Numerical Procedures

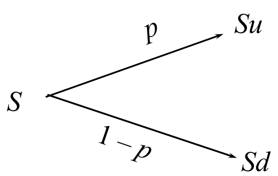
Chapter 18

Binomial Trees

- Binomial trees are frequently used to approximate the movements in the price of a stock or other asset
- In each small interval of time the stock price is assumed to move up by a proportional amount *u* or to move down by a proportional amount *d*

Movements in Time δt

(Figure 18.1)



1. Tree Parameters for a Nondividend Paying Stock

- We choose the tree parameters *p*, *u*, and *d* so that the tree gives correct values for the mean & standard deviation of the stock price changes in a risk-neutral world

$$e^{r\,\delta t} = pu + (1-p)d$$

$$\sigma^2 \delta t = pu^2 + (1-p)d^2 - [pu + (1-p)d]^2$$

- A further condition often imposed is $u = 1/d$

2. Tree Parameters for a Nondividend Paying Stock

(Equations 18.4 to 18.7)

When δt is small, a solution to the equations is

$$u = e^{\sigma\sqrt{\delta t}}$$

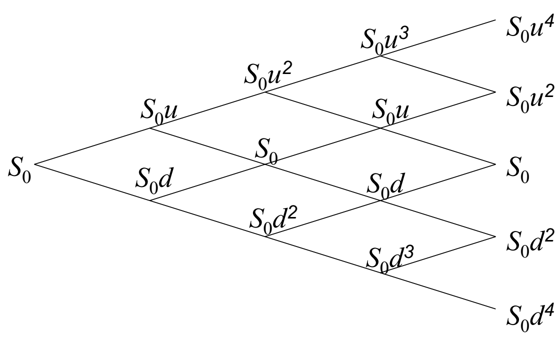
$$d = e^{-\sigma\sqrt{\delta t}}$$

$$p = \frac{a-d}{u-d}$$

$$a = e^{r\,\delta t}$$

The Complete Tree

(Figure 18.2)



Backwards Induction

- We know the value of the option at the final nodes
- We work back through the tree using risk-neutral valuation to calculate the value of the option at each node, testing for early exercise when appropriate

Example: Put Option

$S_0 = 50$; $X = 50$; $r = 10\%$; $\sigma = 40\%$;

$T = 5$ months $= 0.4167$;

$\delta t = 1$ month $= 0.0833$

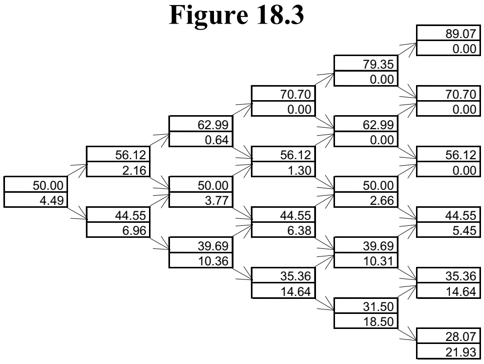
The parameters imply

$u = 1.1224$; $d = 0.8909$;

$a = 1.0084$; $p = 0.5076$

Example (continued)

Figure 18.3



Calculation of Delta

Delta is calculated from the nodes at time δt

$$\Delta = \frac{2.16 - 6.96}{56.12 - 44.55} = -0.41$$

Calculation of Gamma

Gamma is calculated from the nodes at time $2\delta t$

$$\Delta_1 = \frac{0.64 - 3.77}{62.99 - 50} = -0.24; \quad \Delta_2 = \frac{3.77 - 10.36}{50 - 39.69} = -0.64$$

$$\text{Gamma} = \frac{\Delta_1 - \Delta_2}{11.65} = 0.03$$

Calculation of Theta

Theta is calculated from the central nodes at times 0 and $2\delta t$

$$\text{Theta} = \frac{3.77 - 4.49}{0.1667} = -4.3 \text{ per year}$$

or -0.012 per calendar day

Calculation of Vega

- We can proceed as follows
- Construct a new tree with a volatility of 41% instead of 40%.
- Value of option is 4.62
- Vega is

