CH16
Tools for Complicated Derivative Structures

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Outline

1 Introduction

- The types of derivative securities are not always "plain vanilla".
- Recall some assumptions of B-S framework:
  - Non-dividend-paying stock
  - Constant risk-free rate
  - European-type
  - No transaction costs or indivisibilities
- Two aspects of B-S framework are always preserved:
  - No early-exercise
  - Constant risk-free rate

2 New Tools

- Three separate issues:
  - Term structure interest rate and yield curve
    - We use them for motivating the mathematical results, because they are too broad to deal with.
  - Expectation of stochastic process and PDEs
    - Including generator of a stochastic process, Kolmogorov's backward equation, and the Feyman-Kac formula.
  - Stopping time
    - For American-type

2.1 Interest Rate Derivatives

- Bond options
  - Two complications under B-S environment:
    1. Bond price depends on interest rate (B = f(r)). When r is constant, B will be completely predictable (volatility of B is zero).
    2. Bond options are, in general, of the American style.
- Caps and floors
  - Hedge the risk of increasing (for caps) interest rate or decreasing (for floors) interest rate.
- SW options
- For interest rate derivatives, "early exercise possibility" and "stochastic interest rate" must be incorporated in asset pricing.

3 Term Structures of Interest Rates

- When r is constant, the bond price: $B(u, t) = 100e^{-r(u-t)}$ (par value = 100, maturity = u, current time = t).

- For the stochastic interest rate: $B(u, t) = 100E[e^{-\int_t^u r_s ds} | I_t]$ --- (1) (conditional expectation operator = E[·|·], instantaneous rate at future date s = r_s)

- Issue of prob. Measure:

  Q: Why do we use the equivalent martingale measure (EMM) on risk-free asset?

  A: r is stochastic ➔ B with long maturities are subject to more shocks ➔ longer maturities are "riskier" when everything is the same ➔ use EMM to eliminate the corresponding risk premia.

- Consider a 3-year bond in a discrete time framework:

$$B(3, 1) = E\left[\frac{100}{(1+r_1)(1+r_2)(1+r_3)}\middle| I_1\right]$$

(r_1 is current short rate, r_2 and r_3 are unknown short rates)

- From above example, it's useful in interpreting the continuous-time bond price given in (1). We just replace the discrete time discount factor by the exponential function in (1).
- Implication: bond price depends on whole spectrum of future short rates r_s, $t < s < u$

➔ "yield curve" or "the term structure of interest rate" at time t contains all info. concerning future short rates.

DEFINITION: At time t, there exists zero coupon bonds with a full spectrum of maturities $u \in [t, T]$. From those bonds, the spectrum of yields $\{R_t^u, u \in [t, T]\}$ is called the term structure of interest rate.

$$B(u, t) = 100e^{-R_t^u(u-t)}, t < u \text{ ---------------- (2)}$$

where $B = 100E[e^{-\int_t^u r_s ds}|I_t]$ is given by the expection under risk-neutral prob.

$$\Rightarrow R_t^u = \frac{\log B(u, t) - \log(100)}{t - u}$$

- From the term structure of interest rate, we can look at two different changes:

1. At any instant t, we can ask what happens to R_t^u as u changes by du ($\frac{dR_t^u}{du} = g_u$,).
2. As t changes, the yield curve would shift because of random shocks.

3.1 Relating r_s and R_t^u

We can relate future short rate to the yield curve of time t using (1) and (2):

$$e^{-R_t^u(u-t)} = E[e^{-\int_t^u r_s ds}|I_t]$$

$$\Rightarrow R_t^u = \frac{\log E[e^{-\int_t^u r_s ds}|I_t]}{t - u}$$

Finally, we define forward rate:

$$F(t, u, T) = \frac{\log B(u, t) - \log B(T, t)}{T - u}, t < u < T \text{ ------ (3)}$$

let $T \to u$, we get the instantaneous forward rate:

$$f(t, u) = \lim_{T \to u} F(t, u, T)$$

This assumes that bond prices are differentiable. Using (3) and assuming some technique conditions are satisfied:

$$f(t, t) = r_t$$

We can see that the yield curve contains all relevant info. concerning forward rates.

