The Black-Scholes PDE

An Application

- g926420
- 楊子宸(Victor)

Outline

- The Black-Sholes PDE
- PDEs in Asset Pricing:
  1. A Second Factor:Constant dividends
  2. A Random Second Factor
- Solving PDEs in practice:
  1.Closed-Form Solutions
  2.Numerical Solutions
- Exotic Options

The Black-Scholes PDE

- Let F be the price of a derivative written on the underlying asset S_t
- The underlying security did not pay a dividend.
- The risk-free interest rate was assumed to be constant at r.

$$dS = a(S_t,t)dt + \sigma(S_t,t)dW$$

$$a(S_t,t) = \mu \cdot S_t$$

$$\sigma(S_t,t) = \sigma \cdot S_t$$

$$t \in [0,\infty)$$

$$dF = \frac{\partial F}{\partial S}dS + \frac{\partial F}{\partial t}dt + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}(dS)^2$$

By combining the two equation, we can get the following equation.

- By Ito Lemma we can get that

$$-rF + r\frac{\partial F}{\partial S}\cdot S_t + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\cdot \sigma^2 \cdot S_t^{\ 2} = 0$$

$$0 \le S_t, and\, 0 \le t \le T$$

$$F(T) = \max[S_T - K, 0]$$

Solution of the Black-Scholes PDE

- Black and Scholes solve this PDE and obtain the form of the function

$$F(S_t,t)=S_t\cdot N(d_1)-Ke^{-r(T-t)}\cdot N(d_2)$$

$$d_1=\frac{\ln(S_t/K)+(r+\frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2=d_1-\sigma\sqrt{T-t}$$

PDEs in Asset Pricing

Assumptions of Black-Scholes PDE:

1. The underlying asset is a stock.
2. The stock does not pay any dividends.
3. The derivative asset is a European-style call option that cannot be exercised before the expiration date.
4. The risk-free rate is constant

5. There are no indivisibilities or transaction costs such as commissions and bid-ask spreads.

In most applications of derivatives asset pricing

- One or more of these assumptions will be violated.
- If so, in general the Black-Scholes PDE will not apply, and a new PDE should be found.

EX: If the option is American-style.

Constant Dividends

- Portfolio:

$$P_t=\theta_1 F(S_t,t)+\theta_2\cdot S_t$$

$$t\in[0.T]$$

$$\theta_1=1,\theta_2=-F_s$$

$$dP_t=dF+F_s dS_t$$

$$dP_t=F_t dt+\frac{1}{2}F_{ss}\sigma_t^2 dt \quad **$$

Now, the underlying stock pays a known dividend at a rate of δ

Dollars per time that is predictable

- Hence, the capital gains plus the dividends received must equal the earnings of a risk-free portfolio.

$$dP_t + \delta dt = r \cdot P_t \cdot dt$$

Putting this together with **, we get a slightly different PDE:

$$rF - rF_s S_t - \delta - F_t - \frac{1}{2} F_{ss} \sigma_t^{\ 2} = 0$$

There is now a constant term δ. Hence, stocks paying dividends at a constant rate δ do not present a major problem.

A Random Second Factor

- The difficulty with a second factor begins if this factor contains "unpredictable" random components of its own. With a constant rate of dividends, the dividends earnings dD_t at time t are given by

$$dD_t = \delta dt$$

- If dividends depend on some Wiener process, we would instead have:

$$dD_t = a^* dt + \sigma^* dW_t^*$$

Where a^*, σ^* are constants, and where dW_t^* represents increments in a Wiener process W_t^*

- The derivative asset price F could now depend on D_t directly and must be written as

$$F(t) = F(S_t, D_t, t)$$

$$t \in [0, T]$$

$$dF(t) = F_t dt + F_s dS_t + F_D dD_t + \frac{1}{2} F_{ss} (dS_t)^2 + \frac{1}{2} F_{DD} (dD_t)^2 + F_{Ds} dD_t dS_t$$

$$dF(t) = F_t dt + F_s dS_t + F_D dD_t + \frac{1}{2} F_{ss} \sigma_t^{\ 2} dt + \frac{1}{2} F_{DD} \sigma^{*2} dt$$

The increment in the value of the portfolio formed using S_t and the call option will be given by

$$dP_t = \theta_1 dS(t) + \theta_2[F_t dt + F_s dS_t + F_D dD_t$$
$$+ \frac{1}{2}F_{ss}\sigma_t^2 dt + \frac{1}{2}F_{DD}\sigma^{*2} dt]$$

If the choice $\theta_1 = -F_s$

$\theta_2 = 1$

is made, this would eliminate the random term involving dS_t

But dD_t would remain. So this particular choice of Θ_1, Θ_2 is not sufficient to make dP_t nonrandom.

