Martingale and Martingale representation

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Outline

- Definition.
- Properties of martingale trajectories.
- Examples of martingales.
- First Stochastic Integral.

1.Definition

- A stochastic process behaves like a *martingale* if its trajectories display no discernible trends or periodicities.

  A process that, on the average, increases is called a *submartingale*.

  Supermartingale represents process that, on the average, decline.

1.1 Notation

- Let the observed process be noted by

$$\{S_t\ ,\ t \in [0,\infty]\}$$

- Let $\{I_t\ ,\ t \in [0,\infty]\}$ represent a family of information sets that become continuously available to decision maker as time passes.

- With s<t<T , this family of information set will satisfy

$$I_s \subseteq I_t \subseteq I_T$$

The set $\{I_t\ ,\ t \in [0,\infty]\}$ is called a filtration

- If the value of S_t is included in the information set I_t , at each t>0 , then it is said that $\{S_t\ ,\ t \in [0,\infty]\}$ is adapted to $\{I_t\ ,\ t \in [0,\infty]\}$ That is , the value of S_t will be known, given the information set I_t.

1.2 Continuous-Time Martingales

- We say that a process $\{S_t\ ,\ t \in [0,\infty]\}$ is a Martingale with respect to the family of information sets I_t, and with respect to the probability P ,if, for all t>0

1.S_t is known, given I_t .

2.Unconditional forecasts are finite

$$E\,|\,S_t\,|\ < \infty$$

3. $E_t(S_T) = S_t$ For all t < T

- Suppose S_t is a martingale and consider the forecast of the change in S_t over an interval of length u>0

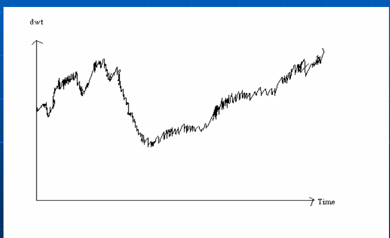
$$E_t[S_{t+u} - S_t] = E_t(S_{t+u}) - E_t(S_t)$$
$$= S_t - S_t$$
$$= 0$$

- So martingales are r.v. whose future variations are completely unpredictable given the information set.

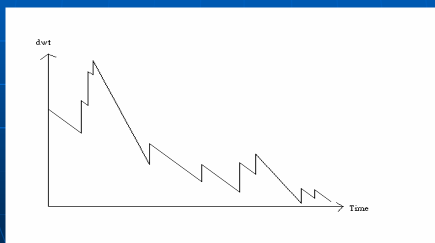
1.3 The use of martingale in asset pricing

- We know that stock prices or bond prices are not completely unpredictable. The price of discount bond is expected to increase over time. If B_t represents the price of discount bond maturing at time T, t<T.
- $$B_t < E_t(B_{t+u})$$

- Irregular trajectories can occur in two different ways. They can be continuous. It leads to *continuous martingale*.



- If they can display jumps. It is called *right continuous martingales.*



- Suppose one is dealing with a continuous martingale X_t that also has a finite second moment

$$E\left(X_t^2\right) < \infty$$

Such a process has finite variance and it is called continuous square integrable martingale.

1.4 Properties of martingale trajectories

- Define the variation of the trajectories

$$V^1 = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|$$

- The quadratic variation is given by

$$V^2 = \sum_{i=1}^{n} \left|X_{t_i} - X_{t_{i-1}}\right|^2$$

- One can similarly define high-order variation. For example, fourth-order variation is

$$V^4 = \sum_{i=1}^{n} \left|X_{t_i} - X_{t_{i-1}}\right|^4$$

- Assume $\{X_t\}$ represents a trajectories of a continuous square integrable martingale. Pick a time interval [0,T], consider the time $\{t_i\}$

$t_0=0 < t_1 < t_2 < \bullet \bullet \bullet \bullet < t_n = T$

- According to the assumption , V^1 ,V^2,V^3,V^4 have some important properties.

$$V^2 = \sum_{i=1}^{n} \left| X_{t_i} - X_{t_{i-1}} \right|^2 < \left[Max \left| X_{t_i} - X_{t_{i-1}} \right| \right] \sum_{i=1}^{n} \left| X_{t_i} - X_{t_{i-1}} \right|$$

$$= \left[Max \left| X_{t_i} - X_{t_{i-1}} \right| \right] V^1$$

As $t_i \to t_{i-1}$ for all t, it impiles that "consecutive" X_{t_i} will get very near each other. At the lmit,

$$Max \left| X_{t_i} - X_{t_{i-1}} \right| \to 0$$

It means that unless V^1 get very large, V^2 will go towards zero. But it is not allowed, because, X_t is a stochastic process with nonzero mean and consequently $V^2 > 0$ even for very fine partitions of [0,T]. So we must have $V^1 \to \infty$

Now consider the same property for high - order variations. For example, consider V^4

$$V^4 < Max \left| X_{t_i} - X_{t_{i-1}} \right|^2 V^2$$

As long as V^2 converges to a well - defined r.v., the right hand side of inequation will go to zero. This means that V^4 will tend go zero.

