Ch 8
The Wiener Process and Rare Events in Financial Market

By
Salih N. Neftci
Presented by
Jay Lin

Outline

- Introduction
- Two Generic Models
- SDE in Discrete Intervals, Again
- Characterizing Rare and Normal Events
- A Model for Rare Events
- Moments That Matter
- Conclusions

1. Introduction

- Every instant of an ordinary trading day (for liquid instrument)
  - Prices go up by one tick
  - Prices decrease by one tick
  - No change
- “rare” or “extreme” events
  - Not the same as “turbulence”
  - Clarify the probabilistic structure of rare events and Contrast them with the behavior of Wiener process
  - Or discontinuity of observed price process

“Rare” vs. “Normal”

- Size and probability of occurrence
  - during a small interval, Var = $E[\sigma_t \Delta W_t]^2 = \sigma_t^2 h$
  -

h → 0	Normal	Rare
Size	smaller	not shrink
Prob.	nonzero	go to zero

  -

h	Normal	Rare
Size	dependent	independent
Prob.	independent	dependent

2. Two Generic Models

- The Wiener Process
  - or Brownian motion
  - continuous stochastic process
  - “extremes” occur infrequently
  - tail areas follow normal dist.
- The Poisson Process
  - discontinuous process
  - modeling systematic jumps

2.1 The Wiener Process

- Random variable ΔW_{t_i} takes one of two possible value $\sqrt{h}$ or $-\sqrt{h}$ at instant

$$\Delta W_{t_i} \overset{iid}{\sim} N(0, \sigma^2 h)$$

(Levy theorem)

- $W_{t_n} = \sum_{i=1}^{n} \Delta W_{t_i}$ will converge weakly to a Wiener process as n goes to infinity

- DEFINITION: A Wiener process W_t, relative to a family of information sets $\{I_t\}$, is a stochastic process such that
  1. The pair I_t , W_t is a square integrable martingale with W_0 =0 and $E[(W_t - W_s)^2] = t - s, \ s \le t$
  2. The trajectories of W_t are continuous over t.
- – W_t has uncorrelated increments
- – W_t has zero mean and variance t
- – The moment of W_t are infinitesimal in infinitesimal intervals

2.1.1 Wiener Process or Brownian Motion ?

- DEFINITION: A random process B_t, $t \in [0, T]$ is a (standard) Brownian motion if:
  1. The process begins at zero, B_0=0
  2. B_t has stationary, independent increments
  3. The B_t is continuous in t
  4. The increments $B_t - B_s$, have a normal distribution with mean zero and variance $|t - s|$:

$$(B_t - B_s) \sim N(0, |t - s|)$$

- – A crucial difference: {W} is a martingale, {B} is a normal dist.
- – Levy theorem: no difference between the two processes

2.2 The Poisson Process

- N_t represents the total number of extreme shocks

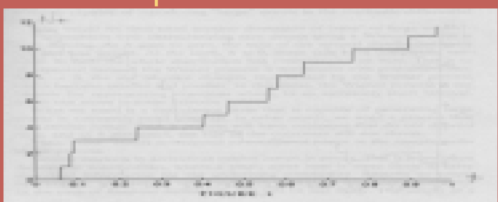
$$dN_t = \begin{cases} 1 & \text{with prob. } \lambda dt \\ 0 & \text{with prob. } 1 - \lambda dt \end{cases}$$

  λ is the rate of occurrence
- The size of Poisson outcomes does not depend on dt, and the prob. with the outcomes are f(dt). (different from {B})
- Define $M_t = N_t - \lambda t$, and {M} will be a discontinuous square integrable martingale, then $E[M_t] = 0$ and $E[M_t]^2 = \lambda t$

{M} and {W}

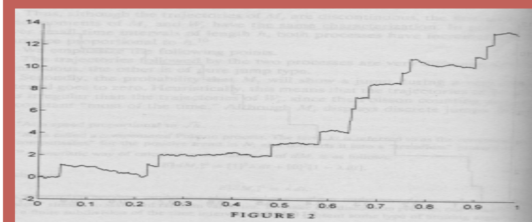
- The first and second moments of {M} and {W} have the same characterization.
- But one is continuous, the other is of pure jump type.
- When dt→0: the prob. of jump→0 (keep constant most of the time)
- {M} will not have unbounded variation.
- When {W} display infinitesimal changes, the variation becomes unbounded.

