Estimating Volatilities and Correlations

Chapter 17

Standard Approach to Estimating Volatility

- Define σ_n as the volatility per day between day n-1 and day n, as estimated at end of day n-1
- Define S_i as the value of market variable at end of day i
- Define $u_i = \ln(S_i/S_{i-1})$

$$\sigma_n^2 = \frac{1}{m-1}\sum_{i=1}^{m}(u_{n-i}-\bar{u})^2$$

$$\bar{u} = \frac{1}{m}\sum_{i=1}^{m}u_{n-i}$$

Simplifications Usually Made

- Define u_i as $(S_i - S_{i-1})/S_{i-1}$
- Assume that the mean value of u_i is zero
- Replace m-1 by m

This gives

$$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m}u_{n-i}^2$$

Weighting Scheme

Instead of assigning equal weights to the observations we can set

$$\sigma_n^2 = \sum_{i=1}^{m}\alpha_i u_{n-i}^2$$

where

$$\sum_{i=1}^{m}\alpha_i = 1$$

ARCH(m) Model

In an ARCH(m) model we also assign some weight to the long-run variance rate, V_L:

$$\sigma_n^2 = \gamma V_L + \sum_{i=1}^{m}\alpha_i u_{n-i}^2$$

where

$$\gamma + \sum_{i=1}^{m}\alpha_i = 1$$

EWMA Model

- In an exponentially weighted moving average model, the weights assigned to the u^2 decline exponentially as we move back through time
- This leads to

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_{n-1}^2$$

Attractions of EWMA

- Relatively little data needs to be stored
- We need only remember the current estimate of the variance rate and the most recent observation on the market variable
- Tracks volatility changes
- RiskMetrics uses $\lambda = 0.94$ for daily volatility forecasting

GARCH (1,1)

In GARCH (1,1) we assign some weight to the long-run average variance rate

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta\sigma_{n-1}^2$$

Since weights must sum to 1

$$\gamma + \alpha + \beta = 1$$

GARCH (1,1) continued

Setting $\omega = \gamma V$ the GARCH (1,1) model is

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta\sigma_{n-1}^2$$

and

$$V_L = \frac{\omega}{1-\alpha-\beta}$$

Example

- Suppose

$$\sigma_n^2 = 0.000002 + 0.13u_{n-1}^2 + 0.86\sigma_{n-1}^2$$

- The long-run variance rate is 0.0002 so that the long-run volatility per day is 1.4%

Example continued

- Suppose that the current estimate of the volatility is 1.6% per day and the most recent percentage change in the market variable is 1%.
- The new variance rate is

$$0.000002 + 0.13 \times 0.0001 + 0.86 \times 0.000256 = 0.00023336$$

The new volatility is 1.53% per day

GARCH (p,q)

$$\sigma_n^2 = \omega + \sum_{i=1}^{p}\alpha_i u_{n-i}^2 + \sum_{j=1}^{q}\beta_j\sigma_{n-j}^2$$

Other Models

- We can design GARCH models so that the weight given to u_i^2 depends on whether u_i is positive or negative
- We do not have to assume that the conditional distribution is normal

Variance Targeting

- One way of implementing GARCH(1,1) that increases stability is by using variance targeting
- We set the long-run average volatility equal to the sample variance
- Only two other parameters then have to be estimated

Maximum Likelihood Methods

- In maximum likelihood methods we choose parameters that maximize the likelihood of the observations occurring

Example 1

- We observe that a certain event happens one time in ten trials. What is our estimate of the proportion of the time, p, that it happens?
- The probability of the outcome is

$$10p(1-p)^9$$

- We maximize this to obtain a maximum likelihood estimate: p=0.1

Example 2

Estimate the variance of observations from a normal distribution with mean zero

Maximize: $$\prod_{i=1}^{n}\left[\frac{1}{\sqrt{2\pi v}}\exp\left(\frac{-u_i^2}{2v}\right)\right]$$

or: $$\sum_{i=1}^{n}\left[-\ln(v)-\frac{u_i^2}{v}\right]$$

This gives: $$v=\frac{1}{n}\sum_{i=1}^{n}u_i^2$$

Application to GARCH

We choose parameters that maximize

$$\sum_{i=1}^{n}\left[-\ln(v_i)-\frac{u_i^2}{v_i}\right]$$

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How Good is the Model?

- The Ljung-Box statistic tests for autocorrelation
- We compare the autocorrelation of the u_i^2 with the autocorrelation of the u_i^2/σ_i^2

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Forecasting Future Volatility

A few lines of algebra shows that

$$E[\sigma_{n+k}^2] = V_L + (\alpha+\beta)^k(\sigma_n^2 - V_L)$$

The variance rate for an option expiring on day m is

$$\frac{1}{m}\sum_{k=0}^{m-1} E\left[\sigma_{n+k}^2\right]$$

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Volatility Term Structures

- The GARCH (1,1) model allows us to predict volatility term structures changes
- It suggests that, when calculating vega, we should shift the long maturity volatilities less than the short maturity volatilities

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Correlations

Define $u_i=(U_i-U_{i-1})/U_{i-1}$ and $v_i=(V_i-V_{i-1})/V_{i-1}$

Also

$\sigma_{u,n}$: daily vol of U calculated on day n-1

$\sigma_{v,n}$: daily vol of V calculated on day n-1

cov_n: covariance calculated on day n-1

The correlation is $\text{cov}_n/(\sigma_{u,n}\,\sigma_{v,n})$

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Correlations continued

Under GARCH (1,1)

$$\text{cov}_n = \omega + \alpha\, u_{n-1}v_{n-1} + \beta\, \text{cov}_{n-1}$$

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Positive Finite Definite Condition

A variance-covariance matrix, Ω, is internally consistent if the positive semi-definite condition

$$\mathbf{w}^{\mathbf{T}}\Omega\,\mathbf{w} \geq 0$$

for all vectors **w**

Example

The variance covariance matrix

$$\begin{pmatrix} 1 & 0 & 0.9 \\ 0 & 1 & 0.9 \\ 0.9 & 0.9 & 1 \end{pmatrix}$$

is not internally consistent