Tools in Probability Theory

g926420
楊子宸

1

Introduction

- We will review some basic notions in probability theory.
- Probability
- Some Important Models
- Convergence of Random Variables
- Conclusions

2

Probability

- The symbol Ω represents all possible states of the world.
- A particular state of the world is denoted by the symbol ω
- The intuitive notion of an event corresponds to a set of elementary ω 's

3

- The set of all possible events is represented by the symbol ξ
- To each event A $\in$ ξ one assigns a probability P(A)

4

Two conditions of consistency are the following:

1. $0 \leq P(A) \leq 1$ any $A \in \xi$

2. $\int_{A \in \xi} dP(A)=1$

5

Example

- Suppose the price of an exchange-traded commodity future during a given day depends only on a harvest report the U.S. Department of Agriculture (USDA) will make public during that day.
- The specifics of the report written by the USDA are equivalent to an ω.

6

- Depending on what is in the report, we can call it either favorable or unfavorable. This constitutes an example of an event. Note that there are several ω's that may lead us to call the harvest report "favorable". It is in this sense that events are collections of ω's.

- Hence, we may want to know the probability of a "favorable report."
- This is given by P(harvest report=favorable).
- Finally, note that in this particular example the Ω is the set of all possible reports that the USDA may make public.

Random Variable

- A random variable X is a function, a mapping, defined on the set ξ.
- Given an event A ξ, a random variable will assume a particular numerical value. Thus, we have X: $\xi \rightarrow$ B
- Where B is the set made of all possible subsets of the real numbers R.

CDF & PDF

- CDF: $G(x)=P(X \leq x)$
- PDF: $g(x)=dG(x)/dx$

Moments

- There are different ways one can classify models of distribution functions. One classification uses the notion of "moments."
- Some random variables can be fully characterized by their first two moments. Others need higher-order moments for a full characterization.

- First Two Moments :

The expected value E(X) of a random variable X, with density f(x), is called the first moment. It is defined by

$$E(X)= \int_{-\infty}^{\infty} xf(x)dx$$

Where f(x) is the corresponding probability density function.

Second moment

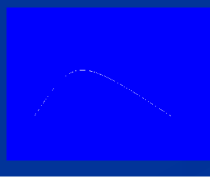
The variance $E(X-EX)^2$

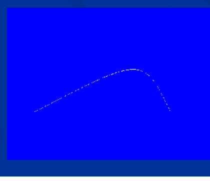
is the second moment around the mean.

Heavy Tail

- A distribution that has heavier tails than normal curve means a higher probability of extreme observations.

For example:





Conditional Expectations

- Conditional probabilty:

$$f(x|y)=\frac{f(x,y)}{f(y)}$$

- Conditional Expectation:

$$E(x|y)=\int_{-\infty}^{\infty} xf(x|y)dx$$

Properties of conditional Expectations:

- Often, the expectation conditional on an information set It is written compactly as

$$E[\bullet \mid It] = Et$$

The t subscript in Et indicates that in the averaging operation one uses all information available up to time t.

- 1. The conditional expectation of the sum of two random variables is the sum of conditional expectations:

$$Eu[St+F(t)] = Et[St] + Eu[F(t)]$$

2.At time t to forecast the expectation that at time (t+T) to forecast (s+T+u)

Et[Et+T(St+T+u)]=Et[St+T+u]

Some important Models

- Binomial Distribution in Financial Markets:

Assume that the price F(t) changes continuously over time, but the trader is assumed to have limited scope of attention and checks the market price every Δ seconds.

We assume that Δ is a small time interval.

More importantly, we assume that at any time t there are two possibilities:

1. There is either an uptick and prices increase according to

$$\Delta F(t) = +a\sqrt{\Delta} \qquad a > 0.$$

2. Or, there is a downtick and prices decrease by

$$\Delta F(t) = -a\sqrt{\Delta} \qquad a > 0$$

$$P(\Delta F(t) = +a\sqrt{\Delta}) = p$$
$$P(\Delta F(t) = -a\sqrt{\Delta}) = (1-p)$$

- If the $\Delta F(t)'s$ are independent of each other, the sequence of increments $\Delta F(t)$, will be called a binomial stochastic process, or simply a binomial process.

