u_2, v) = \alpha_1 B(u_{1,} v) + \alpha_2 B(u_2, v) \tag{8.11}$$
$$\Rightarrow B \text{ is linear in both } u \text{ and } v \tag{8.12}$$

and the *linear form (functional)*

$$F(v) = \int_0^1 fv dx + g_N v(1) \tag{8.13}$$

Step 2. Weak (or Variational or Generalized) Problem

Weak Problem: Given $f \in H^0(\Omega)$, g_D, and g_N, find $u \in H^1_D(\Omega)$ such that

$$B(u,v) = F(v) \qquad \forall v \in H^1_0(\Omega) \tag{8.14}$$

where

$$\begin{aligned}
H^0(\Omega) &= L^2(\Omega) := \{u(x) : \int_0^1 v^2 dx < \infty\} && (8.15)\\
H^1(\Omega) &: = \{v(x) : \int_0^1 [v^2 + (v')^2] dx < \infty\} && (8.16)\\
H^1_0(\Omega) &: = \{v(x) \in H^1(\Omega) : v(0) = 0\} && (8.17)\\
H^0_D(\Omega) &: = \{v(x) \in H^1(\Omega) : v(0) = g_D\} && (8.18)\\
B(u,v) &: = \int_0^1 u'v'dx && (8.19)\\
F(v) &: = \int_0^1 fvdx + g_N v(1) && (8.20)
\end{aligned}$$

Remark 8.1.
(i)

$$\begin{aligned}
(8.1) &\Rightarrow f \in C^0 \Rightarrow u \in C^2 \text{ (stronger space)} && (8.21)\\
(8.14) &\Rightarrow f \in H^0(\Omega) && \\
&\Rightarrow u \in H^1 \text{(weaker space) } C^2 \subset H^1 \text{ (larger)} && (8.22)\\
\text{Larger for } f &\Rightarrow \text{more applications} \Rightarrow \text{ FEM more useful} && (8.23)
\end{aligned}$$

(ii) $B(u,v) = B(v,u)$
$\Rightarrow$ (8.1) is a self-adjoint continuous problem ($-\frac{d^2}{dx^2}$ and $-\Delta$ are self-adjoint operators).
$\Rightarrow A\overrightarrow{x} = \overrightarrow{b}$ is a symmetric discrete problem (A is a symmetric matrix).
(iii) Strong Solution $\underset{\not\Leftarrow}{\Rightarrow}$ Weak Solution
(iv) Most of physical problems belong to (8.14) not to (8.1).

Step 3. Mesh Generation (or Partition or Discretization)
Uniform Mesh (Partition): We partition (discretize) the domain $\overline{\Omega} = [0,1]$ into $N-1$ subintervals (meshes or elements) with uniform mesh size $\Delta x = h = \frac{1}{N-1}$ and N mesh (grid) points (nodes) x_i, $i = 1, \cdots N$. Hence,

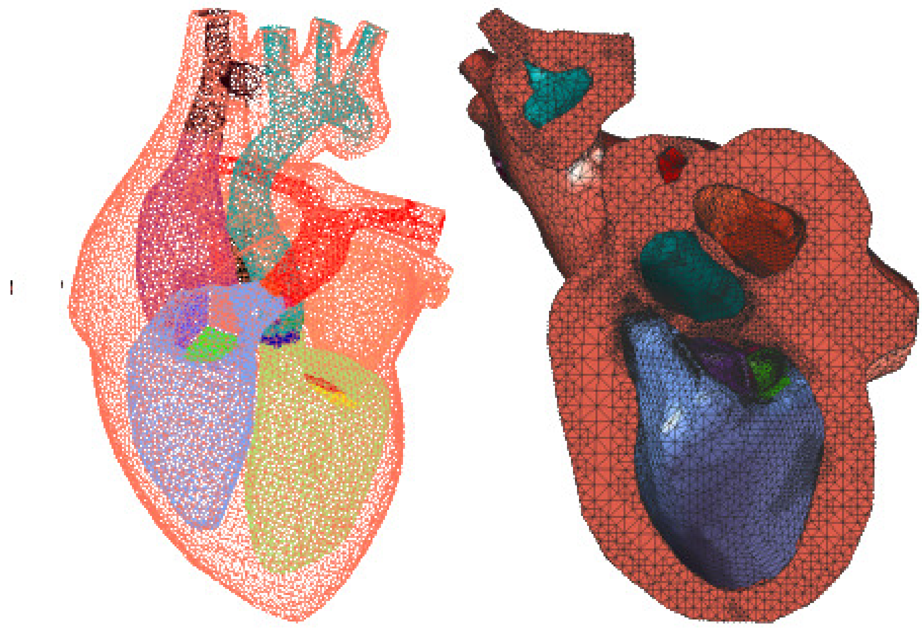

Figure 1: 3D Heart Mesh

$x_i = 0 + (i-1)\Delta x$, $x_{i+1} = 0 + i\Delta x$ etc. 2D or 3D mesh generation is one of very important areas in computational sciences see Figs. 3 and 4. Each subinterval (x_i, x_{i+1}) is called an **element**.

