More on Models and Numerical Procedures

Chapter 20

Models to be Considered

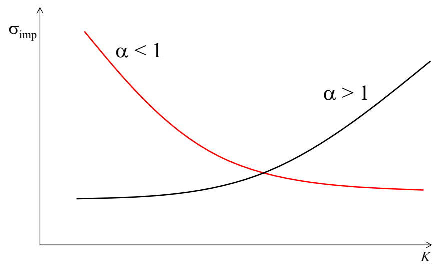
- Constant elasticity of variance (CEV)
- Jump diffusion
- Stochastic volatility
- Implied volatility function (IVF)

CEV Model (p456)

$$dS = (r - q)Sdt + \sigma S^{\alpha} dz$$

- When $\alpha = 1$ we have the Black-Scholes case
- When $\alpha > 1$ volatility rises as stock price rises
- When $\alpha < 1$ volatility falls as stock price rises

CEV Models Implied Volatilities



Jump Diffusion Model (page 457)

- Merton produced a pricing formula when the stock price follows a diffusion process overlaid with random jumps

$$dS / S = (\mu - \lambda k)dt + \sigma dz + dp$$

dp is the random jump
k is the expected size of the jump
$\lambda\, dt$ is the probability that a jump occurs in the next interval of length dt

Jumps and the Smile

- Jumps have a big effect on the implied volatility of short term options
- They have a much smaller effect on the implied volatility of long term options

Time Varying Volatility

- Suppose the volatility is σ_1 for the first year and σ_2 for the second and third
- Total accumulated variance at the end of three years is $\sigma_1^2 + 2\sigma_2^2$
- The 3-year average volatility is

$$3\bar{\sigma}^2 = \sigma_1^2 + 2\sigma_2^2; \quad \bar{\sigma} = \sqrt{\frac{\sigma_1^2 + 2\sigma_2^2}{3}}$$

Stochastic Volatility Models (page 458)

$$\frac{dS}{S} = (r-q)dt + \sqrt{V}\,dz_S$$
$$dV = a(V_L - V)dt + \xi V^{\alpha} dz_V$$

- When *V* and *S* are uncorrelated a European option price is the Black-Scholes price integrated over the distribution of the average variance

The IVF Model (page 460)

The impied volatility function model is designed to create a process for the asset price that exactly matches observed option prices. The usual model

$$dS = (r-q)Sdt + \sigma S dz$$

is replaced by

$$dS = [r(t) - q(t)]dt + \sigma(S,t)Sdz$$

The Volatility Function

The volatility function that leads to the model matching all European option prices is

$$[\sigma(K,t)]^2 = 2\frac{\partial c_{mkt}/\partial t + q(t)c_{mkt} + K[r(t)-q(t)]\partial c_{mkt}/\partial K}{K^2(\partial^2 c_{mkt}/\partial K^2)}$$

Strengths and Weaknesses of the IVF Model

- The model matches the probability distribution of stock prices assumed by the market at each future time
- The models does not necessarily get the joint probability distribution of stock prices at two or more times correct

Numerical Procedures

Topics:

- Path dependent options using trees
- Lookback options
- Barrier options
- Options where there are two stochastic variables
- American options using Monte Carlo

Path Dependence: The Traditional View

- Backwards induction works well for American options. It cannot be used for path-dependent options
- Monte Carlo simulation works well for path-dependent options; it cannot be used for American options

Extension of Backwards Induction

- Backwards induction can be used for some path-dependent options
- We will first illustrate the methodology using lookback options and then show how it can be used for Asian options

Lookback Example (Page 462)

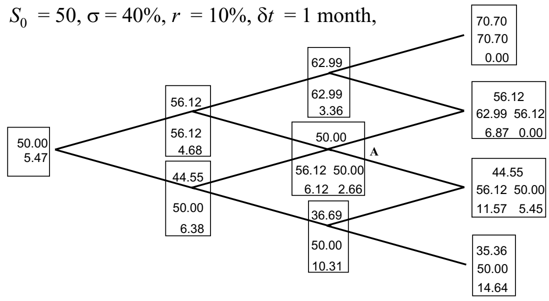
- Consider an American lookback put on a stock where

  S = 50, σ = 40%, r = 10%, δt = 1 month & the life of the option is 3 months
- Payoff is S_{max}-S_T
- We can value the deal by considering all possible values of the maximum stock price at each node

(This example is presented to illustrate the methodology. A more efficient ways of handling American lookbacks is in Section 20.6.)

Example: An American Lookback Put Option (Figure 20.2, page 463)

S_0 = 50, σ = 40%, r = 10%, δt = 1 month,



Why the Approach Works

This approach works for lookback options because

- The payoff depends on just 1 function of the path followed by the stock price. (We will refer to this as a "path function")
- The value of the path function at a node can be calculated from the stock price at the node & from the value of the function at the immediately preceding node
- The number of different values of the path function at a node does not grow too fast as we increase the number of time steps on the tree

Extensions of the Approach

- The approach can be extended so that there are no limits on the number of alternative values of the path function at a node
- The basic idea is that it is not necessary to consider every possible value of the path function
- It is sufficient to consider a relatively small number of representative values of the function at each node