3.1.1 Examples of Yield Curves

Two different ways of proceeding:

1. Assume a functional form for the yield curve (obtain the implied forward rates from spot rate).

$$R(r_t, u, t) = A(u, t) - C(u, t)r_t$$
$$dr = a(r_t, t)dt + \sigma(r_t, t)dW_t$$

Where A and C can be constructed to present upward, downward-sloping, or humped-shaped. The movements of yield curve will be stochastic, because r_t has an unpredictable component.

2. Assume a dynamic behavior for the forward rate to obtain a yield curve.

$$df(t, u) = a(f, t)dt + \sigma(f, t)dW_t^u$$

From here we can use the relationship in (3) to determine the yields.

4 Characterization of Expectations Using PDEs

- The findamental PDE under B-S assumptions:

$$0 = -Fr + F_t + rF_sS_t + \frac{1}{2}F_{ss}\sigma_t^2$$

- The risk-neutral expectation:

$$F(S_t, t) = E^{\tilde{P}}[e^{-r(T-t)}F(S_T, T)] \text{ -------- (4)}$$

- Two questions:
  - Do we get similar PDEs in the case of interest rate derivatives?
  - Given a PDE involving an interest rate derivative, can we obtain a corresponding expectation similar to (4)?

4.1 Risk-Neutral Bond Pricing

$$B(u, t) = E_t^{\tilde{P}}[100e^{-\int_t^u r_s ds}], t < s < u$$

where r_t obeys $dr_t = a(r_t)dt + \sigma(r_t)dW_t$

- Note that the special aspect of the drift and diffusion coefficients. They are not a function of t.
- The instantaneous interest rate is stochastic and changes constantly. The fact that future r are unknown forces us to keep the exponential function inside the expectation operator E.

- But the expectation on above slide is not always easy to calculate.

An alternative representation: $B(u, t) = E_t^{\tilde{P}}[100e^{-\int_t^u r_s ds} f(r_u)]$

where f is some twice differentiable function to stand for some boundary value. And B would automatically satisfy a particular PDE.

5 Random Discount Factors and PDEs

- In order to establish the corresponding PDE, the Feyman-Kac formula would be obtained.
- There are several steps in F-K formula. We'll see the important steps concerning Ito diffusions, generators of Ito diffusions, and Kolmogorov's backward equation.

5.1 Ito Diffusions

- A continuous stochastic process with finite first- and second-order moments is shown to follow the general SDE:

$$dS_t = a(S_t, t)dt + \sigma(S_t, t)dW_t, \; t \in [0, \infty)$$

then we assume that the drift (a) and diffusion (σ) parameters depend on S_t only. The SDE can be written as:

$$dS_t = a(S_t)dt + \sigma(S_t)dW_t, \; t \in [0, \infty)$$

- Processes with this characteristic are called time-homogenous Ito diffusions.
- The drift and diffusion parameters are not dependent on t, in that they are not supposed to vary "too fast."

5.2 The Markov Property

- Let S_t be an Ito diffusion, f(·) be any bounded function, and suppose that the info. set I_t contains all S_u until time t.

- We can say that S_t satisfies the Markov property

if $\quad E[f(S_{t+h}|I_t)] = E[f(S_{t+h})|S_t], \quad h > 0, \quad$ for all t.

- dS is a function of dW (indep. of the present and the past); drift (a) and diffusion (σ) depend on S_t only
$\Rightarrow$ future forecasts are indep. of S_u observed before time t.

5.3 Generator of an Ito Diffusion

- S_t is the Ito diffusion given in (5). $f(S_t)$ is a twice differentiable function of S_t, and suppose the process
- S_t has reached a particular vlaue s_t at time t. Let the operator A be defined as the expected rate of change for $f(S_t)$:

$$Af(s_t) = \lim_{\Delta \to 0} \frac{E[f(S_{t+\Delta})|f(s_t)]-f(s_t)}{\Delta}$$

A is called the generator of the Ito diffusion S_t

Q: The rate is a function of a Wiener process, but the Wiener processes are not differentiable. How can we justify the existence of *A*?

