Interest Rate Derivatives: The Standard Market Models

Chapter 22

Why Interest Rate Derivatives are Much More Difficult to Value Than Stock Options

- We are dealing with the whole term structure of interest rates; not a single variable
- The probabilistic behavior of an individual interest rate is more complicated than that of a stock price

Why Interest Rate Derivatives are Much More Difficult to Value Than Stock Options

- Volatilities of different points on the term structure are different
- Interest rates are used for discounting as well as for defining the payoff

Main Approaches to Pricing Interest Rate Options

- Use a variant of Black's model
- Use a no-arbitrage (yield curve based) model

Black's Model & Its Extensions

- Black's model is similar to the Black-Scholes model used for valuing stock options
- It assumes that the value of an interest rate, a bond price, or some other variable at a particular time T in the future has a lognormal distribution

Black's Model & Its Extensions (continued)

- The mean of the probability distribution is the forward value of the variable
- The standard deviation of the probability distribution of the log of the variable is

$$\sigma\sqrt{T}$$

where σ is the volatility
- The expected payoff is discounted at the T-maturity rate observed today

Black's Model (Eqn 22.1 and 22.2, p 509)

$$c = P(0,T)[F_0N(d_1) - KN(d_2)]$$
$$p = P(0,T)[KN(-d_2) - F_0N(-d_1)]$$
$$d_1 = \frac{\ln(F_0/K) + \sigma^2T/2}{\sigma\sqrt{T}}; d_2 = d_1 - \sigma\sqrt{T}$$

- K : strike price
- F_0 : forward value of variable
- T : option maturity
- s : volatility

The Black's Model: Payoff Later Than Variable Being Observed

$$c = P(0,T^*)[F_0N(d_1) - KN(d_2)]$$
$$p = P(0,T^*)[KN(-d_2) - F_0N(-d_1)]$$
$$d_1 = \frac{\ln(F_0/K) + \sigma^2T/2}{\sigma\sqrt{T}}; d_2 = d_1 - \sigma\sqrt{T}$$

- K : strike price
- F_0 : forward value of variable
- s : volatility
- T : time when variable is observed
- $T*$: time of payoff

Validity of Black's Model

Black's model appears to make two approximations:

1. The expected value of the underlying variable is assumed to be its forward price

2. Interest rates are assumed to be constant for discounting

We will see that these assumptions offset each other

European Bond Options

- When valuing European bond options it is usual to assume that the future bond price is lognormal
- We can then use Black's model (equations 22.1 and 22.2)
- Both the bond price and the strike price should be cash prices not quoted prices

Yield Vols vs Price Vols

The change in forward bond price is related to the change in forward bond yield by

$$\frac{\delta B}{B} \approx -D\,\delta y \text{ or } \frac{\delta B}{B} \approx -Dy\frac{\delta y}{y}$$

where D is the (modified) duration of the forward bond at option maturity

Yield Vols vs Price Vols

continued

- This relationship implies the following approximation

$$\sigma = Dy_0\sigma_y$$

  where σ_y is the yield volatility and σ is the price volatility, y_0 is today's forward yield

- Often σ_y is quoted with the understanding that this relationship will be used to calculate σ

Theoretical Justification for Bond Option Model

Working in a world that is FRN wrt a zero-coupon bond maturing at time T, the option price is

$$P(0,T)E_T[\max(B_T - K, 0)]$$

Also

$$E_T[B_T] = F_0$$

This leads to Black's model

Caps

- A cap is a portfolio of caplets
- Each caplet can be regarded as a call option on a future interest rate with the payoff occurring in arrears
- When using Black's model we assume that the interest rate underlying each caplet is lognormal

Black's Model for Caps

(Equation 22.11, p. 517)

- The value of a caplet, for period $[t_k, t_{k+1}]$ is

$$L\delta_k P(0,t_{k+1})[F_k N(d_1) - R_K N(d_2)]$$

where $d_1 = \dfrac{\ln(F_k / R_K) + \sigma_k^2 t_k / 2}{\sigma_k \sqrt{t_k}}$ and $d_2 = d_1 - \sigma\sqrt{t_k}$

- F_k : forward interest rate for (t_k, t_{k+1})
- L: principal
- R_K : cap rate
- s_k : interest rate volatility
- $d_k = t_{k+1} - t_k$

When Applying Black's Model To Caps We Must ...