The Normal Distribution

$$F(0+\Delta) = \begin{cases} F(0)+a\sqrt{\Delta} & \text{with probability p} \\ F(0)-a\sqrt{\Delta} & \text{with probability 1-p} \end{cases}$$

$$F(0+2\Delta) \begin{cases} F(0)+a\sqrt{\Delta}+a\sqrt{\Delta} & \text{with probability p^2} \\ F(0)-a\sqrt{\Delta}+a\sqrt{\Delta} & \text{with probability 2p(1-p)} \\ F(0)-a\sqrt{\Delta}-a\sqrt{\Delta} & \text{with probability (1-p)^2} \end{cases}$$

- According to the central limit theorem, the distribution of F($n\Delta$) approaches the normal distribution as $n\Delta \rightarrow \infty$, $n \rightarrow \infty$

Assume that p=.5 and that F(0)=0

The approximating density function will be given by

$$g(F(n\Delta)=x)=\frac{1}{\sqrt{2\pi a^2 n\Delta}}e^{-\frac{1}{2a^2 n\Delta}(x)^2}$$

The Poisson Distribution

For a poisson process, the probability of a jump during a small interval Δ will be given approximately by

$P(\Delta N_t=1)=\lambda\Delta$,

where λ is a positive constant called the intensity.

$P(\Delta N_t=0)=1-\lambda\Delta$

The probability that during a finite interval Δ there will be n jumps is given by

$$P(\Delta Nt=n)=\frac{e^{-\lambda\Delta}(\lambda\Delta)^n}{n!},$$

which is the corresponding

Convergence of Random Variables

- In pricing financial securities, a minimum of three different convergence criteria are used.
- The first is mean square convergence.

DEFINITION: Let $X_0, X_1, \ldots, X_n, \ldots$ be a sequence of random variables. Then X_n is said to converge to X in mean square if

$$\lim_{n\to\infty} E[Xn-X]^2=0$$

- According to this definition, the random approximation error ε_n defined by $X_n=X+\varepsilon_n$ will have a smaller and smaller variance as n goes to infinity.

Relevance of Mean Square Convergence

- Mean square(m.s) convergence is important because the Ito integral is defined as the mean square limit of a certain sum. In particular, if one uses other definitions of convergence this limit may not exist.

Consider a more “natural” extension of the notion of limit used in standard calculus

- DEFINITION: A random variable X_n converges to X almost surely if, for arbitrary $\delta > 0$,

$$P(\left|\lim_{n\to\infty} Xn - X\right| > \delta) = 0$$

The definition is a natural extension of the limiting operation used in standard calculus. It says that as n goes to infinity, the difference between the two random variables becomes negligibly small.

Weak Convergence

In the case of weak convergence, what is being approximated is not the value of a random variable X_n, but the probability associated with a sequence $X_0,\ldots,X_n$. Weak convergence is used in approximating the distribution function of families of random variables.

DEFINITION: Let X_n be a random variable indexed by n with probability distribution P_n. We say that X_n converges to X weakly and $\lim_{n\to\infty} Pn = P$,

where P is the probability distribution of X if

$$E^{Pn} f(Xn) \to E^{P} f(X) \;,$$

where f(•) is any bounded, continuous, real-valued function.

Conclusions

The chapter briefly reviewed some basic concepts of probability theory to be needed later.
We spent a minimum of time on the standard definitions of probability. However, we made a number of important points.

- First we characterized normally distribution random variables and poisson process as two basic building blocks.
- Second, we discussed an important binomial process. This example was used to introduce the important notion of convergence of stochastic process.