Value at Risk

Chapter 16

The Question Being Asked in VaR

"What loss level is such that we are *X*% confident it will not be exceeded in *N* business days?"

VaR and Regulatory Capital

- Regulators base the capital they require banks to keep on VaR
- The market-risk capital is *k* times the 10-day 99% VaR where *k* is at least 3.0

VaR vs. C-VaR

(See Figures 16.1 and 16.2)

- VaR is the loss level that will not be exceeded with a specified probability
- C-VaR is the expected loss given that the loss is greater than the VaR level
- Although C-VaR is theoretically more appealing, it is not widely used

Advantages of VaR

- It captures an important aspect of risk in a single number
- It is easy to understand
- It asks the simple question: "How bad can things get?"

Time Horizon

- Instead of calculating the 10-day, 99% VaR directly analysts usually calculate a 1-day 99% VaR and assume

$$\text{10-day VaR} = \sqrt{10} \times \text{1-day VaR}$$

- This is exactly true when portfolio changes on successive days come from independent identically distributed normal distributions

Historical Simulation

(See Table 16.1 and 16.2)

- Create a database of the daily movements in all market variables.
- The first simulation trial assumes that the percentage changes in all market variables are as on the first day
- The second simulation trial assumes that the percentage changes in all market variables are as on the second day
- and so on

Historical Simulation continued

- Suppose we use *m* days of historical data
- Let v_i be the value of a variable on day *i*
- There are *m*-1 simulation trials
- The *i*th trial assumes that the value of the market variable tomorrow (i.e., on day *m*+1) is

$$v_m \frac{v_i}{v_{i-1}}$$

The Model-Building Approach

- The main alternative to historical simulation is to make assumptions about the probability distributions of return on the market variables and calculate the probability distribution of the change in the value of the portfolio analytically
- This is known as the model building approach or the variance-covariance approach

Daily Volatilities

- In option pricing we express volatility as volatility per year
- In VaR calculations we express volatility as volatility per day

$$\sigma_{\text{day}} = \frac{\sigma_{\text{year}}}{\sqrt{252}}$$

Daily Volatility continued

- Strictly speaking we should define σ_{day} as the standard deviation of the continuously compounded return in one day
- In practice we assume that it is the standard deviation of the percentage change in one day

Microsoft Example

- We have a position worth $10 million in Microsoft shares
- The volatility of Microsoft is 2% per day (about 32% per year)
- We use *N*=10 and *X*=99

Microsoft Example continued

- The standard deviation of the change in the portfolio in 1 day is \$200,000
- The standard deviation of the change in 10 days is

$$200{,}000\sqrt{10} = \$632{,}456$$

Microsoft Example continued

- We assume that the expected change in the value of the portfolio is zero (This is OK for short time periods)
- We assume that the change in the value of the portfolio is normally distributed
- Since $N(-2.33)=0.01$, the VaR is

$$2.33 \times 632{,}456 = \$1{,}473{,}621$$

AT&T Example

- Consider a position of \$5 million in AT&T
- The daily volatility of AT&T is 1% (approx 16% per year)
- The S.D per 10 days is
- The VaR is

$$50{,}000\sqrt{10} = \$158{,}144$$

$$158{,}114 \times 2.33 = \$368{,}405$$

Portfolio

- Now consider a portfolio consisting of both Microsoft and AT&T
- Suppose that the correlation between the returns is 0.3

S.D. of Portfolio

- A standard result in statistics states that

$$\sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y}$$

- In this case σ_X = 200,000 and σ_Y = 50,000 and ρ = 0.3. The standard deviation of the change in the portfolio value in one day is therefore 220,227

VaR for Portfolio

- The 10-day 99% VaR for the portfolio is

$$220{,}227 \times \sqrt{10} \times 2.33 = \$1{,}622{,}657$$

- The benefits of diversification are

$$(1{,}473{,}621 + 368{,}405) - 1{,}622{,}657 = \$219{,}369$$

- What is the incremental effect of the AT&T holding on VaR?

The Linear Model

We assume

- The daily change in the value of a portfolio is linearly related to the daily returns from market variables
- The returns from the market variables are normally distributed

The General Linear Model continued (equations 16.1 and 16.2)

$$\delta P = \sum_{i=1}^{n} \alpha_i \delta x_i$$

$$\sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \alpha_i \alpha_j \sigma_i \sigma_j \rho_{ij}$$

$$\sigma_P^2 = \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2 + 2 \sum_{i<j} \alpha_i \alpha_j \sigma_i \sigma_j \rho_{ij}$$

where σ_i is the volatility of variable i

and σ_P is the portfolio's standard deviation

Handling Interest Rates: Cash Flow Mapping

- We choose as market variables bond prices with standard maturities (1mm, 3mm, 6mm, 1yr, 2yr, 5yr, 7yr, 10yr, 30yr)
- Suppose that the 5yr rate is 6% and the 7yr rate is 7% and we will receive a cash flow of $10,000 in 6.5 years.
- The volatilities per day of the 5yr and 7yr bonds are 0.50% and 0.58% respectively

