The Black-Scholes Model

Chapter 12

The Stock Price Assumption

- Consider a stock whose price is S
- In a short period of time of length δt, the return on the stock is normally distributed:

$$\frac{\delta S}{S} \approx \phi\left(\mu \delta t, \sigma \sqrt{\delta t}\right)$$

where μ is expected return and σ is volatility

The Lognormal Property

(Equations 12.2 and 12.3, page 235)

- It follows from this assumption that

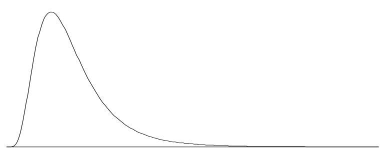
$$\ln S_T - \ln S_0 \approx \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma\sqrt{T}\right]$$

or

$$\ln S_T \approx \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T,\ \sigma\sqrt{T}\right]$$

- Since the logarithm of S_T is normal, S_T is lognormally distributed

The Lognormal Distribution



$$E(S_T) = S_0\, e^{\mu T}$$

$$\text{var}(S_T) = S_0^2 e^{2\mu T}(e^{\sigma^2 T} - 1)$$

Continuously Compounded Return, η

(Equations 12.6 and 12.7), page 236)

$$S_T = S_0\, e^{\eta T}$$

or

$$\eta = \frac{1}{T}\ln\frac{S_T}{S_0}$$

or

$$\eta \approx \phi\left(\mu - \frac{\sigma^2}{2},\ \frac{\sigma}{\sqrt{T}}\right)$$

The Expected Return

- The expected value of the stock price is $S_0 e^{\mu T}$
- The expected return on the stock is

$\mu - \sigma^2/2$

$$E\left[\ln(S_T / S_0)\right] = \mu - \sigma^2 / 2$$

$$\ln\left[E(S_T / S_0)\right] = \mu$$

The Volatility

- The volatility of an asset is the standard deviation of the continuously compounded rate of return in 1 year
- As an approximation it is the standard deviation of the percentage change in the asset price in 1 year

Estimating Volatility from Historical Data (page 239-41)

1. Take observations $S_0, S_1, \ldots, S_n$ at intervals of τ years
2. Calculate the continuously compounded return in each interval as:

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

3. Calculate the standard deviation, s, of the u_i's
4. The historical volatility estimate is:

$$\hat{\sigma} = \frac{s}{\sqrt{\tau}}$$

The Concepts Underlying Black-Scholes

- The option price and the stock price depend on the same underlying source of uncertainty
- We can form a portfolio consisting of the stock and the option which eliminates this source of uncertainty
- The portfolio is instantaneously riskless and must instantaneously earn the risk-free rate
- This leads to the Black-Scholes differential equation

The Derivation of the Black-Scholes Differential Equation

$$\delta S = \mu S\,\delta t + \sigma S\,\delta z$$

$$\delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + ½\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2\right)\delta t + \frac{\partial f}{\partial S}\sigma S\,\delta z$$

We set up a portfolio consisting of

-1: derivative

$+\frac{\partial f}{\partial S}$: shares

The Derivation of the Black-Scholes Differential Equation continued

The value of the portfolio Π is given by

$$\Pi = -f + \frac{\partial f}{\partial S}S$$

The change in its value in time δt is given by

$$\delta \Pi = -\delta f + \frac{\partial f}{\partial S}\delta S$$

The Derivation of the Black-Scholes Differential Equation continued

The return on the portfolio must be the risk-free rate. Hence

$$\delta \Pi = r\,\Pi\,\delta t$$

We substitute for δf and δS in these equations to get the Black-Scholes differential equation:

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + ½\,\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf$$

The Differential Equation

12.13

- Any security whose price is dependent on the stock price satisfies the differential equation
- The particular security being valued is determined by the boundary conditions of the differential equation
- In a forward contract the boundary condition is $f = S - K$ when $t = T$
- The solution to the equation is

$$f = S - K e^{-r(T-t)}$$

Risk-Neutral Valuation

12.14

- The variable μ does not appearin the Black-Scholes equation
- The equation is independent of all variables affected by risk preference
- The solution to the differential equation is therefore the same in a risk-free world as it is in the real world
- This leads to the principle of risk-neutral valuation

Applying Risk-Neutral Valuation

12.15

1. Assume that the expected return from the stock price is the risk-free rate
2. Calculate the expected payoff from the option
3. Discount at the risk-free rate

The Black-Scholes Formulas

12.16

(See pages 246-248)

$$c = S_0 N(d_1) - K e^{-rT} N(d_2)$$

$$p = K e^{-rT} N(-d_2) - S_0 N(-d_1)$$

where $d_1 = \dfrac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

Implied Volatility

12.17

- The implied volatility of an option is the volatility for which the Black-Scholes price equals the market price
- The is a one-to-one correspondence between prices and implied volatilities
- Traders and brokers often quote implied volatilities rather than dollar prices

Causes of Volatility

12.18

- Volatility is usually much greater when the market is open (i.e. the asset is trading) than when it is closed
- For this reason time is usually measured in “trading days” not calendar days when options are valued

12.19

Warrants & Dilution (pages 249-50)

- When a regular call option is exercised the stock that is delivered must be purchased in the open market
- When a warrant is exercised new Treasury stock is issued by the company
- This will dilute the value of the existing stock
- One valuation approach is to assume that all equity (warrants + stock) follows geometric Brownian motion

12.20

Dividends

- European options on dividend-paying stocks are valued by substituting the stock price less the present value of dividends into Black-Scholes
- Only dividends with ex-dividend dates during life of option should be included
- The "dividend" should be the expected reduction in the stock price expected

12.21

American Calls

- An American call on a non-dividend-paying stock should never be exercised early
- An American call on a dividend-paying stock should only ever be exercised immediately prior to an ex-dividend date

12.22

Black's Approach to Dealing with Dividends in American Call Options

Set the American price equal to the maximum of two European prices:

1. The 1st European price is for an option maturing at the same time as the American option
2. The 2nd European price is for an option maturing just before the final ex-dividend date