CH12. Pricing Derivative Product

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Outline

- Formimg risk-free portfolios
- Boundary condition
- Classification of PDE

1. Formimg risk-free portfolios

F_T : the price of derivative contract at maturity T

$F(S_t,t)$: the price of derivative contract at time t.

P_t : dollars be invested in a combination of $F(S_t,t)$ and S_t

$$P_t = \theta_1 F(S_t,t) + \theta_2 S_t$$

θ_1 and θ_2 represent portfolio weights.

Taking θ_1 and θ_2 are constant, we can write this change as

$$dP_t = \theta_1 dF_t + \theta_2 dS_t \qquad (1)$$

Assume:

$$dS_t = a(S_t,t)dt + \sigma(S_t,t)dW_t$$

$$dF_t = F_t dt + \frac{1}{2}F_{ss}\sigma_t^2 dt + F_s dS_t$$

$$\Rightarrow \quad dP_t = \theta_1\left[F_t dt + \frac{1}{2}F_{ss}\sigma_t^2 dt + F_s dS_t\right] + \theta_2 dS_t$$

$$Let \quad \theta_1 = 1 \qquad \theta_2 = -F_s$$

$$\Rightarrow \quad dP_t = F_t dt + \frac{1}{2}F_{ss}\sigma_t^2 dt$$

Given the information set I_t, in this expression, there is no random term. It means that the portfolio P_t is risk-free.

Assuming that the risk-free interest rate is given by r, the expected capital gains must equal

$$rP_t dt$$

$$\Rightarrow \quad rP_t dt = F_t dt + \frac{1}{2}F_{ss}\sigma_t^2 dt$$

$$\Rightarrow \quad r\left[F(S_t,t) - F_s S_t\right] = F_t + \frac{1}{2}F_{ss}\sigma_t^2$$

$$\Rightarrow \quad F_t + rF_s S_t + \frac{1}{2}F_{ss}\sigma_t^2 - rF = 0 \qquad (2)$$

In addition, we know at expiration that the price of the derivative product is given by

$$F(S_T,T) = G(S_T,T)$$

where $G(\cdot)$ is a known function of S_T and T. For example, in the case of a call option, $G(\cdot)$, the expiration price of the call with a strike price K is

$$G(S_T,T) = Max[S_T - K] \qquad \rightarrow (3)$$

Equation (2) is known as a partial differential equation (PDE). Equation (3) is an associated boundary condition.

2.Boundary condition

In physics, boundary conditions are initial or terminal states of some physical phenomenon that envoles over time according to the PDE.
In the case of options, option prices must satisify the equation (3).
If there are no boundary conditions, then finding price function $F(S_t,t)$ that satisfy a given PDE will, in general, not be possible.

3.Classification of PDE

First of all, PDEs can be linear or nonlinear. If an equation is a linear combination of F and its partial derivatives, it is called a linear PDE.
The second type of classfication has to do with the order of differentiation.
The third type of classficationis is specific to PDEs. It can also be classified as elliptic, parabolic, or hyperbolic. The PDE we encounter in finance are similar to parabolic PDEs.

4.1 Example 1:Linear First-order PDE

$F_t + F_s = 0$

What does this function $F(S_t, t)$look like?

We can guess a solution:

Let

$$\begin{cases} F_t = -\alpha \\ F_s = \alpha \end{cases}$$

$\Rightarrow \quad F(S_t, t) = \alpha S_t - \alpha t + \beta$

if we know at time t=5, we have $F(S_5, 5) = 6 - 2S_5$

$\Rightarrow \quad \alpha S_5 - 5\alpha + \beta = 6 - 2S_5$

$\therefore \quad \alpha = -2 \quad , \beta = -4.$

Example2. Linear Second-order PDE

Now consider a second-order PDE

$$\frac{\partial^2 F}{\partial t^2} = 0.3\frac{\partial^2 F}{\partial S_t^2}$$

What could this function be?

Consider ths formula

$$\mathrm{F}(\mathrm{S}_t,\mathrm{t}) = \frac{1}{2}\alpha\left(S_t - S_0\right)^2 + \frac{0.3}{2}(t-t_0)^2 + \beta\left(S_t - S_0\right)(t-t_0)$$

Given the boundary condition:

$\mathrm{F}(10,\mathrm{t}) = 100 + t^2 \quad , \quad \mathrm{F}(\mathrm{S}_0, 0) = 50 + S_0^2$

$\Rightarrow \quad \alpha = -20, \ \beta = 0, \ \mathrm{S}_0 = 4, \ \mathrm{t}_0 = 2$

5. Bivariate, second-degree Equations

In this section we briefly review this aspect of analytical geometry, since it relates to this terminology concerning PDEs.
Let x, y denote two deterministic variables.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

The equation is of the second degree because the highest power of x or of y, is a square.

5.1 Example

Consider the second-degree equaion

$$9x^2 + 16y^2 - 54x - 64y + 3455 = 0$$

$$\Rightarrow 9(x-3)^2 + 16(y-2)^2 = 3600$$

This is the formula of an ellipse.

5.1 Example2

if

$$Ax^2 + Dx + Ey + F = 0$$

It is the general equation for a parabola.

if

$$(x - x_0)^2 + (y - y_0)^2 = R$$

It is the general equation for a circle in the x, y plane.

6.conclusion

We discussed briefly various forms of PDEs and introduced the related terminology.
This chapter also showed that the relationship between finanical derivatives and the underlying asset can be exploited to obtain PDEs that derivative asset prices must satisfy.