Chapter 10

Ito's Lemma

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Types of Derivatives

1. partial derivatives of $F(S_t,t)$

$$F_s=\frac{\partial F(S_t,t)}{\partial S_t},\qquad F_t=\frac{\partial F(S_t,t)}{\partial t}$$

$\rightarrow$ Simple multipliers , there is no difference between stochastic and deterministic variables.

2. total derivatives

$$dF_t=F_s dS_t+F_t dt$$

$\rightarrow$ Assume that both time t and the underlying asset price S_t change, and then calculate the total response of $F(S_t,t)$.

3. chain rule $\quad \dfrac{dF(S_t,t)}{dt}=F_s\dfrac{dS_t}{dt}+F_t$

1. In classical calculus ---

   The chain rule expresses the rate of change in a variable as a chain effect of some initial variation.

2. In stochastic calculus ---

   The operations such as dF_t/dt , dS_t/dt can't be defined for continuous-time square integrable martingales, or Brownian motion.

But, a stochastic equivalent of the chain rule can be formulated in terms of absolute changes such as dF_t, dS_t, dt , and the Ito integral can be used to justify these terms.

Thus, in stochastic calculus, the term "chain rule" will refer to the way stochastic differentials relate to each other.

→ a stochastic version of total differentiation is developed.

Ito's Lemma

The stochastic version of the chain rule is known as Ito's Lemma.

$\rightarrow$ Expand the $F(S_k,k)$ around $S_{k-1},\ k-1$ by Taylor's formula.

$$F(S_k,k)=F(S_{k-1},k-1)+F_s\Delta S_k+F_t[h]$$
$$+\tfrac{1}{2}F_{ss}[\Delta S_k]^2+\tfrac{1}{2}F_{tt}[h]^2+F_{st}[h\Delta S_k]+R$$

while $F(S_k,k)-F(S_{k-1},k-1)=\Delta F(k)$

$$\rightarrow \Delta F(k)=F_s\Delta S_k+F_t[h]+\tfrac{1}{2}F_{ss}[\Delta S_k]^2+\tfrac{1}{2}F_{tt}[h]^2$$
$$+F_{st}[h\Delta S_k]+R$$

Assume S_k obeys $\Delta S_k=a_k h+\sigma_k\Delta W_k$

Ito's Lemma

$$\Delta F(K)=F_s[a_k h+\sigma_k\Delta W_k]+F_t[h]+\tfrac{1}{2}F_{ss}[a_k h+\sigma_k\Delta W_k]^2$$
$$+\tfrac{1}{2}F_{tt}[h]^2+F_{st}[h][a_k h+\sigma_k\Delta W_k]+R$$

In order to obtain a chain rule in stochastic environments, the right-hand side terms will be classified as negligible and nonnegligible.

The notion of "Size" in Stochastic Calculus

Convention :

Given a function $g(\Delta W_k,h)$ depending on the increments of the Wiener process W_k, and on the time increment,

consider the ratio $\dfrac{g(\Delta W_k,h)}{h}$.

(1) If the ratio vanishes as $h\rightarrow 0$

$g(\Delta W_k,h)$ is negligible in small intervals.

(2) Otherwise, $g(\Delta W_k,h)$ is nonnegligible.

The notion of "Size" in Stochastic Calculus

$$\Delta F(K) = F_s[a_k h + \sigma_k \Delta W_k] + F_t[h] + \tfrac{1}{2} F_{ss}[a_k h + \sigma_k \Delta W_k]^2 + \tfrac{1}{2} F_{tt}[h]^2 + F_{st}[h][a_k h + \sigma_k \Delta W_k] + R$$

- first order terms...

$$* \lim_{h\to 0} \frac{F_s a_k h}{h} = F_s a_k$$

$$* \lim_{h\to 0} \frac{F_t h}{h} = F_t$$

$$* \frac{F_s \Delta W_k}{h} \uparrow \text{ as } h \downarrow \quad (\because \Delta W_k = f(h^{\frac{1}{2}}))$$

As a result, all first-order terms are nonnegligible.

