Interest Rate Derivatives: Models of the Short Rate

Chapter 23

Term Structure Models

- Black's model is concerned with describing the probability distribution of a single variable at a single point in time
- A term structure model describes the evolution of the whole yield curve

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Use of Risk-Neutral Arguments

- The process for the instantaneous short rate, *r,* in the traditional risk-neutral world defines the process for the whole zero curve in this world
- If $P(t, T)$ is the price at time t of a zero-coupon bond maturing at time T

$$P(t,T) = \hat{E}\left[e^{-\bar{r}(T-t)}\right]$$

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Equilibrium Models

Rendleman & Bartter:

$$dr = \mu r\, dt + \sigma r\, dz$$

Vasicek:

$$dr = a(b-r)\, dt + \sigma dz$$

Cox, Ingersoll, & Ross (CIR):

$$dr = a(b-r)\, dt + \sigma\sqrt{r}\, dz$$

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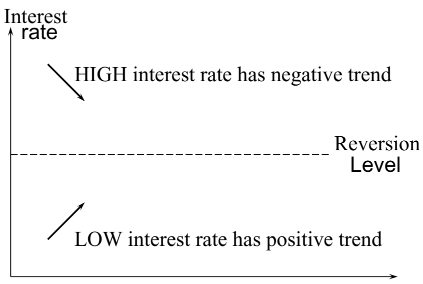
Mean Reversion

(Figure 23.1, page 539)



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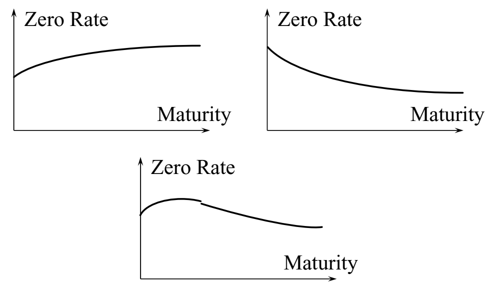
Alternative Term Structures in Vasicek & CIR

(Figure 23.2, page 540)



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Equilibrium vs No-Arbitrage Models

- In an equilibrium model today's term structure is an output
- In a no-arbitrage model today's term structure is an input

Developing No-Arbitrage Model for *r*

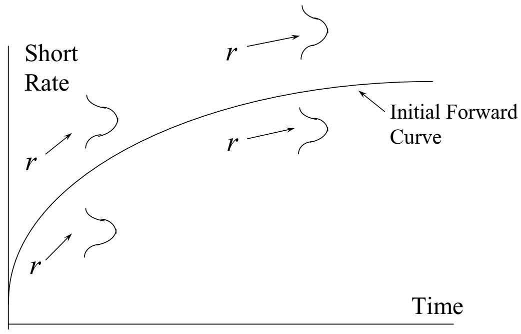
A model for r can be made to fit the initial term structure by including a function of time in the drift

Ho and Lee

$$dr = \theta(t)dt + \sigma dz$$

- Many analytic results for bond prices and option prices
- Interest rates normally distributed
- One volatility parameter, σ
- All forward rates have the same standard deviation

Diagrammatic Representation of Ho and Lee

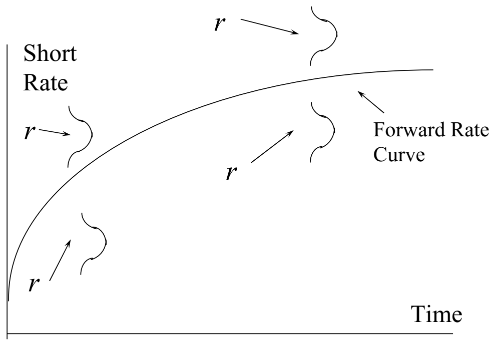


Hull and White Model

$$dr = [\theta(t) - ar]dt + \sigma dz$$

- Many analytic results for bond prices and option prices
- Two volatility parameters, a and σ
- Interest rates normally distributed
- Standard deviation of a forward rate is a declining function of its maturity

Diagrammatic Representation of Hull and White



Options on Coupon Bearing Bonds

- A European option on a coupon-bearing bond can be expressed as a portfolio of options on zero-coupon bonds.
- We first calculate the critical interest rate at the option maturity for which the coupon-bearing bond price equals the strike price at maturity
- The strike price for each zero-coupon bond is set equal to its value when the interest rate equals this critical value