Working Forward

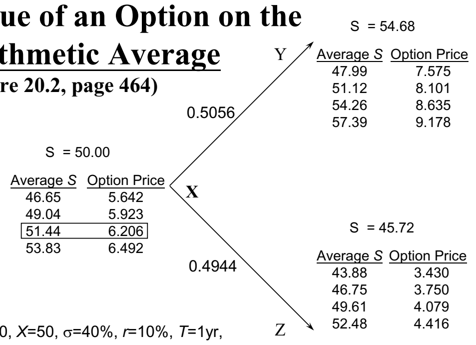
- First work forwards through the tree calculating the max and min values of the "path function" at each node
- Next choose representative values of the path function that span the range between the min and the max
  - Simplest approach: choose the min, the max, and *N* equally spaced values between the min and max

Backwards Induction

- We work backwards through the tree in the usual way carrying out calculations for each of the alternative values of the path function that are considered at a node
- When we require the value of the derivative at a node for a value of the path function that is not explicitly considered at that node, we use linear or quadratic interpolation

Part of Tree to Calculate Value of an Option on the Arithmetic Average

(Figure 20.2, page 464)



S=50, X=50, σ=40%, r=10%, T=1yr, δt=0.05yr. We are at time $4\delta t$

Part of Tree to Calculate Value of an Option on the Arithmetic Average (continued)

Consider Node **X** when the average of 5 observations is 51.44

Node **Y**: If this is reached, the average becomes 51.98. The option price is interpolated as 8.247

Node **Z**: If this is reached, the average becomes 50.49. The option price is interpolated as 4.182

Node **X**: value is

$(0.5056\times8.247 + 0.4944\times4.182)e^{-0.1\times0.05} = 6.206$

A More Efficient Approach for Lookbacks **(Section 20.6, page 465)**

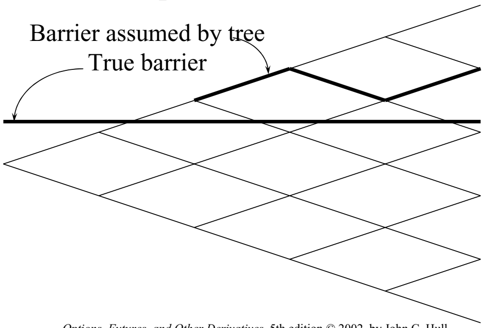
- Define $Y(t) = \frac{F(t)}{S(t)}$
  where $F(t)$ is the MAX stock price
- Construct a tree for $Y(t)$
- Use the tree to value the lookback option in "stock price units" rather than dollars

Using Trees with Barriers

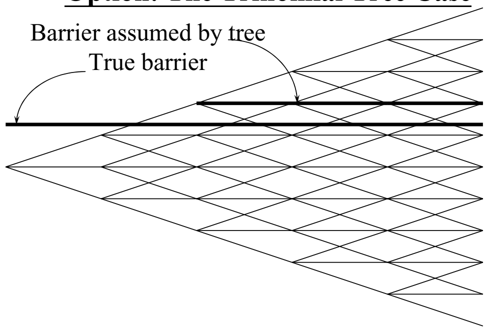
(Section 20.7, page 467)

- When trees are used to value options with barriers, convergence tends to be slow
- The slow convergence arises from the fact that the barrier is inaccurately specified by the tree

True Barrier vs Tree Barrier for a Knockout Option: The Binomial Tree Case



True Barrier vs Tree Barrier for a Knockout Option: The Trinomial Tree Case



Alternative Solutions to the Problem

- Ensure that nodes always lie on the barriers
- Adjust for the fact that nodes do not lie on the barriers
- Use adaptive mesh

In all cases a trinomial tree is preferable to a binomial tree

Modeling Two Correlated Variables

(Section 20.8, page 472)

APPROACHES:

1. Transform variables so that they are not correlated & build the tree in the transformed variables
2. Take the correlation into account by adjusting the position of the nodes
3. Take the correlation into account by adjusting the probabilities

Monte Carlo Simulation and American Options

- Two approaches:
  - The least squares approach
  - The exercise boundary parameterization approach
- Consider a 3-year put option where the initial asset price is 1.00, the strike price is 1.10, the risk-free rate is 6%, and there is no income

Sampled Paths

Path	t=0	t=1	t=2	t=3
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	0.93	0.97	0.92
5	1.00	1.11	1.56	1.52
6	1.00	0.76	0.77	0.90
7	1.00	0.92	0.84	1.01
8	1.00	0.88	1.22	1.34

The Least Squares Approach
(page 474)

- We work back from the end using a least squares approach to calculate the continuation value at each time
- Consider year 2. The option is in the money for five paths. These give observations on S of 1.08, 1.07, 0.97, 0.77, and 0.84. The continuation values are 0.00, $0.07e^{-0.06}$, $0.18e^{-0.06}$, $0.20e^{-0.06}$, and $0.09e^{-0.06}$

The Least Squares Approach
continued

- Fitting a model of the form $V=a+bS+cS^2$ we get a best fit relation

$$V=-1.070+2.983S-1.813S^2$$

  for the continuation value V
- This defines the early exercise decision at t=2. We carry out a similar analysis at t=1

The Least Squares Approach
continued

In practice more complex functional forms can be used for the continuation value and many more paths are sampled

The Early Exercise Boundary Parametrization Approach (page 477)

- We assume that the early exercise boundary can be parameterized in some way
- We carry out a first Monte Carlo simulation and work back from the end calculating the optimal parameter values
- We then discard the paths from the first Monte Carlo simulation and carry out a new Monte Carlo simulation using the early exercise boundary defined by the parameter values.

Application to Example

- We parameterize the early exercise boundary by specifying a critical asset price, S^*, below which the option is exercised.
- At t=1 the optimal S^* for the eight paths is 0.88. At t=2 the optimal S^* is 0.84
- In practice we would use many more paths to calculate the S^*