The notion of "Size" in Stochastic Calculus

$$\Delta F(K) = F_s[a_k h + \sigma_k \Delta W_k] + F_t[h] + \tfrac{1}{2} F_{ss}[a_k h + \sigma_k \Delta W_k]^2 + \tfrac{1}{2} F_{tt}[h]^2 + F_{st}[h][a_k h + \sigma_k \Delta W_k] + R$$

- Second order terms...

$$* \lim_{h\to 0} \frac{F_{tt}[h]^2}{2h} = \lim_{h\to 0} \frac{F_{tt}[h]}{2} = 0$$ negligible

$$* \tfrac{1}{2} F_{ss}[a_k h + \sigma_k \Delta W_k]^2 \text{ divided by } h...$$

$$= \tfrac{1}{2} F_{ss}\left[\frac{a_k^2 h^2}{h} + \frac{(\sigma_k \Delta W_k)^2}{h} + \frac{2 a_k \sigma_k h \Delta W_k}{h}\right] \cong \tfrac{1}{2} F_{ss} a_k^2$$

small — small

The notion of "Size" in Stochastic Calculus

$$\Delta F(K) = F_s[a_k h + \sigma_k \Delta W_k] + F_t[h] + \tfrac{1}{2} F_{ss}[a_k h + \sigma_k \Delta W_k]^2 + \tfrac{1}{2} F_{tt}[h]^2 + F_{st}[h][a_k h + \sigma_k \Delta W_k] + R$$

- the Cross Products terms

$$\lim_{h\to 0} \frac{F_{st}[h][a_k h + \sigma_k \Delta W_k]}{h}$$

$$= \lim_{h\to 0} \frac{F_{st}[h][\sigma_k h]}{h} + \lim_{h\to 0} \frac{F_{st}[h][\sigma_k \Delta W_k]}{h}$$

$$= 0 + \lim_{h\to 0} F_{st}[\sigma_k \Delta W_k] \qquad Var(\Delta W_k) = h$$

$$= 0 \qquad E(\Delta W_k) = 0$$

The notion of "Size" in Stochastic Calculus

$$\Delta F(K) = F_s[a_k h + \sigma_k \Delta W_k] + F_t[h] + \tfrac{1}{2} F_{ss}[a_k h + \sigma_k \Delta W_k]^2 + \tfrac{1}{2} F_{tt}[h]^2 + F_{st}[h][a_k h + \sigma_k \Delta W_k] + R$$

- the remaining terms

  According to the convention, if the unpredictable shocks are of "normal" type-- i.e., there are no "rare events", powers of ΔW_k greater than two will be negligible.

The notion of "Size" in Stochastic Calculus

$$\Delta F(K) = F_s[a_k h + \sigma_k \Delta W_k] + F_t[h] + \tfrac{1}{2} F_{ss}[a_k h + \sigma_k \Delta W_k]^2 + \tfrac{1}{2} F_{tt}[h]^2 + F_{st}[h][a_k h + \sigma_k \Delta W_k] + R$$

$$\Rightarrow \quad \Delta F(K) = F_s[a_k h + \sigma_k \Delta W_k] + F_t[h] + \tfrac{1}{2} F_{ss} \sigma_k^2$$

Ito's Lemma

Let $F(S_t, t)$ be a twice-differentiable function..

1. The random process $dS_t = a_t dt + \sigma_t dW_t$,

2. Thus, $dF_t = \frac{\partial F}{\partial S_t} dS_t + \frac{\partial F}{\partial t} dt + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} \sigma_t^2 dt,$

$$= \left[\frac{\partial F}{\partial S_t} a_t + \frac{\partial F}{\partial t} + \frac{1}{2} \frac{\partial^2 F}{\partial S_t^2} \sigma_t^2\right] dt + \frac{\partial F}{\partial S_t} \sigma_t dW_t,$$

Use of Ito's Lemma

1. Ito's Lemma provides a tool for obtaining stochastic differentials for functions of random processes.
   -- Given the exact formula for $F(S_t, t)$, one can then take the partial derivatives explicitly and replace them in the formula $dF(S_t,t) = F_s dS_t + F_t dt + \frac{1}{2} F_{ss}\sigma_t^2 dt$. and then gets the stochastic differential, $dF(S_t, t)$.

2. Ito's Lemma is useful in evaluating Ito intergrals.

Ito's Formula in More Complex settings

Ito's Lemma may not be applied in some cases…

1. The function F(·) may depend on more than a single stochastic variable St.

   → A multivariate version of the Ito's Lemma should be used.

2. Since the financial market is affected by rare events, so it's not appropriate to consider error terms made of Wiener processes only.

   → The jump processes to the SDEs should be added.

Conclusions

1. Given movements in the underlying assets, Ito's Lemma helps to determine stochastic differentials for financial derivatives.
2. Ito's Lemma is completely dependent on the definition of Ito integral.
3. Ito's Lemma used in deterministic calculus gives significantly different results than standard formulas.

$\dfrac{(\sigma_k \Delta W_k)^2}{h}$

$(\Delta W_k)^2$ is the square of a random variable with mean zero.

