Model of the Behavior of Stock Prices

Chapter 11

Categorization of Stochastic Processes

- Discrete time; discrete variable
- Discrete time; continuous variable
- Continuous time; discrete variable
- Continuous time; continuous variable

Modeling Stock Prices

- We can use any of the four types of stochastic processes to model stock prices
- The continuous time, continuous variable process proves to be the most useful for the purposes of valuing derivatives

Markov Processes (See pages 216-7)

- In a Markov process future movements in a variable depend only on where we are, not the history of how we got where we are
- We assume that stock prices follow Markov processes

Weak-Form Market Efficiency

- This asserts that it is impossible to produce consistently superior returns with a trading rule based on the past history of stock prices. In other words technical analysis does not work.
- A Markov process for stock prices is clearly consistent with weak-form market efficiency

Example of a Discrete Time Continuous Variable Model

- A stock price is currently at $40
- At the end of 1 year it is considered that it will have a probability distribution of $\phi(40,10)$ where $\phi(\mu,\sigma)$ is a normal distribution with mean μ and standard deviation σ.

Questions

11.7

- What is the probability distribution of the stock price at the end of 2 years?
- ½ years?
- ¼ years?
- δt years?

Taking limits we have defined a continuous variable, continuous time process

11.8

Variances & Standard Deviations

- In Markov processes changes in successive periods of time are independent
- This means that variances are additive
- Standard deviations are not additive

11.9

Variances & Standard Deviations (continued)

- In our example it is correct to say that the variance is 100 per year.
- It is strictly speaking not correct to say that the standard deviation is 10 per year.

11.10

A Wiener Process (See pages 218)

- We consider a variable z whose value changes continuously
- The change in a small interval of time δt is δz
- The variable follows a Wiener process if

1. $\delta z = \varepsilon\sqrt{\delta t}$ where ε is a random drawing from $\phi(0,1)$

2. The values of δz for any 2 different (non-overlapping) periods of time are independent

11.11

Properties of a Wiener Process

- Mean of $[z(T) - z(0)]$ is 0
- Variance of $[z(T) - z(0)]$ is T
- Standard deviation of $[z(T) - z(0)]$ is

$$\sqrt{T}$$

11.12

Taking Limits . . .

- What does an expression involving dz and dt mean?
- It should be interpreted as meaning that the corresponding expression involving δz and δt is true in the limit as δt tends to zero
- In this respect, stochastic calculus is analogous to ordinary calculus

Generalized Wiener Processes

(See page 220-2)

- A Wiener process has a drift rate (i.e. average change per unit time) of 0 and a variance rate of 1
- In a generalized Wiener process the drift rate and the variance rate can be set equal to any chosen constants

Generalized Wiener Processes

(continued)

The variable x follows a generalized Wiener process with a drift rate of a and a variance rate of b^2 if

$$dx = a\,dt + b\,dz$$

Generalized Wiener Processes

(continued)

$$\delta x = a\,\delta t + b\,\varepsilon\sqrt{\delta t}$$

- Mean change in x in time T is aT
- Variance of change in x in time T is b^2T
- Standard deviation of change in x in time T is

$$b\sqrt{T}$$

The Example Revisited

- A stock price starts at 40 and has a probability distribution of $\phi(40,10)$ at the end of the year
- If we assume the stochastic process is Markov with no drift then the process is

$$dS = 10dz$$

- If the stock price were expected to grow by $8 on average during the year, so that the year-end distribution is $\phi(48,10)$, the process is

$$dS = 8dt + 10dz$$

Ito Process **(See pages 222)**

- In an Ito process the drift rate and the variance rate are functions of time

$$dx = a(x,t)dt + b(x,t)dz$$

- The discrete time equivalent

$$\delta x = a(x,t)\delta t + b(x,t)\varepsilon\sqrt{\delta t}$$

is only true in the limit as δt tends to zero

Why a Generalized Wiener Process is not Appropriate for Stocks

- For a stock price we can conjecture that its expected percentage change in a short period of time remains constant, not its expected absolute change in a short period of time
- We can also conjecture that our uncertainty as to the size of future stock price movements is proportional to the level of the stock price

11.19

An Ito Process for Stock Prices

(See pages 222-3)

$$dS = \mu S dt + \sigma S dz$$

where μ is the expected return σ is the volatility.