2.3 Examples



- Positive slope ({N} is not a martingale)
- Equal jumps of size 1
- Trajectory is constant between these jumps
- 14 jumps, close to the mean (λ =13.4)

Mixture from Poisson and Wiener



- Jumps due to Poisson component
- Between jumps, fluctuations due to Wiener component
- The noise introduced by Wiener process is much smaller than the jumps due to Poisson process

2.4 Back to Rare Events

- Normal dist. has tails the extend to infinity
  - ➔ with small but nonzero h
  - ➔ a positive prob. and large price change will occur
  - ➔ Wiener process seems to be capable of introducing "large" events
  - ➔ do we need to discuss rare events ?

- The problem with characterizing rare event
  - ➔ h ➔ 0
  - ➔ the weight of tail is getting smaller
  - ➔ the size of price changes become smaller and smaller
  - ➔ in short interval, Wiener process is not suitable for extreme events
  - ➔ we need a process, whose outcomes will not depend on h

3. SDE in Discrete Intervals, Again

- Consider the SDE

$$S_k - S_{k-1} = a(S_{k-1},k)h + \sigma(S_{k-1},k)\Delta W_k, \quad k = 1,2,\ldots,n$$

- We add Assumption 4:

$$\sigma_k \Delta W_k = \begin{cases} \omega_1 & \text{with prob. } p_1 \\ \omega_2 & \text{with prob. } p_2 \\ \vdots & \quad \vdots \\ \omega_m & \text{with prob. } p_m \end{cases}$$

- Two reasons:
  - Reason1: distinguish from normal and rare events when possibilities are finite
  - Reason2: in practical situations, the possible states like tree

4. Characterizing Rare and Normal Events

- According to Assumption 4:

$$\mathrm{Var}[\sigma_k \Delta W_k] = \sum_{i=1}^{m} p_i \omega_i^2 = \sigma_k^2 h$$

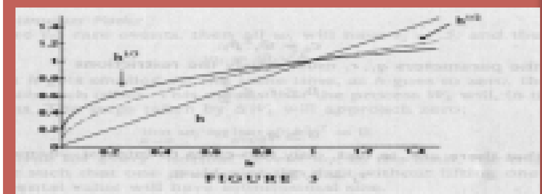
- Then suppose $p_i\omega_i^2 = c_i h$ where c>0
- $p_i\omega_i^2$ are linear function of h, we visualize the p and w as two functions of h: $p_i = p_i(h)$, $\omega_i = \omega_i(h)$ ➔ $p_i(h)\omega_i(h)^2 = c_i h$
- We follow Merton (1990) to assume the exp. forms for p, w:

$$\omega_i(h) = \overline{\omega}_i h^{r_i}$$
$$p_i(h) = \overline{p}_i h^{q_i}$$

where r and q are nonnegative

$\overline{\omega}_i, \overline{p}_i$ are constants and indep. of h

Three examples of h



- For small h, $h^{r_i} > h$
- From above slide, the "size" and the "prob." of the event depend on the interval length h.
- When h gets larger, the price change (w) and prob. (p) will get larger.