And $Var(\sigma_k \Delta W_k) = \sigma_k^2 h$

in the mean square sense... $dW_t^2 = dt$ (i.e., $\Delta W_k^2 = h$)

Thus, ΔW_k^2 is a nonnegligible term.

And the nonzero variance of ΔS_k implies.. $\sigma_k > 0$

Example ..

Given $F(W_t, t) = 3 + t + e^{W_t}$.

where W_t is a Wiener Process with $a_t = 0$, $\sigma_t = 1$

By Ito's Lemma..

$$dF_t = [\frac{\partial F}{\partial W_t} a_t + \frac{\partial F}{\partial t} + \frac{1}{2}\frac{\partial^2 F}{\partial W_t^2}\sigma_t^2]dt + \frac{\partial F}{\partial W_t}\sigma_t dW_t$$

$$= [1 + \tfrac{1}{2} e^{W_t}]dt + e^{W_t} dW_t$$

Then we get the SDE for $F(W_t, t)$

with I_t-dependent drift rate $a(I_t, t) = [1 + \frac{1}{2} e^{W_t}]$

and diffusion rate $\sigma(I_t, t) = e^{W_t}$.

Ito's Lemma as an Integration tool

Example ..

(suppose one needs to evaluate the Ito integral $\int_0^t W_s dW_s$)

Define $F(W_t, t) = \frac{1}{2} W_t^2$

By Ito's Lemma..$dF_t = F_W dW_t + F_t dt + \frac{1}{2} F_{ww}\sigma_t^2 dt$

$= W_t dW_t + 0 + \frac{1}{2} dt$

The corresponding integral equation is..

$$F(W_t, t) = \int_0^t W_s dW_s + \tfrac{1}{2}\int_0^t ds$$

$$\Rightarrow \tfrac{1}{2} W_t^2 = \int_0^t W_s dW_s + \tfrac{1}{2} t$$

$$\Rightarrow \int_0^t W_s dW_s = \tfrac{1}{2} W_t^2 - \tfrac{1}{2} t$$

Bivariate case

$$dS_1(t) = a_1(t)dt + \left[\sigma_{11}(t)dW_1(t) + \sigma_{12}(t)dW_2(t)\right]$$

$$dS_2(t) = a_2(t)dt + \left[\sigma_{21}(t)dW_1(t) + \sigma_{22}(t)dW_2(t)\right]$$

Where $W_1(t)$ and $W_2(t)$ are two independent Wiener processes

Suppose S_t is a 2×1 vector of stochastic process obeying the following SDE :

$$\begin{pmatrix} dS_1(t) \\ dS_2(t) \end{pmatrix} = \begin{pmatrix} a_1(t) \\ a_2(t) \end{pmatrix} dt + \begin{pmatrix} \sigma_{11}(t) & \sigma_{12}(t) \\ \sigma_{21}(t) & \sigma_{22}(t) \end{pmatrix} \begin{pmatrix} dW_1(t) \\ dW_2(t) \end{pmatrix}$$

Suppose now we have a continuous, twice differentiable function of $S_1(t)$ and $S_2(t)$ that we denote by $F(S_1(t), S_2(t), t)$.

How can we write the stochastic differential dF_t ?

$\Rightarrow$ use the multivariate form of Ito's lemma,

$$dF_t = F_t dt + F_{S_1} dS_1 + F_{S_2} dS_2$$
$$+\frac{1}{2}\left(F_{s_1 s_1} dS_1^2 + F_{s_2 s_2} dS_2^2 + 2F_{s_2 s_1} dS_1 dS_2\right) \quad \to \text{(A)}$$

We already know that dt^2 and cross product such as $dtdW_1(t)$ and $dtdW_2(t)$ are equal to zero in the mean square sense.

Since

$$E\left[\Delta W_1(t)\Delta W_2(t)\right] = E[\Delta W_1(t)]E[\Delta W_2(t)] = 0$$

hence, in the mean square sense,

$$dW_1(t)dW_2(t) = 0$$

$$dS_1(t)^2 = \left[\sigma_{11}(t)^2 + \sigma_{12}(t)^2\right]dt \qquad \text{(in the mean square sense..}$$

$$dS_2(t)^2 = \left[\sigma_{21}(t)^2 + \sigma_{22}(t)^2\right]dt \qquad dW_t^2 = dt.)$$

The cross-product term is given by

$$dS_1(t)dS_2(t) = \left[\sigma_{11}(t)\sigma_{21}(t) + \sigma_{12}(t)\sigma_{22}(t)\right]dt$$

These expressions can be substituted into the bivariate Ito formula in (A) to eliminate $dS_1(t)^2$, $dS_2(t)^2$ and $dS_1(t)dS_2(t)$