Step 4. Finite Element Subspaces

Let $S^h \subset H^1(\Omega)$ be a finite element subspace so that its *basis functions* (*shape* functions) are defined by

$$\phi_i(x) = \begin{cases} 1 & \text{when} \quad x = x_i \\ 0 & \text{when} \quad x \notin (x_{i-1}, x_{i+1}) \end{cases} \tag{8.24}$$

which in general are chosen as polynomials (linear (Fig. 5), quadratic, cubic, etc.). The open interval (x_{i-1}, x_{i+1}) is called the *support* of the basis function ϕ_i.

In implementation, these functions are constructed via the *standard shape functions* (ψ_1 and ψ_2) defined on a *standard* (*reference*) element $(-1, 1)$, i.e.,

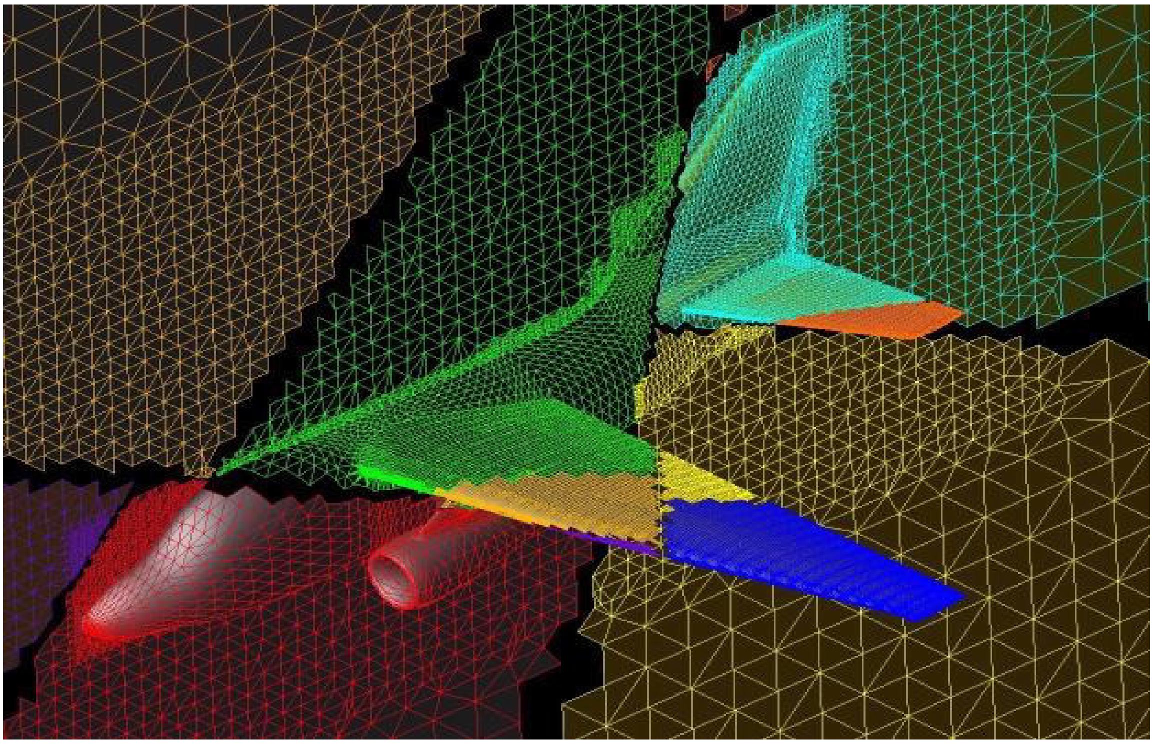

Figure 2: 3D Aircraft Mesh

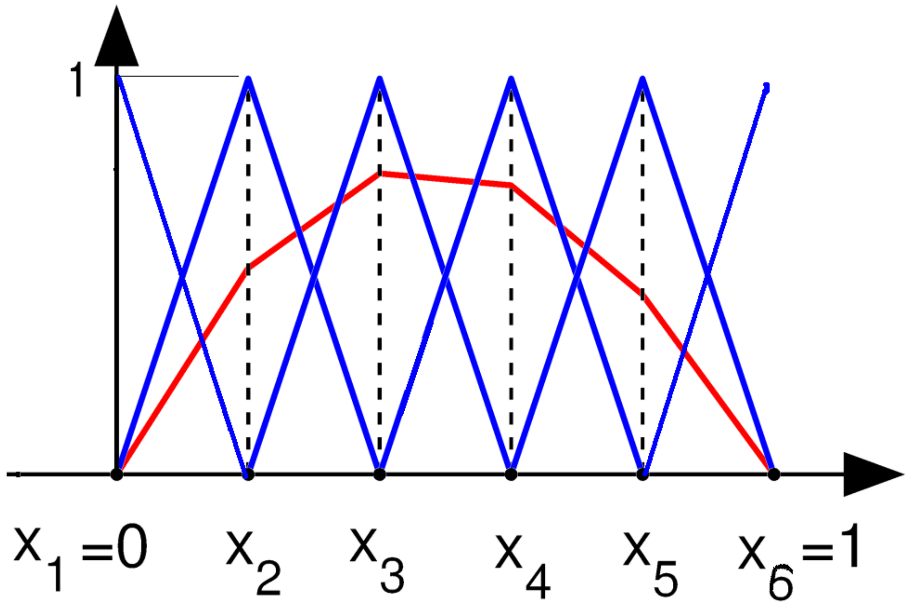

Figure 3: 1D Linear FE Basis Functions

there exists a linear transformation

$$\frac{x - x_i}{h_i} = \frac{\xi + 1}{2}, \quad \xi = \frac{2(x - x_i)}{h_i} - 1, \quad h_i = x_{i+1} - x_i \tag{8.25}$$

between the standard element $(-1, 1)$ and any element (x_i, x_{i+1}) such that