A: *A* doesn't deal with the actual rate of change in f(S). It's an expected rate of change. The expected change in f(S) will be a smoother function

5.4 A Representation for *A*

Univariate stochastic process:

$$Af = a_t \frac{\partial f}{\partial S} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial S^2}$$

Ito's lemma:

$$df(S_t) = [a_t \frac{\partial f}{\partial S} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial S^2}]dt + \sigma_t \frac{\partial f}{\partial S} dW_t$$

- The difference between *A* and Ito's lemma is at two points:
  - dW term in Ito's formula is replaced by it's drift, which is zero.
  - The remaining part of Ito's is divided by dt.
- These differences are consistent with the definition of *A*.

5.4.1 The multivariate Case

Let X_t be a k-dimensional Ito diffusion given by th SDE:

$$\begin{bmatrix} dX_{1t} \\ \vdots \\ dX_{kt} \end{bmatrix} = \begin{bmatrix} a_{1t} \\ \vdots \\ a_{kt} \end{bmatrix} dt + \begin{bmatrix} \sigma_t^{11} & \vdots & \sigma_t^{1k} \\ \cdots & \cdots & \cdots \\ \sigma_t^{k1} & \cdots & \sigma_t^{kk} \end{bmatrix} \begin{bmatrix} dW_{1t} \\ \vdots \\ dW_{kt} \end{bmatrix}$$

in symbolic form:

$$dX_t = a_t dt + \sigma_t dW_t, \quad t \in [0, \infty)$$

a_t is a $k \times 1$ vector, σ_t is a $k \times k$ matrix

The corresponding A operator will be given by

$$Af = \sum_{i=1}^{k} a_{it} \frac{\partial f}{\partial X_i} + \sum_{i=1}^{k}\sum_{j=1}^{k} \frac{1}{2}(\sigma_t \sigma_t^T)^{ij} \frac{\partial^2 f}{\partial X_i \partial X_j}$$

This expression is the infinitesimal generator of $f(\cdot)$.

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5.5 Kolmogorov's Backward Equation

- Consider the expectation: $\tilde{f}(S^-, t) = E[f(S_t)|S^-]$, for all $t \geq 0$ --- (6)

S^- is the latest value observed before time t.

- Use A to characterize how the $\tilde{f}$ may change over time.

Kolmogorov's backward equaiton (KBE): $\frac{\partial \tilde{f}}{\partial t} = A\tilde{f}$ ----------- (7)

where $A\hat{f} = a_t \frac{\partial \tilde{f}}{\partial S} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 \tilde{f}}{\partial S^2} \Rightarrow \tilde{f}_t = a_t \tilde{f}_s + \frac{1}{2}\sigma_t^2 \tilde{f}_{ss}$

- The correspondence can be stated in two different ways:

1. $\tilde{f}$ staisfies the PDE in (7)
2. Given the PDE in (7), we can find an $\tilde{f}$ such that (6) is satisfied.

- This result means that $\tilde{f}$ is a soluation for the PDE in (7). KBE gives the first correspondence between an expectation of a stochastic process and PDEs.
- It's not very useful in financial market, because f depends on S only, and a random discount factor is not allowed.

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5.5.1 Example

Consider the function $p(S_t, S_0, t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(S_t - S_0)^2}{2t}}$

- a conditional density function of a Wiener process with zero drift and variance t.

- The SDE for this process: $dS_t = dW_t$
(drift papameter = 0, diffusion paprmeter = 1)

- Apply KBE to this density. The twice-differentiable funtion $\tilde{f}$ of S_t will be $\tilde{f}_t = \frac{1}{2}\tilde{f}_{ss}$.

- For the conditional density $p(S_t, S_0, t)$, we take the first partial derivative with respect to t and the second partial derivative with respect to S_t. The equaiton will be satisfied (Wiener process satisfies KBE).

- This PDE tells us how the prob. associated with S_t and "initial" condition S_0, will evlove as time passes.