  4.62 – 4.49 = 0.13

  per 1% change in volatility

Trees and Dividend Yields

- When a stock price pays continuous dividends at rate q we construct the tree in the same way but set $a = e^{(r-q)\delta t}$
- As with Black-Scholes:
  - For options on stock indices, q equals the dividend yield on the index
  - For options on a foreign currency, q equals the foreign risk-free rate
  - For options on futures contracts $q = r$

Binomial Tree for Dividend Paying Stock

- Procedure:
  - Draw the tree for the stock price less the present value of the dividends
  - Create a new tree by adding the present value of the dividends at each node
- This ensures that the tree recombines and makes assumptions similar to those when the Black-Scholes model is used

Extensions of Tree Approach

- Time dependent interest rates
- The control variate technique

Alternative Binomial Tree

Instead of setting $u = 1/d$ we can set each of the 2 probabilities to 0.5 and

$$u = e^{(r-\sigma^2/2)\delta t + \sigma\sqrt{\delta t}}$$

$$d = e^{(r-\sigma^2/2)\delta t - \sigma\sqrt{\delta t}}$$

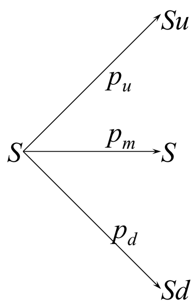
Trinomial Tree (Page 409)

$u = e^{\sigma\sqrt{3\delta t}} \qquad d = 1/u$

$$p_u = \sqrt{\frac{\delta t}{12\sigma^2}}\left(r - \frac{\sigma^2}{2}\right) + \frac{1}{6}$$

$$p_m = \frac{2}{3}$$

$$p_d = -\sqrt{\frac{\delta t}{12\sigma^2}}\left(r - \frac{\sigma^2}{2}\right) + \frac{1}{6}$$



Adaptive Mesh Model

- This is a way of grafting a high resolution tree on to a low resolution tree
- We need high resolution in the region of the tree close to the strike price and option maturity

Monte Carlo Simulation

When used to value European stock options, this involves the following steps:

1. Simulate 1 path for the stock price in a risk neutral world
2. Calculate the payoff from the stock option
3. Repeat steps 1 and 2 many times to get many sample payoff
4. Calculate mean payoff
5. Discount mean payoff at risk free rate to get an estimate of the value of the option

Sampling Stock Price Movements

(Equations 18.13 and 18.14, page 411)

- In a risk neutral world the process for a stock price is

$$dS = \hat{\mu} S\ dt + \sigma S\ dz$$

- We can simulate a path by choosing time steps of length δt and using the discrete version of this

$$\delta S = \hat{\mu} S\ \delta t + \sigma S\ \varepsilon\sqrt{\delta t}$$

where ε is a random sample from $\phi(0,1)$

A More Accurate Approach

(Equation 18.15, page 411)

Use $d \ln S = \left(\hat{\mu} - \sigma^2/2\right) dt + \sigma\, dz$

The discrete version of this is

$$\ln S(t+\delta t) - \ln S(t) = \left(\hat{\mu} - \sigma^2/2\right)\delta t + \sigma\varepsilon\sqrt{\delta t}$$

or

$$S(t+\delta t) = S(t)\, e^{\left(\hat{\mu} - \sigma^2/2\right)\delta t + \sigma\varepsilon\sqrt{\delta t}}$$

Extensions

When a derivative depends on several underlying variables we can simulate paths for each of them in a risk-neutral world to calculate the values for the derivative

Sampling from Normal Distribution (Page 412)

- One simple way to obtain a sample from $\phi(0,1)$ is to generate 12 random numbers between 0.0 & 1.0, take the sum, and subtract 6.0

To Obtain 2 Correlated Normal Samples

Obtain independent normal samples ε_1 and ε_2 and set

$$\varepsilon_1 = x_1$$

$$\varepsilon_2 = \rho x_1 + x_2\sqrt{1-\rho^2}$$

A procedure known a Cholesky's decomposition can be used when samples are required from more than two normal variables

Standard Errors in Monte Carlo Simulation

The standard error of the estimate of the option price is the standard deviation of the discounted payoffs given by the simulation trials divided by the square root of the number of observations.