Example continued

- We interpolate between the 5yr rate of 6% and the 7yr rate of 7% to get a 6.5yr rate of 6.75%
- The PV of the $10,000 cash flow is

$$\frac{10{,}000}{1.0675^{6.5}} = 6{,}540$$

Example continued

- We interpolate between the 0.5% volatility for the 5yr bond price and the 0.58% volatility for the 7yr bond price to get 0.56% as the volatility for the 6.5yr bond
- We allocate α of the PV to the 5yr bond and (1- α) of the PV to the 7yr bond

Example continued

- Suppose that the correlation between movement in the 5yr and 7yr bond prices is 0.6
- To match variances
  $$0.56^2 = 0.5^2\alpha^2 + 0.58^2(1-\alpha)^2 + 2\times 0.6\times 0.5\times 0.58\times\alpha(1-\alpha)$$
- This gives α=0.074

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Example continued

The value of 6,540 received in 6.5 years

$$6{,}540 \times 0.074 = \$484$$

in 5 years and by

$$6{,}540 \times 0.926 = \$6{,}056$$

in 7 years.

This cash flow mapping preserves value and variance

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When Linear Model Can be Used

- Portfolio of stocks
- Portfolio of bonds
- Forward contract on foreign currency
- Interest-rate swap

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The Linear Model and Options

Consider a portfolio of options dependent on a single stock price, S. Define

$$\Delta = \frac{\delta P}{\delta S}$$

and

$$\delta x = \frac{\delta S}{S}$$

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Linear Model and Options continued

(equations 16.3 and 16.4)

- As an approximation

$$\delta P = \Delta\,\delta S = S\Delta\,\delta x$$

- Similar when there are many underlying market variables

$$\delta P = \sum_i S_i \Delta_i\,\delta x_i$$

where Δ_i is the delta of the portfolio with respect to the ith asset

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Example

- Consider an investment in options on Microsoft and AT&T. Suppose the stock prices are 120 and 30 respectively and the deltas of the portfolio with respect to the two stock prices are 1,000 and 20,000 respectively
- As an approximation

$$\delta P = 120 \times 1{,}000\,\delta x_1 + 30 \times 20{,}000\,\delta x_2$$

where δx_1 and δx_2 are the percentage changes in the two stock prices

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Skewness

(See Figures 16.3, 16.4 , and 16.5)

The linear model fails to capture skewness in the probability distribution of the portfolio value.

Quadratic Model

For a portfolio dependent on a single stock price it is approximately true that

$$\delta P = \Delta \delta S + \frac{1}{2}\Gamma(\delta S)^2$$

this becomes

$$\delta P = S\Delta\,\delta x + \frac{1}{2}S^2\Gamma(\delta x)^2$$

Moments of δP for one market variable

$$E(\delta P) = \frac{1}{2}S^2\gamma\sigma^2$$

$$E(\delta P^2) = S^2\delta^2\sigma^2 + \frac{3}{4}S^4\gamma^2\sigma^4$$

$$E(\delta P^3) = \frac{9}{2}S^4\delta^2\gamma\sigma^4 + \frac{15}{8}S^6\gamma^3\sigma^6$$

Quadratic Model continued

With many market variables and each instrument dependent on only one

$$\delta P = \sum_{i=1}^{n} S_i\Delta_i\delta x_i + \sum_{i=1}^{n}\frac{1}{2}S_i^2\Gamma_i(\delta x_i)^2$$

where Δ_i and Γ_i are the delta and gamma of the portfolio with respect to the *i*th variable

Quadratic Model continued

When the change in the portfolio value has the form

$$\delta P = \sum_{i=1}^{n}\alpha_i\delta x_i + \sum_{i=1}^{n}\beta_i(\delta x_i)^2$$

we can calculate the moments of ΔP analytically if the δx_i are assumed to be normal

Quadratic Model continued

Once we have done this we can use the Cornish Fisher expansion to calculate fractiles of the distribution of δP

Monte Carlo Simulation

To calculate VaR using M.C. simulation we

- Value portfolio today
- Sample once from the multivariate distributions of the δx_i
- Use the δx_i to determine market variables at end of one day
- Revalue the portfolio at the end of day

Monte Carlo Simulation

- Calculate δP
- Repeat many times to build up a probability distribution for δP
- VaR is the appropriate fractile of the distribution times square root of N
- For example, with 1,000 trial the 1 percentile is the 10th worst case.

Speeding Up Monte Carlo

Use the quadratic approximation to calculate δP

Stress Testing

- This involves testing how well a portfolio performs under some of the most extreme market moves seen in the last 10 to 20 years

Back-Testing

- Tests how well VaR estimates would have performed in the past
- We could ask the question: How often was the actual 10-day loss greater than the 99%/10 day VaR?

Principal Components Analysis for Interest Rates

- The first factor is a roughly parallel shift (83.1% of variation explained)
- The second factor is a twist (10% of variation explained)
- The third factor is a bowing (2.8% of variation explained)