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5.6 The Feyman-Kac Formula

- In more general setting: $\tilde{f}(t, r_t) = E[e^{-\int_t^u q(r_s)ds} f(r_u)|r_t]$ ----- (8)
the formula permits a stochastic interest rate to be used.

- In fact, the a close examination of (8) indicates that with $q(r_s) = r_s$ and $f(\cdot)$ selected as expiration payoff at time u.

- Fetman-Kac (F-K) formula is an extension of KBE.

DEFINITION: Given $\tilde{f}(t, r_t) = E[e^{-\int_t^u q(r_s)ds} f(r_u)|r_t]$, all $t \geq 0$

we have $\frac{\partial \tilde{f}}{\partial t} = A\tilde{f} - q(r_t)\tilde{f}$ where $A\tilde{f} = a_t \frac{\partial \tilde{f}}{\partial r_t} + \frac{1}{2}\sigma_t^2 \frac{\partial^2 \tilde{f}}{\partial r_t^2}$

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5.6.1 A PDE for Bond Prices

Consider the time t price of a discout bond that maturites at time u:

$$B(u, t) = E[e^{-\int_t^u r_s ds} 100|r_t], \quad t \in [0, u]$$

The r satisfies:

$$dr_t = a(r_t)dt + \sigma(r_t)dW_t, \quad t \in [0, \infty)$$

In F-K formula, $B(t, u, r_t)$ must satisfy:

$$\frac{\partial B}{\partial t} = AB - r_t B$$

Substituting for the A, the PDE will be:

$$B_t = a_t B_r + \frac{1}{2}\sigma_t^2 B_{rr} - r_t B, \quad r \geq 0, 0 \leq t \leq u$$

the boundary condition: $B(u, u) = 100$

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6 American Securities

6.1 Stopping Times

- Stopping times are special types of random variables.
- It means the "possible date" that we may exercise on the particular time period (on or before the expiration date).

DEFINITION: A stopping time (τ) is an I_t-measurable nonnegative random variable such that

1. Given I_t we can tell if $\tau \leq t$ or not.
2. We have $P(\tau < \infty) = 1$

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6.2 Use of Stopping Times

- If one wait until expiraiton, a call option will be worth:

$$F(S_t, t)^T = E_t^{\tilde{p}}[e^{-r(T-t)}\max\{S_T - K, 0\}],$$

where $\tilde{p}$ is EMM

- If the option can be exercised early:

$$F(S_t, t)^* = \sup_{\tau \in \Phi_{t,T}} [E_t^{\tilde{p}}[e^{-r(T-t)}F(S_t, t, \tau)]],$$

where $\Phi_{t,T}$ is the set of all possible stopping opportunities

- At time t, we calculate a spectrum of possible prices $F(S_t, t, \tau)$ indexed by τ using the possible values for the stopping time τ.

7 Extending the Results to Stopping Times

7.1 Martingales

- Suppose M_t represents a continuous-time martingale with

$$E[M_{t+u}|I_t] = M_t, \; u > 0$$

- Let τ_1 and τ_2 be two indep. stopping times with respect to I_t and satisfying $P(\tau_1 < \tau_2) = 1$.

- Then the martingale property will still hold: $E[M_{\tau_2}|I_{\tau_1}] = M_{\tau_1}$

- The fact that the randomly selected stopping time does not preclude the use of EMM.

7.2 Dynkin's Formula

- Let B be a process: $dB_t = a(B_t)dt + \sigma(B_t)dW_t, \; t \geq 0$.

  $f(B_t)$ is a twice-differentiable bounded function

- Consider a stoppoing time τ such that $E[\tau] < \infty$

- Then we have $E[f(B_\tau)|B_0] = f(B_0) + E[\int_0^\tau Af(B_s)ds|B_0]$.

8 Conclusions

- We showed that there was an important equivalence between some expectation of stochastic process and certain class of PDEs
- These results enable the practitioner to choose the more convenient method.
- This chapter also introduced the notion of stopping times in pricing American-style derivatives.

Many Thanks for your Patience