An Exception

$$dD_t = a_t^* dt + \sigma_t^* dW_t$$
$$dS_t = a_t dt + \sigma_t dW_t$$
$$dP_t = \theta_1 a_t dt + \theta_2[F_t + F_s a_t + F_D a_t^* + \frac{1}{2}F_{DD}\sigma_t^{*2} + F_{sD}\sigma_t\sigma_t^*]dt +$$
$$[\sigma_t(\theta_1 + F_s\theta_2) + \theta_2\sigma_t^* F_D]dW_t$$

Let $\Theta_2 = 1$ $\Theta_1 = -\frac{(F_s\sigma_t + \sigma_t^* F_D)}{\sigma_t}$

$$dP_t = \theta_1 a_t dt + \theta_2[F_t + F_s a_t + F_D a_t^* + \frac{1}{2}F_{DD}\sigma_t^{*2} + F_{sD}\sigma_t\sigma_t^*]dt$$

- It will again convert dP_t into a nonrandom increment, since these choices for Θ_1, Θ_2 will eliminate all terms involving the random dW_t.

Solving PDEs in Practice

- Closed-Form Solutions

  The first method is similar to the one used by Black and Scholes, which involves solving the PDE for a closed-form formula. It turns out that the PDEs describing the behavior of derivative prices cannot in every case be solved for closed forms.

Numerical Solutions

To solve this PDE numerically, one assumes that the PDE is valid for finite increments in S_t and t.

1. A grid size for ΔS must be selected as a minimum increment in the price of the underlying security.
2. Time t is the second variable in $F(S_t,t)$. Hence, a grid size for Δt is needed as well. Needless to say, Δt, ΔS must be “small.”

3. Next one has to decide on range of possible values for S_t. To be more precise one selects, a priori, the minimum Smin and maximum Smax as possible values of S_t. These extreme values should be selected so that observed prices remain within the range Smin $\leqq S_t \leqq$ Smax.
4. The boundary conditions must be determined.
5. Assuming that for small but noninfinitesimal ΔS_t and Δt the same PDE is valid, the values of $F(S_t,t)$ at the grid points should be determined.

$$-rF + r\frac{\partial F}{\partial S}\cdot St + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial S^2}\cdot\sigma^2\cdot St^2 = 0$$

$$0 \le St, 0 \le t \le T$$

$$F(T) = \max[S_T - K, 0]$$

$$\frac{\Delta F}{\Delta t} + rS\frac{\Delta F}{\Delta S} + \frac{1}{2}\sigma^2 S^2\frac{\Delta^2 F}{\Delta S^2} = rF$$

Where the first-order partial derivatives are approximated by the corresponding differences.

For first partials we can use the backward difference

$$\frac{\Delta F}{\Delta t} \simeq \frac{F_{ij} - F_{i,j-1}}{\Delta t}$$

$$rS\frac{\Delta F}{\Delta S} \simeq rS_j\frac{Fij - F_{i-1,j}}{\Delta S}$$

- Or we can use forward differences

$$rS\frac{\Delta F}{\Delta S} \simeq rS_j\frac{F_{i+1,j} - F_{ij}}{\Delta S}$$

For the second-order partials we use the approximations

$$\frac{\Delta^2 F}{\Delta S^2} = [\frac{F_{i+1,j} - F_{ij}}{\Delta S} - \frac{F_{ij} - F_{i-1,j}}{\Delta S}]\frac{1}{\Delta S}$$

Exotic Options

- Lookback Options:

  Case: In a floating lookback call option,the payoff is the difference S_T-S_{min}, where S_{min} is the minimum price of the underlying asset observed during the life of the option.

- Ladder Options:

  A ladder option has several thresholds, such that if the underlying price reaches these thresholds, the return of option is "locked in."

- Trigger or Knock-in Options

  A down-and-in option gives its holder a European option if the spot price falls below a barrier during the life of the option. If the barrier is not reached, the option expires with some rebate as a payoff.

- Knock-out Options

  Knock-out options are European options that expire immediately if, for example, the underlying asset price falls below a barrier during the life of the option.

- The option pays a rebate if the barrier is reached. Otherwise, it is a "standard" European option. Such an option is called "down-and-out."