Summarize the three properties of the trajectories.

- 1. The variation V^1 will converge to infinity in some probability sense and the continuous martingale will be very irregular.
- 2.The quadratic variation V^2 will converge to some well-defined r.v.
- 3.All high-order variation will vanish in some probability sense.

1.5 Examples of martingales

- In this section, we will learn how to convert $\{X_t\}$ which is not a martingale into a martingale.

1.We can subtract an expected trend.

2.Doob-Meyer decomposition.

Example1

Suppose X_t represents a continuous process whose increments are normally distributed. Such a process is called Brownian motion. We observer a value $\{X_t\}$ for each t . Incremental changes in X_t are assumed to be independent across time.

Under these conditions , if Δ is a small interval, the increments ΔXt during ΔX_t will have a normal distribution with mean $\mu\Delta$ and variance $\sigma^2\Delta$

$$\Delta X_t \sim N(\mu\Delta\ ,\ \sigma^2\Delta)$$

The fact that increments are uncorrelated can be expressed as

$E[(\Delta X_t - \mu\Delta)(\Delta X_u - \mu\Delta)] = 0$

Is X_t a martingale ?

The process X_t is the "accumulaton" of infintesimal increments over time, that is,

$$X_{t+T} = X_0 + \int_0^{t+T} dX_u$$

$$E_t[X_{t+T}] = E_t[X_t + \int_t^{t+T} dX_u] = X_t + \mu T$$

So $\{X_t\}$ is not a martingale.

Consider a new process

$$Z_t = X_t - \mu t$$

Then

$$E_t[Z_{t+T}] = E_t[X_{t+T} - \mu(t+T)]$$
$$= X_t + \mu T - \mu(t+T)$$
$$= X_t - \mu t$$
$$= Z_t$$

That is, Z_t is a martingale

Example2

Suppose a trader observes at time t_i,

$$t_0 < t_1 < \cdots\cdots < t_{k-1} < t_k = T$$

the price of a financial asset S_t.

$$\Delta S_{t_i} = \begin{cases} 1 & \textit{with probability } p \\ -1 & \textit{with probability } (1-p) \end{cases}$$

For example, a typical sample path can be

$$\{\Delta S_1 = -1, \cdots\cdots, \Delta S_t = 1\}$$

Another assumption that simplified this task was the independence of successive price changes.

$$S_{t_k} = S_{t_0} + \sum_{i=1}^{k}(S_{t_i} - S_{t_{i-1}})$$

the highest possible value for S_{t_k} is $S_{t_0} + k$

$$P(S_{t_k} = S_{t_0} + k) = P^k$$

the lowest possible value of S_{t_k} is $S_{t_0} - k$

$$P(S_{t_k} = S_{t_0} - k) = (1-p)^k$$

In general, the price would be somewhere between these two extremes. Of the k incremental changes observed, m would be made of +1's and k-m made of -1's, with m-k.

$$S_{t_k} = S_{t_0} + m - (k - m) = S_{t_0} + 2m - k$$

$$P(S_{t_k} = S_{t_0} + 2m - k) = \binom{k}{m} p^m (1-p)^{k-m}$$

Is S_{t_k} a Martingale ?

$$E_{k-1}[S_{t_k} \mid S_{t_0},\ \triangle S_{t_1},\cdots\cdots,\triangle S_{t_{k-1}}] = S_{t_{k-1}} + [(+1)p + (-1)(1-p)]$$
$$= S_{t_{k-1}} + 2p - 1$$

$p = \frac{1}{2} \Rightarrow S_{t_k}$ is a martingale.

$p \neq \frac{1}{2} \Rightarrow S_{t_k}$ is not a martingale.

Define :

$$Z_{t_k} = S_{t_k} - (2p-1)(k+1)$$
$$\Rightarrow Z_{t_k} = S_{t_k} + (1-2p)(k+1)$$

$$E_k(Z_{t_{k+1}}) = E_k[S_{t_{k+1}} + (1-2p)(k+2)]$$
$$= E_k[S_{t_{k+1}}] + (1-2p)(k+2)$$
$$= S_{t_k} + 2p - 1 + (1-2p)(k+2)$$
$$= S_{t_k} + (1-2p)(k+1)$$
$$= Z_{t_k}$$

So, $\{Z_{t_k}\}$ is a martingale with respect to I_{t_k}.

