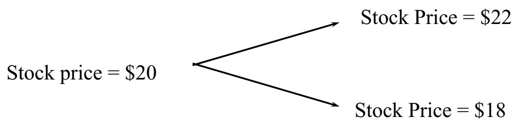
Introduction to Binomial Trees

Chapter 10

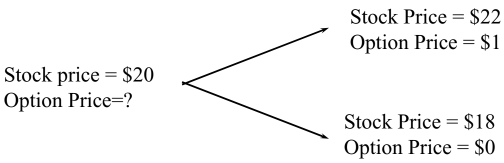
A Simple Binomial Model

- A stock price is currently \$20
- In three months it will be either \$22 or \$18



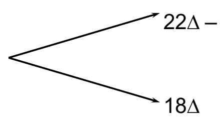
A Call Option (Figure 10.1, page 200)

A 3-month call option on the stock has a strike price of 21.



Setting Up a Riskless Portfolio

- Consider the Portfolio: long Δ shares
  short 1 call option



- Portfolio is riskless when $22\Delta - 1 = 18\Delta$ or $\Delta = 0.25$

Valuing the Portfolio (Risk-Free Rate is 12%)

- The riskless portfolio is:

  long 0.25 shares
  short 1 call option

- The value of the portfolio in 3 months is
  $22 \times 0.25 - 1 = 4.50$
- The value of the portfolio today is
  $4.5e^{-0.12 \times 0.25} = 4.3670$

Valuing the Option

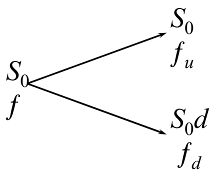
- The portfolio that is

  long 0.25 shares
  short 1 option

  is worth 4.367
- The value of the shares is
  5.000 (= 0.25×20)
- The value of the option is therefore
  0.633 (= $5.000 - 4.367$)

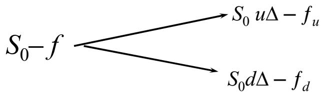
Generalization (Figure 10.2, page 202)

- A derivative lasts for time T and is dependent on a stock



Generalization (continued)

- Consider the portfolio that is long Δ shares and short 1 derivative



- The portfolio is riskless when $S_0u\Delta - f_u = S_0d\,\Delta - f_d$ or

$$\Delta = \frac{f_u - f_d}{S_0u - S_0d}$$

Generalization (continued)

- Value of the portfolio at time T is
  $S_0u\,\Delta - f_u$
- Value of the portfolio today is
  $(S_0u\,\Delta - f_u)e^{-rT}$
- Another expression for the portfolio value today is $S_0\Delta - f$
- Hence
  $f = S_0\Delta - (S_0u\,\Delta - f_u)e^{-rT}$

Generalization (continued)

- Substituting for Δ we obtain

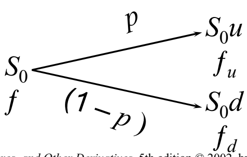
$$f = [\,p\,f_u + (1-p)f_d\,]e^{-rT}$$

where

$$p = \frac{e^{rT} - d}{u - d}$$

Risk-Neutral Valuation

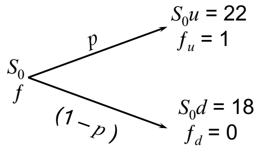
- $f = [\,p\,f_u + (1-p)f_d\,]e^{-rT}$
- The variables p and $(1-p)$ can be interpreted as the risk-neutral probabilities of up and down movements
- The value of a derivative is its expected payoff in a risk-neutral world discounted at the risk-free rate



Irrelevance of Stock's Expected Return

When we are valuing an option in terms of the underlying stock the expected return on the stock is irrelevant

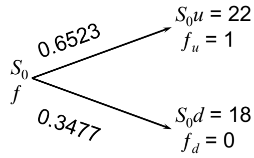
Original Example Revisited



- Since p is a risk-neutral probability $20e^{0.12\times 0.25} = 22p + 18(1 - p)$; $p = 0.6523$
- Alternatively, we can use the formula

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12\times 0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

Valuing the Option



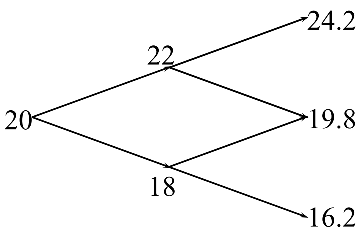
The value of the option is

$e^{-0.12\times 0.25}[0.6523\times 1 + 0.3477\times 0]$

$= 0.633$

A Two-Step Example

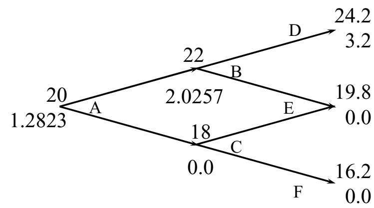
Figure 10.3, page 205



- Each time step is 3 months

Valuing a Call Option

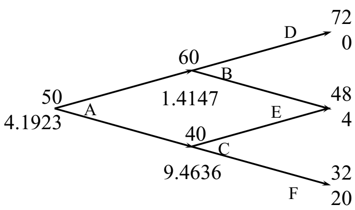
Figure 10.4, page 206



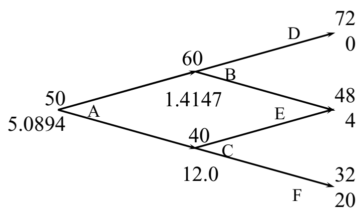
- Value at node B
  $= e^{-0.12\times 0.25}(0.6523\times 3.2 + 0.3477\times 0) = 2.0257$
- Value at node A
  $= e^{-0.12\times 0.25}(0.6523\times 2.0257 + 0.3477\times 0)$
  $= 1.2823$

A Put Option Example; *K*=52

Figure 10.7, page 208



What Happens When an Option is American (Figure 10.8, page 209)



Delta

- Delta (Δ) is the ratio of the change in the price of a stock option to the change in the price of the underlying stock
- The value of Δ varies from node to node

Choosing *u* and *d*

One way of matching the volatility is to set

$$u = e^{\sigma\sqrt{\delta t}}$$
$$d = e^{-\sigma\sqrt{\delta t}}$$

where σ is the volatility and δ*t* is the length of the time step. This is the approach used by Cox, Ross, and Rubinstein