Interest Rate Trees vs Stock Price Trees

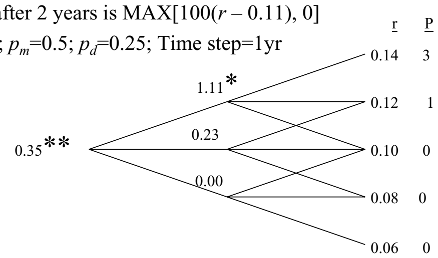
- The variable at each node in an interest rate tree is the δt-period rate
- Interest rate trees work similarly to stock price trees except that the discount rate used varies from node to node

Two-Step Tree Example

(Figure 23.6, page 551))

Payoff after 2 years is MAX[100(r – 0.11), 0]

p_u=0.25; p_m=0.5; p_d=0.25; Time step=1yr

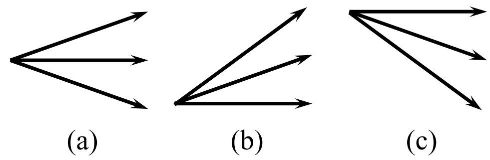


*: $(0.25\times3 + 0.50\times1 + 0.25\times0)e^{-0.12\times1}$

**: $(0.25\times1.11 + 0.50\times0.23 + 0.25\times0)e^{-0.10\times1}$

Alternative Branching Processes in a Trinomial Tree

(Figure 23.7, page 552)



An Overview of the Tree Building Procedure

$dr = [\theta(t) - ar]dt + \sigma dz$

1. Assume $\theta(t) = 0$ and $r(0) = 0$
2. Draw a trinomial tree for r to match the mean and standard deviation of the process for r
3. Determine $\theta(t)$ one step at a time so that the tree matches the initial term structure

Example

$\sigma = 0.01$

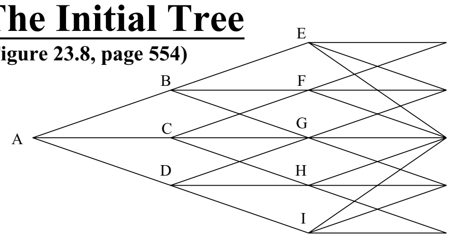
$a = 0.1$

$\delta t = 1$ year

The zero curve is as shown in Table 23.1 on page 556

The Initial Tree

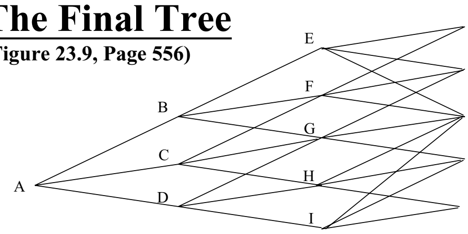
(Figure 23.8, page 554)



Node	A	B	C	D	E	F	G	H	I
r	0.000%	1.732%	0.000%	-1.732%	3.464%	1.732%	0.000%	-1.732%	-3.464%
p_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

The Final Tree

(Figure 23.9, Page 556)



Node	A	B	C	D	E	F	G	H	I
r	3.824%	6.937%	5.205%	3.473%	9.716%	7.984%	6.252%	4.520%	2.788%
p_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

Extensions

The tree building procedure can be extended to cover more general models of the form:

$$df(r) = [\theta(t) - a\,f(r)]dt + \sigma dz$$

Other Models

Black, Derman, and Toy :

$$d\ln r = \left[\theta(t) + \frac{\sigma'(t)}{\sigma(t)}\ln(r)\right]dt + \sigma(t)dz$$

Black and Karasinski :

$$d\ln r = [\theta(t) - a(t)\ln(r)]dt + \sigma(t)dz$$

- These models allow the initial volatility environment to be matched exactly
- But the future volatility structure may be quite different from the current volatility structure

Calibration: *a* and σ constant

- The volatility parameters a and σ are chosen so that the model fits the prices of actively traded instruments such as caps and European swap options as closely as possible
- We can choose a global best fit value of a and imply σ from the prices of actively traded instruments. This creates a volatility surface for interest rate derivatives similar to that for equity option or currency options (see Chapter 15)

Calibration: *a* and σ functions of time

- We minimize a function of the form

$$\sum_{i=1}^{n}(U_i - V_i)^2 + P$$

where U_i is the market price of the ith calibrating instrument, V_i is the model price of the ith calibrating instrument and P is a function that penalizes big changes or curvature in a and σ