1.6 Doob-Meyer Decomposition

If $X_t,\ 0 \leq t \leq \infty$ is a right continous submartingale with respect to $\{I_t\}$, and if $E[X_t] < \infty$, for all t , then, X_t admits the decomposition

$$X_t = M_t + A_t$$

where M_t is a right continuous martingale, and A_t is an increase process measurable with respect to I

Example3.

Let

$$S_{t_k} = S_{t_0} + 2m - k$$
$$\Rightarrow E_{k-1}[S_{t_k} \mid S_{t_0},\ \triangle S_{t_1},\cdots\cdots,\triangle S_{t_{k-1}}]$$
$$= S_{t_{k-1}} + 2p - 1$$

if

$$\frac{1}{2} < p < 1$$
$$\Rightarrow E_{k-1}[S_{t_k} \mid S_{t_0},\ \triangle S_{t_1},\cdots\cdots,\triangle S_{t_{k-1}}] > S_{t_{k-1}}$$

So, $\{S_{t_k}\}$ *is a submartingale.*

Then we can write

$$S_{t_k} = -(1-2p)(k+1) + Z_{t_k}$$

where Z_{t_k} *is a martingale.*

1.7 The First Stochastic Integral

Let $H_{t_{i-1}}$ be any r.v adapted to $I_{t_{i-1}}$. Let Z_t be any martingale with respect to I_t.

Define a new martingale M_t

$$M_{t_k} = M_{t_0} + \sum_{i=1}^{k} H_{t_{i-1}}[Z_{t_i} - Z_{t_{i-1}}]$$

$$E_{t_0}[M_{t_k}] = M_{t_0} + E_{t_0}[\sum_{i=1}^{k} E_{t_{i-1}}[H_{t_{i-1}}(Z_{t_i} - Z_{t_{i-1}})]]$$

$$(\quad \because E_{t_0}[\bullet] = E_{t_0}[E_{t_{i-1}}[\bullet]] \quad)$$

and

$$E_{t_{i-1}}[H_{t_{i-1}}(Z_{t_i} - Z_{t_{i-1}})]$$
$$= H_{t_{i-1}} E_{t_{i-1}}(Z_{t_i} - Z_{t_{i-1}})$$
$$= 0$$

$\therefore\ E_{t_0}[M_{t_k}] = M_{t_0}$

M_t has the martingale property.

It is the first example of a stochastic integral

$As\ t_i - t_{i-1} \to 0$

Using some analogy, can We obtain an expression such as

$$M_{t_k} = M_{t_0} + \int_0^t H_u\, dZ_u$$

Where dZ_u represents an infinitesimalstochastic increment with zero mean given the information at time t ?

1.7.1 Application to Finance

We consider a decision maker who invest in a riskless and risky security at t_i

$$0 < t_1 < t_2 < \cdots < t_n = T$$

Let $\alpha_{t_{i-1}}$ and $\beta_{t_{i-1}}$ be the number of shares of riskless and risky securities held by the investor before time t_i trading begins. Clearly, these random variables will be I_{t_i} adapted.

α_{t0} and β_{t0} are nonrandom initial holdings.

Let B_{t_i} and S_{t_i} denote the prices of the riskless and risky security at time t_i.

Suppose at time t_i investments are financed solely from the proceeds of time t_{i-1} holdings.

$$\therefore \quad \alpha_{t_{i-1}} B_{t_i} + \beta_{t_{i-1}} S_{t_i} = \alpha_{t_i} B_{t_i} + \beta_{t_i} S_{t_i}$$

$$where\ i = 1, 2, 3, \cdots\cdots, n$$

Use $\quad B_{t_i} = B_{t_{i-1}} + B_{t_i} - B_{t_{i-1}}$

$\quad S_{t_i} = S_{t_{i-1}} + S_{t_i} - S_{t_{i-1}}$

$$\Rightarrow \quad \alpha_{t0} B_0 + \beta_{t0} S_0 + \sum_{j=1}^{i-1} [\alpha_{t_j}[B_{t_{j+1}} - B_{t_j}] + \beta_{t_j}[S_{t_{j+1}} - S_{t_j}]$$

$$= \alpha_{t_i} B_{t_i} + \beta_{t_i} S_{t_i}$$

The left-hand side has exactly the same setup as the stochastic integral discussed in the previous section.

1.8 Conclusion

- 1.Martingales were introduced as processes with no recognizable time trends.
- 2.We also introduced ways of obtaining martingales from processes that had positive (or negative) time trends.

Thank you

For

your listening