Pricing Derivative Product

Equivalent Martingale Measure

By Wen Lai

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Outline

Introduction

- Two methods used in derivative pricing

  1. PDEs
  2. Equivalent Martingale Measure

Introduction

- PDEs

  This is done by constructing a arbitrage-free portfolio.
- Equivalent Martingale Measure

  This is done by transforming the underlying probability distributions using the tool provided by Girsanov theorem

Probability Measure

- Denote probability measure as $dP(Z_t)$ or dP.
- *Z_t is a normally distributed random variable,* then for *a small interval* Δ, the integration over the interval centered on $\bar{Z}$ can be approximated as

$$P(\bar{Z}-\frac{1}{2}\Delta < Z_t < \bar{Z}+\frac{1}{2}\Delta) = \int_{\bar{Z}-\frac{1}{2}\Delta}^{\bar{Z}+\frac{1}{2}\Delta} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z_t^2} dZ_t \cong \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\bar{Z}_t^2} \int_{\bar{Z}-\frac{1}{2}\Delta}^{\bar{Z}+\frac{1}{2}\Delta} dZ_t = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\bar{Z}_t^2} \Delta$$

Probability Measure

- For infinitesimal Δ, which we write as dZ_t, the probability measure is expressed as

$$dP(Z_t) = P(\bar{Z}-\frac{1}{2}dZ_t < Z_t < \bar{Z}+\frac{1}{2}dZ_t)$$

$$= \int_{\bar{Z}-\frac{1}{2}dZ_t}^{\bar{Z}+\frac{1}{2}dZ_t} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z_t^2} dZ_t \cong \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z_t^2} \int_{\bar{Z}-\frac{1}{2}dZ_t}^{\bar{Z}+\frac{1}{2}dZ_t} dZ_t = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z_t^2} dZ_t$$

Transformation of Probability Distribution

- Moving the location of the distribution
  : changing the mean of the distribution.
- Changing the shape of the distribution
  : changing the variance of the distribution.

☆ Modern methods for pricing derivatives utilize the way of changing means

Changing Means

- Method 1: Operating on possible values

Assume a random variable Zt with mean Zero. One simply adds a constant μ to Zt to obtain a new variable $\tilde{z}_t$ with mean μ. That is,

$$E[Z_t] = 0$$

$$E[\tilde{Z}_t] = E[Z_t + \mu] = \mu$$

Changing Means

- Method 2: Operating on probabilities

This is done by transforming the probability measure so that the mean is changed, while leaving the variance unchanged.

Changing Means

- Example 1

Suppose the random variable Z is defined as follows.
A die is rolled and the value of Z is set according to the rule.

$$Z = \begin{cases} 10 & roll\ \ of\ \ 1\ or\ 2 \\ -3 & roll\ \ of\ \ 3\ or\ 4 \\ -1 & roll\ \ of\ \ 5\ or\ 6 \end{cases}$$

$$E[Z] = \frac{1}{3}[10] + \frac{1}{3}[-3] + \frac{1}{3}[-1] = 2$$

If we would like to change the mean to equal 1, then

Method 1

A new variable can be set as: $\tilde{Z} = Z - 1$ and $E[\tilde{Z}]$ can be calculated as

$$E[\tilde{Z}] = \frac{1}{3}[10-1] + \frac{1}{3}[-3-1] + \frac{1}{3}[-1-1] = 1$$

Changing Means

Method 2

Consider the following transformation of the original probability associated with rolling the die.

$$P(getting\ \ 1\ or\ 2) = \frac{1}{3} \rightarrow \tilde{P}(getting\ \ 1\ or\ 2) = \frac{122}{429}$$

$$P(getting\ \ 3\ or\ 4) = \frac{1}{3} \rightarrow \tilde{P}(getting\ \ 3\ or\ 4) = \frac{22}{39}$$

$$P(getting\ \ 5\ or\ 6) = \frac{1}{3} \rightarrow \tilde{P}(getting\ \ 5\ or\ 6) = \frac{5}{33}.$$

$$E^{\tilde{P}}[Z] = [\frac{122}{429}][10] + [\frac{22}{39}][-3] + [\frac{5}{33}][-1] = 1$$

Changing Means

- Example 2

Let S_t: the risky asset price.
R_t: the return of St.
r_t: the risk-free interest rate.
μ : risk premium.

