The Dynamics of Derivative Prices

Stochastic Differential Equations

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Overall Review

- Introduction
- Type of SDE Solutions
- A discussion of Strong Solution
- Major Models of SDEs
- Conclusions

Introduction

$$dS_t = a(S_t,t)dt + \sigma(S_t,t)dW_t \ , \ t \in [0,\infty)$$

- $a(S_t, t)$, $\sigma(S_t, t)$, W_t are I_t-adapted.
- Conditions on a_t and σ_t

$$P(\int_0^t | a(S_u,u) | du, < \infty) = 1$$
$$P(\int_0^t \sigma(S_u,u)^2 du, < \infty) = 1$$

→ The drift and diffusion parameters do not vary too much.

Type of SDE Solutions

- If a continuous time process S_t satisfies the equation

$$\int_0^t dS_u = \int_0^t a(S_u,u)du + \int_0^t \sigma(S_u,u)dW_u$$

for all t >0, then we say that S_t is the solution of

$$dS_t = a(S_t,t)dt + \sigma(S_t,t)dW_t \ , \ t \in [0,\infty)$$

Type of SDE Solutions

Strong Solution	**Weak Solution**
Similar to ODE	Specific to SDE
W_t is known given the I_t	The distribution of $\tilde{W}_t$ is known given H_t
I_t-adapted	H_t-adapted,

Type of SDE Solutions

Which Solution Is To Be Preferred?

- The use of a strong solution implies knowledge of error process W_t.
- Often, in pricing derivatives, one works with weak solution for not knowing the exact process W_t

A discussion of Strong Solution

- How to verify a solution to ODE?

→Differentiate it.

Ex: the solution to an ODE $dX_t/dt = aX_t$ with X_0 given is

$$X_t = X_0 e^{at}$$

Differentiate the function X_t with respect to t, we obtain the ODE above

A discussion of Strong Solution

How to verify a solution to SDE?

→Ito's Lemma

Ex: By using Ito's Lemma we can verify the following candidate

$$S_t = S_0 e^{\{(a-0.5\sigma^2)t+\sigma W_t\}}$$

is a solution to the SDE $dS_t = \mu S_t dt + \sigma S_t dW_t$

Major Models of SDEs

- Linear Constant Coefficient SDEs

$$dS_t = \mu dt + \sigma dW_t$$

<Features>

1. The asset price fluctuates randomly around a straight line (linear trend.)
2. A good approximation to asset price with no systematic jumps.

Major Models of SDEs

- Geometric SDEs

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

<Features>

1. The drift and the diffusion coefficients change proportionally with S_t.
2. The asset price fluctuates randomly around an exponential trend.

Major Models of SDEs

- Square Root Process

$$dS_t = \mu S_t dt + \sigma \sqrt{S_t} dW_t$$

<Features>

1. The asset price follows an exponential trend.
2. The asset price volatility does not increase "too much" as St increases.

Major Models of SDEs

- Mean Reverting Process

$$dS_t = \lambda(\mu - S_t)dt + \sigma S_t dW_t$$

μ : mean value

λ : the rate pulling asset price back to μ .

<Features>

1. The asset price can take an excursion away from the long-run trend but eventually revert back to the trend.
2. This process is often used to model interest rate dynamics.

Major Models of SDEs

- Ornstein-Uhlenbeck Process

$$dS_t = -\mu S_t dt + \sigma_t dW_t$$

<Feature>

A special case of "mean reverting SDE", which reverts back to the long-run mean value of zero.

Conclusions

- The strong solution is similar to the case of ordinary differential equations, the weak solution is specific to stochastic differential equations.
- The asset price is modeled with some major types of SDEs.

Appendix: Matlab Code

```
S1 = zeros(1000,1); S1(1)=100;
S2 = zeros(1000,1); S2(1)=100;
S3 = zeros(1000,1); S3(1)=100;
S4 = zeros(1000,1); S4(1)=0.1;
S5 = zeros(1000,1); S5(1)=0.1;
S6 = zeros(1000,1); S6(1)=0.1;
h = 0.001;
for i = 2:1000
    z = randn;
    S1(i) = S1(i-1)+10*h+30*sqrt(h)*z;   % constant coefficient
    S2(i) = S2(i-1)+0.15*S2(i-1)*h+0.3*S(i-1)*sqrt(h)*z;   % geometric
    S3(i) = S3(i-1)+0.15*S2(i-1)*h+0.3*sqrt(S(i-1))*sqrt(h)*z;   % sqart root
    S4(i) = S4(i-1)+0.5*(0.05-S4(i-1))*h+0.8*sqrt(h)*z;   % mean reverting
    S5(i) = S5(i-1)+(-0.5)*S5(i-1)*h+0.8*sqrt(h)*z;   % Ornstein-Uhlenbeck
    S6(i) = S6(i-1)+(-0.9)*S6(i-1)*h+0.8*sqrt(h)*z;   % Ornstein-Uhlenbeck
end
```