Chapter 4

Pricing Derivatives

Models and Notation

Introduction

- By simplifying assumptions, we can express **the arbitrage-free price of a derivative** as a function of some "basic" securities, and then obtain a set of formulas that can be used to price the asset without any linkages to other financial markets or to the real side of the economy.

Approaches to pricing derivatives

- Method of equivalent **martingale** measures.
- Constructs a risk-free portfolio, and obtains a **partial differential equation (PDE).**
  - Problem – hard to find a **closed-form formula.**

Pricing Functions- -Forwards

- **Cash-and-carry goods.**
  -- long futures, short spot.
- ***Securing one unit of physical gold at time T.***

1. Buying directly now.

TC = $S_t * e^{r_t(T-t)} + (T-t)C$

Pricing Functions- -Forwards

- ***Securing one unit of physical gold at time T.***

2. Buying a forward contract.

→ **Outcome is the same as (1)**

→ **cost must be identical** ☺

→ $F(S_t, t) = S_t * e^{r_t(T-t)} + (T-t)C$

Pricing Functions-Forwards

- Forward contracts – linear products.
  (linear in S_t)

$$F(S_t, t) = S_t * e^{r_t(T-t)} + (T-t)C$$

- Optionlike contracts – nonlinear products.

$$c = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Boundary Conditions

- **expiration date gets nearer.**

(concept of limit) $\lim_{t \to T} e^{r_t (T-t)} = 1$

(problem) r_t **– random variable.**

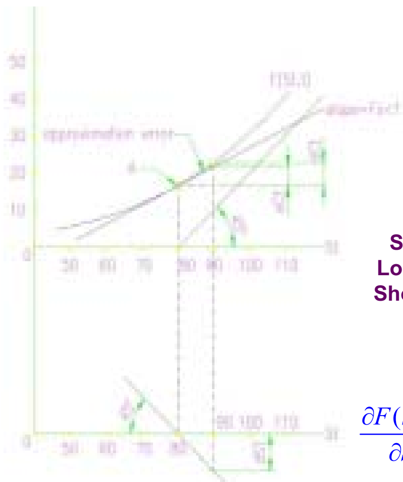
$$F(S_t, t) = S_t * e^{r_t (T-t)} + (T-t)C$$

$$S_T = F(S_T, T)$$

Options

$$C_t = F(S_t, t)$$

- Under simplifying conditions, S_t will be the only determinant affecting the option's price, unpredictable movements in S_t can be offset by opposite positions taken simultaneously in C_t.



Strategy…
Long 1@ call
Short stock…

$$\frac{\partial F(S_t, t)}{\partial S_t} = F_s$$

Options - delta hedge.

Definition : $d[F_s S_t] + d[F(S_t, t)] = g(t)$

offsetting changes in C_t by taking the opposite position in F_s units of the underlying asset.

portfolio : delta neutral

$$\frac{\partial F(S_t, t)}{\partial S_t} = F_s$$ → delta ratio

$\partial C_t \cong dC_t$ (as dSt is large, it fails.)

Application - -
Another Pricing Method

- Summarize the pricing method that uses **PDE**s.

1. Given $F(S_t, t)$、S_t we'd like to calculate $\frac{\partial F(S_t, t)}{\partial S_t}$

2. Total differential of F(·)

$$dF(S_t, t) = F_s dS_t + F_t dt \quad \text{----(A)}$$

$$F_s = \frac{\partial F}{\partial S_t}, F_t = \frac{\partial F}{\partial t}$$

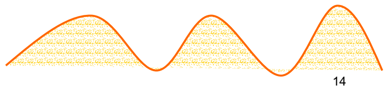
$dF(S_t, t) = F_s dS_t + F_t dt$ ----- (A)

3. Equation (A) can be used as the partial derivatives F_s, F_t are evaluated numerically.
(it requires that the functional form of $F(S_t, t)$ be known)

- *What if the underlying variables are continuous-time stochastic processes?*

4. By delta-hedging and risk-free portfolios, we can get relationships among $dF(S_t, t)$, dS_t, and dt.

5. Then, we can get PDE. And if we have enough boundary condition and if a closed-form solution exists, $F(S_t, t)$ could be solved.



Example

- **trivial PDE** $F_x = b \qquad x \in (0, X)$

$$F(x) = a + bx$$

- **(Question) the parameter a is unknown….**

- **solution…**
  using "boundary condition."

Example (con.)

- Assume we know that at the boundary x = X , $F(X) = 10$

$$F(x) = a + bx$$

$$a = 10 - bX$$

$$F(x) = 10 + b(x - X)$$

The problem…

$$dF(t) = F_s dS_t + F_r dr_t + F_t d_t \ ??$$

- It works only during infinitesimal intervals.
- Since time is continuous, we observe uncountable many random variables as time passes.
- (i.e., $F(S_t, t)$, S_t , r_t are all continuous-time stochastic processes.)
- Financial mkt data are not **deterministic**

Multivariate Taylor serious expansion…

$$dF(x) = F_s dS_t + F_r dr_t + F_t d_t + \tfrac{1}{2} F_{ss} dS_t^2 + \tfrac{1}{2} F_{rr} dr_t^2 + \tfrac{1}{2} F_{tt} dt + F_{sr} dS_t dr_t + F_{st} dS_t dt + F_{rt} dr_t dt + \ldots\ldots\ldots$$

Assume $(dS_t)^2$, $(dr_t)^2$ and $(dt)^2$ are small enough…

$$dF(t) = F_s dS_t + F_r dr_t + F_t d_t$$

Actually, they are random during small intervals, and have nonzero variances, so we use positive number for the average values of $(dS_t)^2$, $(dr_t)^2$

Hence, we have to study "stochastic calculus"