$$\begin{aligned} \psi_1(\xi) &= \frac{1 - \xi}{2} = \frac{1 - \frac{2(x - x_i)}{h_i} + 1}{2}, \qquad \forall\, x \in (x_i, x_{i+1}) \\ &= 1 - \frac{x - x_i}{h_i} = \frac{x_{i+1} - x_i - x + x_i}{h_i} \\ &= \frac{x_{i+1} - x}{h_i} = \phi_i(x) \end{aligned} \tag{8.26}$$

$$\psi_2(\xi) = \frac{1 + \xi}{2} = \frac{x - x_i}{h_i} = \phi_{i+1}(x), \qquad \forall\, x \in (x_{i-1}, x_i) \tag{8.27}$$

Hence, the basis functions are constructed element-by-element via standard shape functions on $(-1, 1)$ and the linear transformation (8.25), i.e., we construct $\phi_{i-1}(x)$, $\phi_i(x)$ on (x_{i-1}, x_i) and then $\phi_i(x)$, $\phi_{i+1}(x)$ on (x_i, x_{i+1})

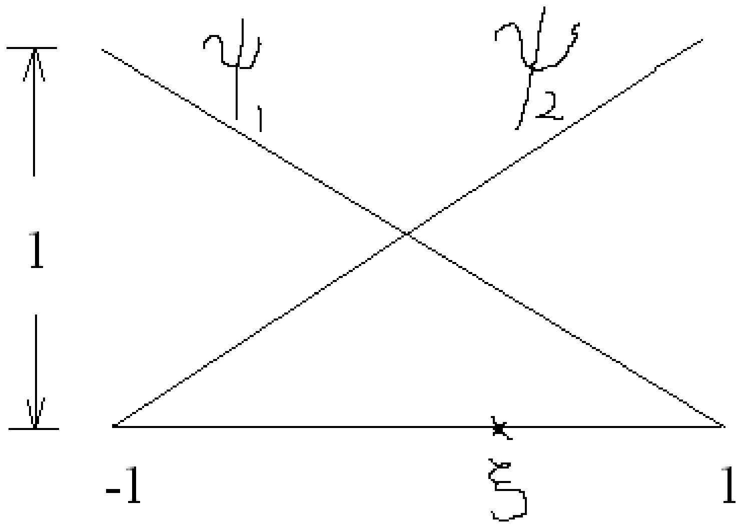

Figure 4: 1D Standard Element

etc. Therefore, we have the linear finite element subspace

$$S^h := \text{span}\,\{\phi_i\}_{i=1}^N \subset H^1(0,1) \tag{8.28}$$

since it can be easily verified that

$$\int_0^1 [\phi_2(x)]^2 + [\phi_2'(x)]^2 dx < \infty \quad ? \tag{8.29}$$

For any $v^h(x) \in S^h$, we can write

$$v^h(x) = \sum_{i=1}^N V_i \phi_i(x) \tag{8.30}$$

where V_i are scalars

$$S_0^h \;:\; = \{v^h(x) \in S^h : v(0) = 0\} \subset H_0^1(\Omega) \tag{8.31}$$

$$S_D^h \;:\; = \{v^h(x) \in S^h : v(0) = g_D\} \subset H_D^1(\Omega) \tag{8.32}$$