Use r and q to characterize events

- r governs how fast the size of the event goes to zero as the interval, h, decreases.
- q governs how fast the prob. goes to zero as the interval, h, decreases.
- We know "$p_i\omega_i^2 = \overline{p}_i h^{q_i} \overline{\omega}_i^2 h^{2r_i} = c_i h$" is proportional to h, implies $q_i + 2r_i = 1$ and $c_i = \overline{\omega}_i^2 \overline{p}_i$
- The q and r satisfy $0 \le r_i \le \frac{1}{2}$, $0 \le q_i \le 1$
- We can call "normal" events (r=0.5,q=0)
  "rare" events (r=0, q=1)

4.1 Normal Events

- The condition of “normal” events is

$$0.5 \geq r_i > 0$$

- The size and prob. will be

$$\omega_i = \bar{\omega}_i \sqrt{h}$$
$$p_i = \bar{p}_i$$

4.1.1 Continuous Paths

- When h gets smaller, the size $\omega_i = \bar{\omega}_i \sqrt{h}$ will shrink.
- As h goes to zero, $\lim_{h\to 0} \omega_i = \lim_{h\to 0} \bar{\omega}_i h^{0.5} = 0$ and $p_i = \bar{p}_i$ (q=0), as h→0
  ➔ {W} will be continuous.

4.1.2 Smoothness of Sample Paths

- Smoothness: a deterministic function, f(x) if $\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$ exists ➔ f(x) is smooth at x_0
- Is the smoothness valid for W (nondeterministic function) ?
- Explanation:
  - A finite number m of possible value w
  - Sizes are proportional to $\sqrt{h}$
- We can say $\frac{W_{t+h}-W_t}{h} = \frac{\omega_i}{h}$ and take limits $\lim_{h\to 0} \frac{W_{t+h}-W_t}{h} = \lim_{h\to 0} \frac{\omega_i}{h} = \lim_{h\to 0} \bar{\omega}_i \frac{h^{0.5}}{h} \to \infty$

- When h gets smaller, the W starts to change at an infinite rate.
- Asset prices will behave continuously but erratically.
- We assumed without any loss of generality, that $\bar{\omega}_i$ was positive

4.2 Rare Events

- The size and prob. will be $p_i = \bar{p}_i h$, $\omega_i = \bar{\omega}_i$ (r=0, q=1)
- 4.2.1 Sample Paths
  - Discontinuous
  - Size will be indep. of h, but Prob. will depend on h.
  - Need a model other than Wiener process to capture the behavior.
- 4.2.2 Further Comments
  - If $0<r<0.5$, $0<q<1$, the paths will be continuous but nonsmooth. (Wiener process)
  - h→0, w→0
    ➔ not rare events
  - h→0, p→0, but size will get smaller
    ➔ not qualified rare events

5. A Model for Rare Events

- Two components:
  - One is predictable given info. at time t
  - Another is unpredictable
- That is $S_k - S_{k-1} = a(S_{k-1}, k)h + \sigma(S_{k-1}, k)\Delta W_k, \quad k = 1, 2, \ldots, n$
- Continuous-time version: $dS_t = a(S_t, t)dt + \sigma(S_t, t)dW_t$
- We split the error term into two:
  - ΔW, normal events (continuous fashion)
  - ΔN, rare events (jumps)

$$\Delta N_k = N_k - N_{k-1} = \begin{cases} 1 & \text{with prob. } \lambda h \\ 0 & \text{with prob. } 1-\lambda h \end{cases}$$

λ = the rate of occurrence

- Use Poisson process to model N, because the process has following properties:
  - h → 0, at most one event occur with prob. 1
  - The info. up to time t does not help to predict the occurrence of the event (memoryless)
  - The events occur at a constant rate, λ
- Two modifications:
  - Rate of occurrence may change over time
  - The mean of ΔN is not zero (SDE only deals with zero mean), we consider $J_t = (N_t - \lambda t)$

- We multiply ΔJ_k by the size of jumps $\sigma_2(S_{k-1}, k)$ (time-dependent).
- The SDE will be written as
  $$S_k - S_{k-1} = a(S_{k-1}, k)h + \sigma_1(S_{k-1}, k)\Delta W_k + \sigma_2(S_{k-1}, k)\Delta J_k, \quad k = 1, 2, ..., n$$
- As h gets smaller
  $$dS_t = a(S_t, t)dt + \sigma_1(S_t, t)dW_t + \sigma_2(S_t, t)dJ_t$$
- Above SDE can handle "normal" and "rare" events simultaneously.
- Jump component dJ(t) and Wiener component dW(t) are statistically indep.