It's easy to know that the expected return of S_t can be expressed approximately as

$$E[R_t] \cong r_t + \mu$$

Changing Means

- Example 2

  There are two ways to calculate *St*.

  $$A.\ S_t = E_t[\frac{1}{1+R_t} S_{t+1}]$$

  Problem: this requires a knowledge of the distribution of *Rt*, which requires knowing the risk premium of μ.

  $$B.\ S_t = E_t^{\tilde{P}}[\frac{1}{1+r_t} S_{t+1}]$$

  Where $\tilde{P}$ denotes another distribution with mean equal to r_t.

The Girsanov Theorem

- The Girsonov theorem provides the general framework for transforming the probability measure into another "equivalent " measure under continuous-time stochastic process.
- We introduce the theorem using the following special cases with growing complexity

The Girsanov Theorem

Case 1 :A Normally Distributed Random Variable

- The original probability measure is

$$dP(Z_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Z_t^2} dZ_t$$

- By finding the function $\xi(Z_t)$, we can change the original probability measure to a new measure ($d\tilde{P}$) with mean μ.

$$\xi(Z_t) = e^{Z_t\mu - \frac{1}{2}\mu^2} \Longrightarrow d\tilde{P}(Z_t) = dP(Z_t)\xi(Z_t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(Z_t-\mu)^2} dZ_t$$

The Girsanov Theorem

- The Radon-Nikodim Derivative

  The $\xi(Z_t)$ can be rewritten as

  $$\xi(Z_t) = \frac{d\tilde{P}(Z_t)}{dP(Z_t)}$$

- Implication:

  If the Radon-Nikodim Derivative of $\tilde{P}$ with respect to P exists, then we can use the result $\xi(Z_t)$ to transform the mean of *Zt* by leaving its variance unchanged.

The Girsanov Theorem

- Case 2 :A Continuous Stochastic Process— the formal version of the Girsanov theorem.

  Given a information set I_t over a period [0, T], we define a random process:

  $$\xi_t = e^{(\int_0^t X_u dW_u - \frac{1}{2}\int_0^t X_u^2 du)}$$

  Where X_t is an I_t –measurable process. The W_t is a Wiener process with probability distribution P.

The Girsanov Theorem

- We can prove that ξ_t is a martingale with E[ξ_t]=1 (see p291 for more details), then a new Wienner process $\tilde{W}_t$ and probability measure $\tilde{P}_t$ can be given by the following theorem.
- Theorem: (p291)

$$\tilde{W}_t = W_t - \int_0^t X_u du$$

$$\tilde{P}_t(A) = E^P[1_A \xi_T]$$

A: an event in I_t.

1_A: indicator function of the event A .

Application to SDEs

- Consider the following SDE

$$ds_t = \mu dt + \sigma dW_t$$
$$\Rightarrow S_t = \mu t + \sigma W_t$$

Clearly, S_t cannot be a martingale under probability measure P. We could easily switch to an equivalent measure $\tilde{P}$ using Girsanov theorem.

Application to SDEs

- Define $\xi(t)$

$$\xi(S_t) = e^{-\frac{1}{\sigma^2}[\mu S_t - \frac{1}{2}\mu^2 t]}$$

- Transform the probability measure form P to $\tilde{P}$

$$d\tilde{P}(S_t) = \xi(S_t) dP(S_t)$$
$$= e^{-\frac{1}{\sigma^2}[\mu S_t - \frac{1}{2}\mu^2 t]} \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{1}{2\sigma^2 t}(S_t - \mu t)^2} dS_t$$
$$= \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{1}{2\sigma^2 t}(S_t)^2} dS_t$$

Note : the pdf of S_t *is*

$$f_s = \frac{1}{\sqrt{2\pi\sigma^2 t}} e^{-\frac{1}{2\sigma^2 t}(S_t - \mu t)^2}$$

Application to SDEs

- After the previous transformation, S_t is a martingale under probability measure $\tilde{P}$.
- We can write the increments of S_t in terms of a new error term $\tilde{W}_t$:

$$dS_t = \sigma d\tilde{W}_t$$

Conclusion

- The transformation was done by switching the distribution of S_t from P to $\tilde{P}$. This was accomplished by using a new error term $\tilde{W}_t$.
- The new error term $\tilde{W}_t$ still had the same variance.
- The transformation was used to convert S_t into a martingale.