- EITHER
  - Use forward volatilities
  - Volatility different for each caplet
- OR
  - Use flat volatilities
  - Volatility same for each caplet within a particular cap but varies according to life of cap

Theoretical Justification for Cap Model

Working in a world that is FRN wrt a zero-coupon bond maturing at time t_{k+1} the option price is

$$P(0,t_{k+1})E_{k+1}[\max(R_k - R_X, 0)]$$

Also

$$E_{k+1}[R_k] = F_k$$

This leads to Black's model

European Swaptions

- When valuing European swap options it is usual to assume that the swap rate is lognormal
- Consider a swaption which gives the right to pay s_K on an n-year swap starting at time T. The payoff on each swap payment date is

$$\frac{L}{m}\max(s_T - s_K, 0)$$

where L is principal, m is payment frequency and s_T is market swap rate at time T

European Swaptions continued

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(Equation 22.13, page 545)

The value of the swaption is

$$LA[s_0 N(d_1) - s_K N(d_2)]$$

where $d_1 = \dfrac{\ln(s_0/s_K) + \sigma^2 T/2}{\sigma\sqrt{T}}; d_2 = d_1 - \sigma\sqrt{T}$

s_0 is the forward swap rate; σ is the swap rate volatility; t_i is the time from today until the i th swap payment; and

$$A = \frac{1}{m}\sum_{i=1}^{mn} P(0, t_i)$$

Theoretical Justification for Swap Option Model

22.20

Working in a world that is FRN wrt the annuity underlying the swap, the option price is

$$LAE_A[\max(s_T - s_K, 0)]$$

Also

$$E_A[s_T] = s_0$$

This leads to Black' s model

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Relationship Between Swaptions and Bond Options

- An interest rate swap can be regarded as the exchange of a fixed-rate bond for a floating-rate bond
- A swaption or swap option is therefore an option to exchange a fixed-rate bond for a floating-rate bond

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Relationship Between Swaptions and Bond Options (continued)

- At the start of the swap the floating-rate bond is worth par so that the swaption can be viewed as an option to exchange a fixed-rate bond for par
- An option on a swap where fixed is paid and floating is received is a put option on the bond with a strike price of par
- When floating is paid and fixed is received, it is a call option on the bond with a strike price of par

Convexity Adjustments

22.23

- We define the forward yield on a bond as the yield calculated from the forward bond price
- There is a non-linear relation between bond yields and bond prices
- It follows that when the forward bond price equals the expected future bond price, the forward yield does not necessarily equal the expected future yield
- What is known as a convexity adjustment may be necessary to convert a forward yield to the appropriate expected future yield

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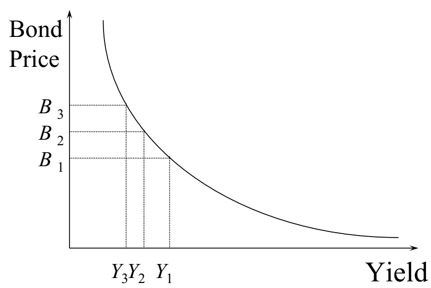
Relationship Between Bond Yields and Prices (Figure 22.4, page 525)



Analytic Approximation for Convexity Adjustment (Eqn 22.15, p. 525)

- Suppose a derivative depends on a bond yield, y_T observed at time T. Define:
- $G(y_T)$: price of the bond as a function of its yield

  y_0 : forward bond yield at time zero

  σ_y : forward yield volatility
- The convexity adjustment that should be made to the forward bond yield is

$$-\frac{1}{2} y_0^2 \sigma_y^2 T \frac{G''(y_0)}{G'(y_0)}$$

Convexity Adjustment for Swap Rate

The same formula gives the convexity adjustment for a forward swap rate. In this case $G(y)$ defines the relationship between price and yield for a bond that pays a coupon equal to the forward swap rate

Example 22.5 (page 526)

- An instrument provides a payoff in 3 years equal to the 1-year zero-coupon rate multiplied by \$1000
- Volatility is 20%
- Yield curve is flat at 10% (with annual compounding)
- The convexity adjustment is 10.9 bps so that the value of the instrument is $101.09/1.1^3 = 75.95$

Example 22.6 (Page 527)

- An instrument provides a payoff in 3 years = to the 3-year swap rate multiplied by \$100
- Payments are made annually on the swap
- Volatility is 22%
- Yield curve is flat at 12% (with annual compounding)
- The convexity adjustment is 36 bps so that the value of the instrument is $12.36/1.12^3 = 8.80$

Timing Adjustments

When a variable is observed at time T_1 and the resultant payoff occurs at time T_2 rather than T_1, the growth rate of the variable should be increased by

$$-\frac{\rho\sigma_F\sigma_R R_0(T_2-T_1)}{1+R_0/m}$$

where R is the forward interest rate between T_1 and T_2 expressed with a compounding frequency of m, σ_R is the volatility of R, R_0 is the value of R today, F is the forward value of the variable for a contract maturing at time T_1,, σ_F is the volatility of F, and ρ is the correlation between R and F

When is a Convexity or Timing Adjustment Necessary

- A convexity or timing adjustment is necessary when the payoff from a derivative does not incorporate the natural time lags between an interest rate being set and the interest payments being made
- They are not necessary for a vanilla swap, a cap or a swap option

Deltas of Interest Rate Derivatives

Alternatives:

- Calculate a DV01 (the impact of a 1bps parallel shift in the zero curve)
- Calculate impact of small change in the quote for each instrument used to calculate the zero curve
- Divide zero curve (or forward curve) into buckets and calculate the impact of a shift in each bucket
- Carry out a principal components analysis. Calculate delta with respect to each of the first few factors factors