Application of Monte Carlo Simulation

- Monte Carlo simulation can deal with path dependent options, options dependent on several underlying state variables, and options with complex payoffs
- It cannot easily deal with American-style options

Determining Greek Letters

For Δ:

1. Make a small change to asset price
2. Carry out the simulation again using the same random number streams
3. Estimate Δ as the change in the option price divided by the change in the asset price

Proceed in a similar manner for other Greek letters

Variance Reduction Techniques

- Antithetic variable technique
- Control variate technique
- Importance sampling
- Stratified sampling
- Moment matching
- Using quasi-random sequences

Representative Sampling Through the Tree

- We can sample paths randomly through a binomial or trinomial tree to value an option
- An alternative is to choose representative paths
- Paths are representative if the proportion of paths through each node is approximately equal to the probability of the node being reached

Finite Difference Methods

- Finite difference methods aim to represent the differential equation in the form of a difference equation
- Define $f_{i,j}$ as the value of f at time $i\delta t$ when the stock price is $j\delta S$

Finite Difference Methods (continued)

In $\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = r?$

we set $\frac{\partial f}{\partial S} = \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta S}$

$$\frac{\partial^2 f}{\partial S^2} = \left(\frac{f_{i,j+1} - f_{i,j}}{\Delta S} - \frac{f_{i,j} - f_{i,j-1}}{\Delta S} \right) \Big/ \Delta S \quad \text{or}$$

$$\frac{\partial^2 f}{\partial S^2} = \frac{f_{i,j+1} + f_{i,j-1} - 2f_{i,j}}{\Delta S^2}$$

Implicit Finite Difference Method (Equation 18.25, page 420)

If we also set $\frac{\partial f}{\partial t} = \frac{f_{i+1,j} - f_{i,j}}{\Delta t}$

we obtain the implicit finite difference method. This involves solving simultaneous equations of the form:

$$a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j}$$

Explicit Finite Difference Method (Equation 18.32, page 422)

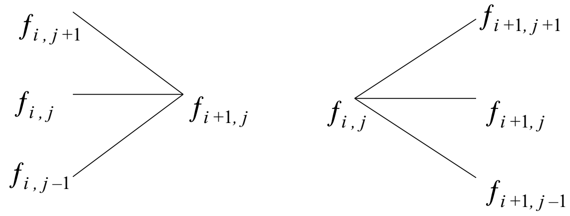
If $\partial f / \partial S$ and $\partial^2 f / \partial S^2$ are assumed to be the same at the $(i+1, j)$ point as they are at the (i, j) point we obtain the explicit finite difference method. This involves solving equations of the form:

$$f_{i,j} = a_j^* f_{i+1,j-1} + b_j^* f_{i+1,j} + c_j^* f_{i+1,j+1}$$

Implicit vs Explicit Finite Difference Method

- The explicit finite difference method is equivalent to the trinomial tree approach
- The implicit finite difference method is equivalent to a multinomial tree approach

Implicit vs Explicit Finite Difference Methods (Figure 18.16, page 422)



Implicit Method Explicit Method

Other Points on Finite Difference Methods

- It is better to have ln S rather than S as the underlying variable
- Improvements over the basic implicit and explicit methods:
  - –Hopscotch method
  - –Crank-Nicolson method

The Barone Adesi & Whaley Analytic Approximation for American Call Options

Appendix 18A, page 433)

$$C(S)=\begin{cases} c(S)+A_2\left(\frac{S}{S^*}\right)^{\gamma_2} & \text{when } S<S^* \\ S - X & \text{when } S\geq S^* \end{cases}$$

where A_2 & γ_2 are easily calculated constants & S^* is the solution to

$$S^*-X=c(S^*)+\left\{1-e^{-q(T-t)}N\left[d_1(S^*)\right]\right\}\frac{S^*}{\gamma_2}$$

The Barone Adesi & Whaley Analytic Approximation for American Put Options

$$P(S)=\begin{cases} p(S)+A_1\left(\frac{S}{S^{**}}\right)^{\gamma_1} & \text{when } S>S^{**} \\ X-S & \text{when } S\leq S^{**} \end{cases}$$

where A_1 & γ_1 are easily calculated constants & S^{**} is the solution to

$$X-S^{**}=p(S^{**})-\left\{1-e^{-q(T-t)}N\left[-d_1(S^{**})\right]\right\}\frac{S^{**}}{\gamma_1}$$