Step 5. Finite Element Problem

Finite Element Problem: Given $f \in H^0(\Omega)$, g_D, and g_N, find $u^h(x) \in S_D^h$ such that

$$B(u^h, v^h) = F(v^h) \qquad \forall v^h \in S_0^h \tag{8.33}$$

where

$$u^h(x) \;=\; \sum_{i=1}^N U_j \phi_j(x), \quad U_j : \text{unknown scalars} \tag{8.34}$$

$$v^h(x) \;=\; \sum_{i=1}^N V_i \phi_i(x), \quad V_i : \text{ arbitrary scalars} \tag{8.35}$$

$$\begin{aligned}
v^h(x) &\in S_0^h \Rightarrow \\
v^h(x_1) &= v^h(0) \\
&= V_1\phi_1(x_1) + V_2\phi_2(x_1) + V_3\phi_3(x_1) + \ldots + V_N\phi_N(x_1) \\
&= V_1 = 0
\end{aligned}$$

$$
\begin{aligned}
(8.11) \Rightarrow B(u^h, \sum_{i=2}^{N} V_i \phi_i(x)) &= F(\sum_{i=2}^{N} V_i \phi_i(x)) \\
\sum_{i=2}^{N} V_i B(u^h, \phi_i) &= \sum_{i=2}^{N} V_i F(\phi_i) \\
\Longleftrightarrow B(u^h, \phi_i) = F(\phi_i) &\quad \forall i = 2, \ldots, N
\end{aligned}
$$

Theorem 8.1.

$$
u^h(x) \in S_D^h \Rightarrow u^h(0) = u^h(x_1) = U_1 = g_D \tag{8.36}
$$

$$
\begin{aligned}
B(u^h,\ v^h) = F(\ v^h) &\qquad \forall\ \ v^h \in S_0^h \\
\Longleftrightarrow B(u^h, \phi_i) = F(\phi_i) &\quad \forall i = 2, \cdots, N
\end{aligned} \tag{8.37}
$$

Step 6. Matrix Formulation

From (8.36), we obtain the first equation $U_1 = g_D$ for $i = 1$. For $i = 2, \cdots N$, we have

$$
\begin{aligned}
B(u_h, \phi_i) &= F(\phi_i) \qquad \Rightarrow \\
B(\sum_{j=1}^{N} U_j \phi_j, \phi_i) &= F(\phi_i)\ \ \forall\ i\ \Rightarrow
\end{aligned} \tag{8.39}
$$

$$
\begin{cases}
\sum_j U_j B(\phi_j, \phi_2) = F(\phi_2) \\
\sum_j U_j B(\phi_j, \phi_3) = F(\phi_3) \\
\qquad \vdots \\
\sum_j U_j B(\phi_j, \phi_N) = F(\phi_N)
\end{cases} \tag{8.40}
$$

The matrix formulation

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ B(\phi_1,\phi_2) & B(\phi_2,\phi_2) & B(\phi_3,\phi_2) & \cdots & B(\phi_N,\phi_2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ B(\phi_1,\phi_i) & \cdots & B(\phi_i,\phi_i) & \cdots & B(\phi_N,\phi_i) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ B(\phi_1,\phi_i) & \vdots & \vdots & \vdots & B(\phi_N,\phi_i) \end{pmatrix} \begin{pmatrix} U_1 \\ U_2 \\ \vdots \\ U_i \\ \vdots \\ U_N \end{pmatrix}$$

$$= \begin{pmatrix} g_D \\ F(\phi_2) \\ \vdots \\ F(\phi_i) \\ \vdots \\ F(\phi_N) \end{pmatrix} \Longrightarrow A\overrightarrow{U} = \overrightarrow{b} \tag{8.41}$$

where

$$a_{ij} = B(\phi_j,\phi_i) = B(\phi_i,\phi_j) = a_{ji} \tag{8.42}$$

$$\Rightarrow A \text{ is symmetric for the part of } 2 \le i, j \le N-1 \text{ ?} \tag{8.43}$$

$$\overrightarrow{U} = \begin{pmatrix} U_1 \\ \vdots \\ U_i \\ \vdots \\ U_N \end{pmatrix} \qquad \overrightarrow{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_N \end{pmatrix} \tag{8.44}$$

$$U_j \approx u(x_j) = u_j \text{ unknown scalars} \tag{8.45}$$

Step 7. Solve $A\overrightarrow{U} = \overrightarrow{b}$

Example 8.1: Let $f(x) = 2$, $g_D = g_N = 0$. Write down the linear system (8.41) for $N = 5$.

