# Ch15 Equivalent Martingale Measures-*Applications*

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## Outline

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## Introduction

There are two ways of calculating the arbitrage-free price of a European call $C_t$ option written on a stock $S_t$ that does not pay any dividends.

1. The original Black-Scholes approach, where (1) a **riskless portfolio** is formed, (2) a **partial differential equation** in $F(S_t,t)$ is obtained, and (3)the PDE is solved either directly or numerically.

2. The martingale methods, where one finds a **"synthetic" probability** $\tilde{P}$ under which $S_t$ becomes a martingale. One then calculates

$$C_t = E^{\tilde{P}} e^{-r(T-t)}[\max(S_T - K,0)]$$

Again, either analytically or numerically.

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## The Moment-Generating Function

- Now let $Y_t$ be a continuous-time process

$$Y_t \sim N(\mu t, \sigma^2 t)$$

with $Y_0$ given.

- We define $S_t$ as the geometric process

$$S_t = S_0 e^{Y_t}$$

- The moment-generating function of $Y_t$ is

$$M(\lambda) = E[e^{Y_t \lambda}] = e^{\lambda \mu t + \frac{1}{2}\sigma^2 t \lambda^2}$$

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## Converting Asset Prices into Martingales

- Recall that the "**true**" distribution of $S_t$ is determined by the distribution of $Y_t$ .Hence, the probability P is given by $Y_t \sim N(\mu t, \sigma^2 t)$ $,t \in [0,\infty)$.
- We know that because the asset $S_t$ is risky when discounted by the risk-free rate, **it cannot be a martingale**.In fact, because of the existence of a **risk premium**, in general, we have

$$E^P[e^{-rt}S_t \mid S_u, u<t] > e^{-ru}S_u$$

- And, under the "true" probability measure P, the discounted process $Z_t$ defined by $Z_t = e^{-rt}S_t$ cannot be a martingale.

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## Converting Asset Prices into Martingales (cont.)

●Yet, the ideas introduced in Chapter 14 can be used to change the drift of $Z_t$ and convert it into a martingale.

●Now, define a new probability $\tilde{P}$ by $N(\rho t, \sigma^2 t)$ ,such that the equality $E^{\tilde{P}}[e^{-rt}S_t \mid S_u, u<t] = e^{-ru}S_u$ is satisfied.

$$E^{\tilde{P}}[e^{-rt}S_t \mid S_u, u<t] = e^{-rt}S_u e^{\rho(t-u)+\frac{1}{2}\sigma^2(t-u)}$$

$$= e^{-ru}S_u e^{(-r+\rho+\frac{1}{2}\sigma^2)(t-u)}$$

Thus, we can find $\rho = r - \frac{1}{2}\sigma^2$

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## The Implied SDEs

● Under the "**true**" probability P, the asset price $S_t$ satisfies an SDE :

$$dS_t = [\mu S_t + \frac{1}{2}\sigma^2 S_t]dt + \sigma S_t dW_t \quad t \in [0,\infty).$$

● Under the "**synthetic**" probability P, the asset price satisfies an SDE :

$$dS_t = rS_t dt + \sigma S_t d\tilde{W}_t \quad t \in [0,\infty).$$

● This is an interesting result. The probability that makes $S_t$ a martingale, switches the drift parameter of the original SDE to the risk-free interest rate r.

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## Comparing Martingale and PDE approaches

- Suppose the underlying asset price follows the stochastic differential equation:

$$dS_t = \mu(S_t)dt + \sigma(S_t)dW_t \quad t \in [0,\infty).$$

- Applying Ito's lemma to $e^{-rt}S_t$ , we can calculate the SDE followed by $e^{-rt}S_t$ .

$$d[e^{-rt}S_t] = S_t de^{-rt} + e^{-rt}dS_t \quad t \in [0,\infty).$$

$$d[e^{-rt}S_t] = e^{-rt}[\mu_t - rS_t]dt + e^{-rt}\sigma_t dW_t \quad t \in [0,\infty).$$

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## Comparing Martingale and PDE approaches (cont.)

- The Girsanov theorem says that we can find an $I_t$ – adapted process $\tilde{W}_t$ and a new Wiener process $X_t$ such that $d\tilde{W}_t = dX_t + dW_t$

$$d[e^{-rt}S_t] = e^{-rt}[\mu_t - rS_t]dt + e^{-rt}\sigma_t[d\tilde{W}_t - dX_t] \quad t \in [0,\infty).$$

Grouping the terms:

$$d[e^{-rt}S_t] = e^{-rt}[\mu_t - rS_t]dt - e^{-rt}dX_t + e^{-rt}\sigma_t d\tilde{W}_t \quad t \in [0,\infty).$$

- If we define this SDE under the new probability $\tilde{P}$, $\tilde{W}_t$ will be a standard Wiener process. In addition, $\tilde{P}$ will be a martingale measure if we equate the drift term to zero.

$$dX_t = [\frac{\mu_t - rS_t}{\sigma_t}]dt$$

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## Comparing Martingale and PDE approaches (cont.)

● Converting $e^{-rt}F(S_t,t)$ into a Martingale:

$$d[e^{-rt}F(S_t,t)] = e^{-rt}[-rFdt] + e^{-rt}[F_t dt + F_s dS_t + \frac{1}{2}F_{ss}\sigma_t^2 dt]$$

Rearranging:

$$d[e^{-rt}F(S_t,t)] = e^{-rt}[-rF + F_t + F_s\mu_t + \frac{1}{2}F_{ss}\sigma_t^2]dt + e^{-rt}\sigma_t F_s dW_t$$

We know $d\tilde{W}_t = dX_t + dW_t$

$$d[e^{-rt}F(S_t,t)] = e^{-rt}[-rF + F_t + F_s\mu_t + \frac{1}{2}F_{ss}\sigma_t^2 - \sigma_t F_s\left(\frac{\mu_t - rS_t}{\sigma_t}\right)]dt + F_s e^{-rt}\sigma_t F_s d\tilde{W}_t$$

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## Comparing Martingale and PDE approaches (cont.)

$$d[e^{-rt}F(S_t,t)] = e^{-rt}[-rF + F_t + F_s rS_t + \frac{1}{2}F_{ss}\sigma_t^2]dt + F_s e^{-rt}\sigma_t F_s d\tilde{W}_t$$

- in order for $e^{-rt}F(S_t,t)$ to be a martingale under the pair $\tilde{W}_t$ , $\tilde{P}$ the drift term of this SDE must be zero. This is the desired result:

$$-rF + F_t + F_s rS_t + \frac{1}{2}F_{ss}\sigma_t^2 = 0$$

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## Conclusions

- There is a certain equivalence between the **martingale approach** to pricing derivative assets and the one that uses **PDEs**.
- It was shown that the martingale approach implies the same PDEs utilized by the PDE methodology. The difference is that, in the martingale approach, the PDE is **a consequence of risk-neutral asset pricing**, whereas in the PDE method, one **begins with the PDEs to obtain risk-free prices**.

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