Ch9 Integration in Stochastic Environments

-The Ito Integral

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Introduction

The fundamental theorem of calculus:

- Integral denotes a sum of increments.
- Derivative denotes a rate of change.

It seems natural to expect that if one adds changes dX_t in a variable X_t , with initial value $X_0=0$, one would obtain the latest value of the variable

$$\int_0^t dX_u = X_t$$

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Introduction

- Now consider the SDE which represents dynamic behavior of some asset price S_t :

$$dS_t = a(S_t,t)dt + \sigma(S_t,t)dW_t, \quad t\in[0,\infty).$$

- After we take integrals on both sides, this equation implies that

$$\int_0^t dS_u = \int_0^t a(S_u,u)du + \int_0^t \sigma(S_u,u)dW_u$$

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The Ito Integral and SDEs

$$S_{t+h}-S_t = \int_t^{t+h} a(S_u,u)du + \int_t^{t+h}\sigma(S_u,u)dW_u$$

where h is some finite time interval.

- If $a(S_t,t)$ and $\sigma(S_t,t)$ are **smooth functions** S_u and u .then we could rewrite this equation as:

$$S_{t+h}-S_t \cong a(S_t,t)\int_t^{t+h} du + \sigma(S_t,t)\int_t^{t+h} dW_u$$

- From here, one can obtain the **finite difference approximation**.

$$S_{t+h}-S_t \cong a(S_t,t)h + \sigma(S_t,t)[W_{t+h}-W_t]$$

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The Ito Integral and SDEs

The representation is an approximation for at least two reasons:

1.The $E_t[S_{t+h}-S_t]$ was set equal to a first-order Taylor series approximation with respect to h:

$$E_t[S_{t+h}-S_t] = a(S_t,t)h$$

2.The $a(S_u,u)$,$\sigma(S_u,u)$, $u\in[t,t+h]$ were approximated by their value at u=t.

Both of these approximations require some smoothness conditions on $a(S_t,t)$ and $\sigma(S_t,t)$.

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The Ito Integral and SDEs

- All of above implies that when we write

$$dS_t = a(S_t,t)dt + \sigma(S_t,t)dW_t, \quad t \in [0,\infty).$$

We in fact mean that in the integral equation,

$$\int_t^{t+h} dS_u = \int_t^{t+h} a(S_u,u)du + \int_t^{t+h} \sigma(S_u,u)dW_u$$

The second integral on the right-hand side is defined in the Ito sense and that as $h \to 0$,

$$\int_t^{t+h} \sigma(S_u,u)dW_u \cong \sigma(S_t,t)dW$$

That is, the diffusion terms of the SDEs are in fact Ito integrals approximated during infinitesimal time intervals.

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The Practical Relevance of Ito Integral

- In practice, the Ito integral is used less frequently than stochastic differential equations. Practitioners almost never use the Ito integral directly to calculate derivative asset prices.
- But a SDE can be defined only in terms of the Ito integral. To understand the real meaning behind the SDEs, one has to have some understanding of the Ito integral. Otherwise, errors can be made in applying SDEs to practical problems.

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The Riemann-Stieltjes Integral

- Suppose we would like to calculate

$$\int_0^T g(x_t)dF(x_t)$$

- Time is partitioned into n smaller intervals:

$$t_0 = 0 < t_1 < ... < t_{n-1} < t_n = T$$

- Then the finite Riemann sum is defined:

$$V_n = \sum_{i=0}^{n-1} g(x_{t_{i+1}})[F(x_{t_{i+1}}) - F(x_{t_i})]$$

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The Riemann-Stieltjes Integral

- If the function g(.) is integrable, then the limit

$$\lim_{\sup_i |t_i - t_{i-1}| \to 0} \sum_{i=0}^{n-1} g(x_{t_{i+1}})[F(x_{t_{i+1}}) - F(x_{t_i})] = \int_0^T g(x_t)dF(x_t)$$

will exists and will be called the **Riemann-Stieltjes integral**.

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Stochastic Integration and Riemann Sums

- Question: Can we use a methodology similar to the Riemann-Stieltjes approach in Stochastic environments???

$$\int_0^T \sigma(S_u,u)dW_u = \lim_{n\to\infty} \sum_{k=1}^{n} \sigma(S_{k-1},k)[\Delta W_k]$$

- Ans: unfortunately, ΔW_k is a random variable.The **deterministic notion of limit** utilized by Riemann-Stieltjes methodology cannot be used here

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Stochastic Integration and Riemann Sums

- Under some conditions, it is possible to define a stochastic integral as the **limit in mean square** of the random sum:

$$\sum_{k=1}^{n} \sigma(S_{k-1},k)[\Delta W_k]$$

- The **mean square convergence**:

$$\lim_{n\to\infty} E[\sum_{k=1}^{n} \sigma(S_{k-1},k)[\Delta W_k] - \int_0^T \sigma(S_u,u)dW_u]^2 = 0$$

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Definition: The Ito Integral

- Definition: Consider the finite interval approximation of the stochastic differential equation

$$S_k - S_{k-1} = a(S_{k-1},k)h + \sigma(S_{k-1},k)[W_k - W_{k-1}] \quad k = 1,2,\ldots,n$$

Let

1. The $\sigma(S_t,t)$ be **non-anticipative**, in the sense that they are independent of the future.(otherwise, there will be no guarantee that the partial sums used to construct will converge in mean square to a meaningful random variable.)
2. The random variables $\sigma(S_t,t)$ be "**non-explosive**":

$$E[\int_0^T \sigma(S_t,t)^2 dt] < \infty$$

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Definition: The Ito Integral

- Then, the Ito integral,

$$\int_0^T \sigma(S_t,t)dW_t$$

Is the mean square limit

$$\sum_{k=1}^{n} \sigma(S_{k-1},k)[\Delta W_k] \to \int_0^T \sigma(S_u,u)dW_u$$

as $n \to \infty$

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Properties of the Ito Integral

- The Ito integral is a martingale

$$E_s[\int_0^t \sigma(S_u,u)dW_u] = \int_0^s \sigma(S_u,u)dW_u \quad 0 < s < t$$

Hence, the existence of unpredictable innovation terms in SDEs coincides well with the martingale property of the Ito integral.

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Properties of the Ito Integral

- Existence:

$$\int_0^t f(S_u,u)dS_u$$

If the function f(.) is **continuous**, and if it is **non-anticipating**, this integral exists.In other words, the finite sums

$$\sum_{i=0}^{n-1} f(S_{t_i},t_i)[S_{t_{i+1}} - S_{t_i}]$$

converge in mean square to "some" random variable that we call the Ito integral.

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Properties of the Ito Integral

- Correlation Properties :

$$E[\int_0^T f(W_t,t)dW_t] = 0$$

$$E[\int_0^t f(W_u,u)dW_u \int_0^t g(W_u,u)dW_u] = \int_0^t E[f(W_u,u)g(W_u,u)]du$$

$$E[\int_0^t f(W_u,u)dW_u]^2 = E[\int_0^t f(W_u,u)^2 du]$$

- Addition :

$$\int_0^T [f(S_t,t) + g(S_t,t)]dS_t = \int_0^T f(S_t,t)dS_t + \int_0^T g(S_t,t)dS_t$$

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Conclusions

- The **error terms** in stochastic differential equations are defined in **the sense of the Ito integral**. Numerical calculations must obey the conditions set by this definition.
- The **stochastic differential equations** routinely used in asset pricing are also defined in **the sense of the Ito integral**.

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