6. Moments That Matter

- The first moment:
  - Expected value, $E[\Delta W_k]$
- The second (centered) moment:
  - Variance, $Var(X_t) = E[X_t - E[X_t]]^2$
- The third (centered) moment:
  - Skewness of the dist., $E[X_t - E[X_t]]^3$
- The forth (centered) moment:
  - Heavy tails, $E[X_t - E[X_t]]^4$
- Higher-order (centered) moment: $E[X_t - E[X_t]]^k$

- Normal events:
  - the first two moments matter, the others are of marginal significance.
- Rare events:
  - all moments need to take into account.
- The first two moments are:
  $$E[\sigma_1\Delta W_k + \sigma_2\Delta J_k] = [p_1\omega_1 + ... + p_m\omega_m] = 0$$
  $$Var[\sigma_1\Delta W_k + \sigma_2\Delta J_k] = [p_1\omega_1^2 + ... + p_m\omega_m^2]$$
  $\Delta W, \Delta J$ are indep.

- If all events are normal events:
  - 1th moment: $E[\sigma_1\Delta W_k] = h^{0.5}[p_1\overline{\omega}_1 + ... + p_m\overline{\omega}_m]$

    the average rate of unexpected changes in prices: $\dfrac{E[\sigma_1\Delta W_k]}{h}$

    as h→0 ➔ $\sqrt{h}$ > h ➔ the rate will be large, and the first moment is not equal to zero.
  - 2nd moment: $Var(\sigma_1\Delta W_k) = h[\sum_{i=1}^{m} p_i\overline{\omega}_i^2]$

    the average rate of var.: $Var(\sigma_1\Delta W_k)/h = \sum_{i=1}^{m} p_i\overline{\omega}_i^2$

    as h→0 ➔ the rate remain constant ➔ for "normal" events, the var. provides significant info.

- We consider higher-order moments:
  $$E[\sigma_1\Delta W_k]^n = [p_1\omega_1^n + ... + p_m\omega_m^n]$$
  - For normal events: $\omega_i^n = \overline{\omega}_i^n(h^{0.5})^n$

    $h^{n/2} < h$, n > 2
  - Then the rate will be: $\dfrac{E[\sigma_1\Delta W_k]^n}{h} = h^{(n-2)/2}\sum_{i=1}^{m}\overline{\omega}_i^n p_i$

    as h→0 ➔ the rate will converge to zero ➔ for normal events, higher-order moments will not carry any useful info.
- Summary: only two parameters (1st and 2nd moments) will be suff. for normal type info.

- If all events are rare events:
  - 2nd moment: $E[\sigma_2 \Delta J_k]^2 = h[\sum_{i=1}^{m} \omega_i^2 \overline{p}_i]$

    the average rate of var.: $E[\sigma_2 \Delta J_k]^2 / h = \sum_{i=1}^{m} \omega_i^2 \overline{p}_i$

    as h→0 ➔ the rate will become indep. of h

    ➔ no difference from Wiener processes.
  - Higher-order moments: $E[\sigma_2 \Delta J_k]^n = h[\sum_{i=1}^{m} \omega_i^n \overline{p}_i]$

    as h→0 ➔ still the same ➔ higher-order info. can not be ignored.
- Summary: when rare events are a systematic part of the data, Wiener process may not be appropriate.

7. Conclusions

- SDE: $dS_t = a(S_t, t)dt + [\sigma_1(S_t, t)dW_t + \sigma_2(S_t, t)dJ(t)]$
  - The first term is expected change in S
  - The second term is the surprise component, unpredictable given the info. at time t.
    - dW captures events of insignificant size that happen regularly.
    - dJ captures "large" events that occur rarely.
  - Given some parameters and some assumptions, dW (Wiener process) and dJ (Poisson process) can represent all types of dist.

Many Thanks for your Patience