The discrete time equivalent is

$$\delta S = \mu S \delta t + \sigma S \varepsilon \sqrt{\delta t}$$

11.20

Monte Carlo Simulation

- We can sample random paths for the stock price by sampling values for ε
- Suppose μ= 0.14, σ= 0.20, and $\delta t = 0.01$, then

$$\delta S = 0.0014S + 0.02S\varepsilon$$

11.21

Monte Carlo Simulation – One Path

(See Table 11.1)

Period	Stock Price at Start of Period	Random Sample for ε	Change in Stock Price, ΔS
0	20.000	0.52	0.236
1	20.236	1.44	0.611
2	20.847	-0.86	-0.329
3	20.518	1.46	0.628
4	21.146	-0.69	-0.262

11.22

Ito's Lemma (See pages 226-227)

- If we know the stochastic process followed by *x*, Ito's lemma tells us the stochastic process followed by some function *G* (*x, t*)
- Since a derivative security is a function of the price of the underlying and time, Ito's lemma plays an important part in the analysis of derivative securities

11.23

Taylor Series Expansion

- A Taylor's series expansion of *G*(*x, t*) gives

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial t}\delta t + ½\frac{\partial^2 G}{\partial x^2}\delta x^2 + \frac{\partial^2 G}{\partial x \partial t}\delta x\,\delta t + ½\frac{\partial^2 G}{\partial t^2}\delta t^2 + \ldots$$

11.24

Ignoring Terms of Higher Order Than δ*t*

In ordinary calculus we have

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial t}\delta t$$

In stochastic calculus this becomes

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial t}\delta t + ½\frac{\partial^2 G}{\partial x^2}\delta x^2$$

because δx has a component which is of order $\sqrt{\delta t}$

11.25

Substituting for δx

Suppose

$$dx = a(x,t)dt + b(x,t)dz$$

so that

$$\delta x = a\,\delta t + b\,\varepsilon\sqrt{\delta t}$$

Then ignoring terms of higher order than δt

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial t}\delta t + ½\frac{\partial^2 G}{\partial x^2}b^2\varepsilon^2\delta t$$

11.26

The $\varepsilon^2\Delta t$ Term

Since $\varepsilon \approx \phi(0,1)\ E(\varepsilon) = 0$

$$E(\varepsilon^2) - [E(\varepsilon)]^2 = 1$$

$$E(\varepsilon^2) = 1$$

It follows that $E(\varepsilon^2\delta t) = \delta t$

The variance of δt is proportional to δt^2 and can be ignored. Hence

$$\delta G = \frac{\partial G}{\partial x}\delta x + \frac{\partial G}{\partial t}\delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2}b^2\delta t$$

11.27

Taking Limits

Taking limits $\quad dG = \frac{\partial G}{\partial x}dx + \frac{\partial G}{\partial t}dt + ½\frac{\partial^2 G}{\partial x^2}b^2 dt$

Substituting $\quad dx = a\,dt + b\,dz$

We obtain $\quad dG = \left(\frac{\partial G}{\partial x}a + \frac{\partial G}{\partial t} + ½\frac{\partial^2 G}{\partial x^2}b^2\right)dt + \frac{\partial G}{\partial x}b\,dz$

This is Ito's Lemma

11.28

Application of Ito's Lemma to a Stock Price Process

The stock price process is

$$dS = \mu S\,dt + \sigma S\,dz$$

For a function G of S and t

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + ½\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2\right)dt + \frac{\partial G}{\partial S}\sigma S\,dz$$

11.29

Examples

1. The forward price of a stock for a contract maturing at time T

$$G = S\,e^{r(T-t)}$$

$$dG = (\mu - r)G\,dt + \sigma G\,dz$$

2. $G = \ln S$

$$dG = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\,dz$$