Solution: Step 3 $\Rightarrow \Delta x = \frac{1}{4} = h \Rightarrow$ Mesh points: $x_1 = 0, x_2 = \frac{1}{4}, x_3 = \frac{1}{2}, x_4 = \frac{3}{4}, x_5 = 1$.

$$a_{32} = B(\phi_2, \phi_3) = \int_0^1 \phi_2' \phi_3' dx = \int_{x_2}^{x_3} \phi_2' \phi_3' dx = \int_{x_2}^{x_3} (-\frac{1}{h})(\frac{1}{h}) dx = -\frac{1}{h} \tag{8.46}$$

$$\begin{aligned} a_{33} &= \int_{x_2}^{x_4} = \int_{x_2}^{x_3} + \int_{x_3}^{x_4} = \int_{x_2}^{x_3} (\phi_3')^2 dx + \int_{x_3}^{x_4} (\phi_3') dx \\ &= \frac{1}{h^2} h + \frac{1}{h^2} h = \frac{2}{h} \end{aligned} \tag{8.47}$$

$$\begin{aligned} b_3 &= \int_{x_2}^{x_3} f(x)\phi_3(x)dx + \int_{x_3}^{x_4} f(x)\phi_3(x)dx \\ &= \int_{x_2}^{x_4} f(x)\phi_3(x)dx \\ &\approx f(x_3) \int_{x_2}^{x_4} \phi_3(x)dx \\ &= hf(x_3) = 2h \end{aligned} \tag{8.48}$$

$$\Rightarrow AU = \overrightarrow{b}, \quad A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{h} & \frac{2}{h} & \frac{-1}{h} & 0 & 0 & 0 \\ 0 & \frac{-1}{h} & \frac{2}{h} & \frac{-1}{h} & 0 & 0 \\ 0 & 0 & \frac{-1}{h} & \frac{2}{h} & \frac{-1}{h} & 0 \\ 0 & 0 & 0 & 0 & -1? & 2? \end{bmatrix}, \quad \overrightarrow{b} = \begin{pmatrix} 0 \\ 2h \\ 2h \\ 2h \\ 0? \end{pmatrix} \tag{8.49}$$

Question 8.1: (a) Answer ? in (8.43). (b) Is the linear system $AU = \overrightarrow{b}$ in (8.49) (from FEM) the same as that in Question 1.1 (from FDM)? (c) Answer ??? in (8.49).

Remarks:

1. For general function $f(x)$, we need to evaluate the integral $b_3 = \int_{x_2}^{x_3} f(x)\ \phi_3(x)\ dx$ by using some numerical integration method. The most frequently used method is **Gaussian Quadrature Rule**, named after Carl Friedrich Gauss. In 1D, an n-point Gaussian quadrature rule is a quadrature rule constructed to yield an exact result for polynomials

of degree $2n-1$, by a suitable choice of the n points x_i and n weights w_i. The domain of integration for such a rule is conventionally taken as $[-1, 1]$, so the rule is stated as

$$\int_{-1}^{1} f(x)dx \approx \sum_{i=1}^{n} w_i f(x_i).$$

For the Gaussian quadrature rules in 2D and 3D, see O. C. **Zienkiewicz** and R. L. **Taylor**, *The finite element methods*, 4th Ed., Vol. 1. MrGraw-Hill, 1989.

2. In implementation, the entries of A and $\overrightarrow{b}$ are usually calculated in an element-by-element way.

Project 8.1. Consider the 1D Poisson Problem (1.1) (with $f(x) = 2$, $g_D = 0$, and $g_N = 0$) and implement the methods FEM and GE. Given a total number of nodes N, the mesh size $\Delta x = h = \frac{1}{N-1}$.

Input: N, A, $\overrightarrow{b}$, TOL (write the input in the program).

Output:

N	k	$E^{\overrightarrow{x}}$	E^u	α
5				
9				
17				
33				
65				
129				