Martingales and Measures

Chapter 21

Derivatives Dependent on a Single Underlying Variable

Consider a variable, θ, (not necessaril y the price of a traded security) that follows the process

$$\frac{d\theta}{\theta} = m\,dt + s\,dz$$

Imagine two derivative s dependent on θ with prices f_1 and f_2. Suppose

$$\frac{df_1}{f_1} = \mu_1\,dt + \sigma_1\,dz$$

$$\frac{df_2}{f_2} = \mu_2\,dt + \sigma_2\,dz$$

Forming a Riskless Portfolio

We can set up a riskless portfolio Π, consisting of

$+\sigma_2 f_2$ of the 1st derivative and

$-\sigma_1 f_1$ of the 2nd derivative

$$\Pi = (\sigma_2 f_2) f_1 - (\sigma_1 f_1) f_2$$

$$\delta\Pi = (\mu_1\sigma_2 f_1 f_2 - \mu_2\sigma_1 f_1 f_2)\delta t$$

Market Price of Risk (Page 485)

Since the portfolio is riskless : $\delta\Pi = r\,\Pi\delta t$

This gives : $\mu_1\sigma_2 - \mu_2\sigma_1 = r\,\sigma_2 - r\,\sigma_1$

or $$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}$$

- This shows that $(\mu - r)/\sigma$ is the same for all derivatives dependent on the same underlying variable, θ
- We refer to $(\mu - r)/\sigma$ as the market price of risk for θ and denote it by λ

Extension of the Analysis to Several Underlying Variables

(Equations 21.12 and 21.13, page 487)

If f depends on several underlying variables with

$$\frac{df}{f} = \mu\,dt + \sum_{i=1}^{n}\sigma_i\,dz_i$$

then

$$\mu - r = \sum_{i=1}^{n}\lambda_i\sigma_i$$

Martingales (Page 488)

- A martingale is a stochastic process with zero drift
- A variable following a martingale has the property that its expected future value equals its value today

Alternative Worlds

In the traditional risk-neutral world

$$df = rf\,dt + \sigma f\,dz$$

In a world where the market price of risk is λ

$$df = (r + \lambda\sigma) f\,dt + \sigma f\,dz$$

A Key Result (Page 489)

If we set λ equal to the volatility of a security g, then Ito's lemma shows that f/g is a martingale for all derivative security prices f (f and g are assumed to provide no income during the period under consideration)

Forward Risk Neutrality

We refer to a world where the market price of risk is the volatility of g as a world that is forward risk neutral with respect to g.

If E_g denotes a world that is FRN wrt g

$$\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right)$$

Aleternative Choices for the Numeraire Security g

- Money Market Account
- Zero-coupon bond price
- Annuity factor

Money Market Account as the Numeraire

- The money market account is an account that starts at $1 and is always invested at the short-term risk-free interest rate
- The process for the value of the account is

$$dg = rg\,dt$$

- This has zero volatility. Using the money market account as the numeraire leads to the traditional risk-neutral world

Money Market Account continued

Since $g_0 = 1$ and $g_T = e^{\int_0^T r\,dt}$, the equation

$$\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right)$$

becomes

$$f_0 = \hat{E}\left[e^{-\int_0^T r\,dt} f_T\right]$$

where $\hat{E}$ denotes expectations in the traditional risk-neutral world

Zero-Coupon Bond Maturing at time T as Numeraire

The equation

$$\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right)$$

becomes

$$f_0 = P(0,T)E_T[f_T]$$

where $P(0,T)$ is the zero-coupon bond price and E_T denotes expectations in a world that is FRN wrt the bond price

Forward Prices

In a world that is FRN wrt $P(0,T)$, the expected value of a security at time T is its forward price

Interest Rates

In a world that is FRN wrt $P(0,T_2)$ the expected value of an interest rate lasting between times T_1 and T_2 is the forward interest rate

Annuity Factor as the Numeraire

The equation

$$\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right)$$

becomes

$$f_0 = A(0)E_A\left[\frac{f_T}{A(T)}\right]$$

Annuity Factors and Swap Rates

Suppose that $s(t)$ is the swap rate corresponding to the annuity factor A.

Then:

$$s(t)=E_A[s(T)]$$

Extension to Several Independent Factors (Page 492)

In the traditional risk-neutral world

$$df(t) = r(t)f(t)dt + \sum_{i=1}^{m}\sigma_{f,i}(t)f(t)dz_i$$

$$dg(t) = r(t)g(t)dt + \sum_{i=1}^{m}\sigma_{g,i}(t)g(t)dz_i$$

For other worlds that are internally consistent

$$df(t) = \left[r(t) + \sum_{i=1}^{m}\lambda_i\sigma_{f,i}(t)\right]f(t)dt + \sum_{i=1}^{m}\sigma_{f,i}(t)f(t)dz_i$$

$$dg(t) = \left[r(t) + \sum_{i=1}^{m}\lambda_i\sigma_{g,i}(t)\right]g(t)dt + \sum_{i=1}^{m}\sigma_{g,i}(t)g(t)dz_i$$

Extension to Several Independent Factors
continued

We define a world that is FRN wrt g as world where $\lambda_i = \sigma_{g,i}$

As in the one-factor case, f/g is a martingale and the rest of the results hold.

Applications
(Section 21.6, page 493)

- Valuation of a European call option when interest rates are stochastic
- Valuation of an option to exchange one asset for another

Change of Numeraire
(Section 21.7, page 495)

When we change the numeraire security from g to h, the drift of a variable v increases by $\rho\sigma_v\sigma_q$ where σ_v is the volatility of v, $w = h/g$, σ_q is the volatility of w, and ρ is the correlation between v and w

Quantos
(Section 21.8, page 497)

- Quantos are derivatives where the payoff is defined using variables measured in one currency and paid in another currency
- Example: contract providing a payoff of $S_T - K$ dollars ($) where S is the Nikkei stock index (a yen number)

Diff Swap

- Diff swaps are a type of quanto
- A floating rate is observed in one currency and applied to a principal in another currency

Quantos continued

When we move from a forward risk neutral world in currency Y to a forward risk neutral world in currency X (both being wrt to zero-coupon bonds maturing at time T), the growth rate of a variable V increases by

$$\rho\sigma_F\sigma_G$$

where σ_F is the volatility of the forward value of V, σ_G is the volatility of the forward exchange rate (units of Y per unit of X), and ρ is the coefficient of correlation between the two

Quantos continued

When we move from the traditional risk neutral world in currency Y to the traditional risk neutral world in currency X, the growth rate of a variable V increases by

$$\rho\sigma_V\sigma_S$$

where σ_V is the volatility of V, σ_S is the volatility of the exchange rate (units of Y per unit of X), and ρ is the coefficient of correlation between the two

Siegel's Paradox

An exchange rate S (units of currency Y per untit of currency X) follows the risk-neutral process

$$dS = [r_Y - r_X]Sdt + \sigma_S Sdz$$

This implies from Ito's lemma that

$$d(1/S) = [r_X - r_Y + \sigma_S^2](1/S)dt - \sigma_S(1/S)dz$$

Given that the process for S has a drift rate of $r_Y - r_X$, we expect the process for $1/S$ to have a drift of $r_X - r_Y$.

What is going on here?