10.1 Introduction to Binomial Trees

Chapter 10

10.2 A Simple Binomial Model

• A stock price is currently $20
• In three months it will be either $22 or $18

Stock Price = $22
Stock Price = $18

10.3 A Call Option (Figure 10.1, page 200)

A 3-month call option on the stock has a strike price of 21.

Stock price = $20
Option Price = $?

Stock Price = $22
Option Price = $1

Stock Price = $18
Option Price = $0

10.4 Setting Up a Riskless Portfolio

• Consider the Portfolio: long Δ shares
  short 1 call option

22Δ − 1
18Δ

• Portfolio is riskless when 22Δ − 1 = 18Δ or Δ = 0.25

10.5 Valuing the Portfolio (Risk-Free Rate is 12%)

• The riskless portfolio is:
  long 0.25 shares
  short 1 call option

• The value of the portfolio in 3 months is
  22'0.25 − 1 = 4.50

• The value of the portfolio today is
  4.5e^{−0.12'0.25} = 4.3670

10.6 Valuing the Option

• The portfolio that is
  long 0.25 shares
  short 1 option

is worth 4.367

• The value of the shares is
  5.000 (= 0.25'20 )

• The value of the option is therefore
  0.633 (= 5.000 − 4.367 )
Generalization (Figure 10.2, page 202)

- A derivative lasts for time $T$ and is dependent on a stock

\[ \begin{align*}
S_0 & \quad \text{uf} \\
\text{uf} & \quad \text{fd}
\end{align*} \]

Generalization (continued)

- Consider the portfolio that is long $\Delta$ shares and short 1 derivative

\[ S_0 - f = S_0 \mu \Delta - f_u \]

- The portfolio is riskless when $S_0 \mu \Delta - f_u = S_0 d \Delta - f_d$

\[ \Delta = \frac{f_u - f_d}{S_0 u - S_0 d} \]

Generalization (continued)

- Value of the portfolio at time $T$ is $S_0 \mu \Delta - f_u$

- Value of the portfolio today is $(S_0 \mu \Delta - f_u) e^{-rT}$

- Another expression for the portfolio value today is $S_0 \Delta - f$

- Hence

\[ f = S_0 \Delta - (S_0 \mu \Delta - f_u) e^{-rT} \]

Risk-Neutral Valuation

- $f = [p f_u + (1 - p) f_d] e^{-rT}$

- The variables $p$ and $(1 - p)$ can be interpreted as the risk-neutral probabilities of up and down movements

- The value of a derivative is its expected payoff in a risk-neutral world discounted at the risk-free rate

\[ \begin{align*}
S_0 & \quad S_0 \mu \\
\text{uf} & \quad f_u \\
\text{fd} & \quad f_d
\end{align*} \]

Irrelevance of Stock’s Expected Return

When we are valuing an option in terms of the underlying stock the expected return on the stock is irrelevant.
Original Example Revisited

Since \( p \) is a risk-neutral probability, \( 20e^{0.12 \times 0.25} = 22p \) + 18(1 – \( p \)); \( p = 0.6523 \)

Alternatively, we can use the formula:

\[ p = \frac{e^{uT} - d}{u - d} = \frac{e^{0.12 \times 3} - 0.9}{1.1 - 0.9} = 0.6523 \]

Valuing the Option

The value of the option is:

\[ e^{-0.12 \times 0.25} [0.6523 \times 1 + 0.3477 \times 0] = 0.633 \]

A Two-Step Example

Figure 10.3, page 205

- Each time step is 3 months

Valuing a Call Option

Figure 10.4, page 206

- Value at node B:
  \[ e^{0.12 \times 0.25} (0.6523 \times 3.2 + 0.3477 \times 0) = 2.0257 \]
- Value at node A:
  \[ e^{0.12 \times 0.25} (0.6523 \times 2.0257 + 0.3477 \times 0) = 1.2823 \]

A Put Option Example; \( K = 52 \)

Figure 10.7, page 208

What Happens When an Option is American (Figure 10.8, page 209)
Delta

- Delta (\(\Delta\)) is the ratio of the change in the price of a stock option to the change in the price of the underlying stock
- The value of \(\Delta\) varies from node to node

Choosing \(u\) and \(d\)

One way of matching the volatility is to set

\[
\begin{align*}
  u &= e^{\frac{\sigma \sqrt{\delta t}}}{ \sqrt{\delta t}} \\
  d &= e^{-\frac{\sigma \sqrt{\delta t}}}{ \sqrt{\delta t}} 
\end{align*}
\]

where \(\sigma\) is the volatility and \(\delta t\) is the length of the time step. This is the approach used by Cox, Ross, and Rubinstein.