Chapter 7

Differentiation in Stochastic Environments

1. Introduction

- This chapter guides and gives some concepts for readers to deal with differentiation in stochastic environment。
- In financial markets, we deal with some stochastic variables instead of deterministic variables. The ways to use calculus techniques might different from traditional calculus.

1. Introduction

- More specifically, in deterministic environment, all variables are known, there is no risk and uncertainty. But asset price is unpredictable as we pricing a derivative. Thus we cannot use traditional calculus to help us compute. We need to use stochastic differential equation (SDE) to help us.

1. Introduction

- This chapter tells us first under what condition a continuous-time process S_t can be interpreted as follows :

$$dS_t = a(S_t,t) + b(S_t,t)dW_t$$

- Second, it introduces some properties of dW_t

2. Motivation

- One wants to know how a derivative changes as the underlying asset changes, “chain rule” needs to be utilized.

$$dF_t = \frac{\partial F}{\partial S} dS_t$$

or

$$dF_t = F_s dS_t$$

2. Motivation

- As discussed in Chapter3, standard differentiation is the limiting operation defined as

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = f(x), \quad f_x < \infty$$

2. Motivation

- Questions: Can we replace X for a random variable X straightforward?
- In general，the answer is ”no”

2. Motivation

- Suppose f(x) is a function of random process x. suppose we want to expand f(x) around a known value of x, say x_0. A Taylor series expansion will yield

$$f(x) = f_x(x_0) + f_x(x_0)[x - x_0] + \frac{1}{2} f_{xx}(x_0)[x - x_0]^2$$
$$+ \frac{1}{3!} f_{xxx}(x_0)[x - x_0]^3 + R(x, x_0)$$

2. Motivation

$$let \quad \Delta x = x - x_0$$

$$f(x_0 + \Delta x) - f(x_0) \cong f_x(\Delta x) + \frac{1}{2} f_{xx}(\Delta x)^2$$
$$+ \frac{1}{3!} f_{xxx}(\Delta x)^3 + R(x, x_0)$$

footnote 4

2. Motivation

- If Δx is very small，we can neglect Δx^2 in general
- If x is a random variable，we cannot neglect

  Ex: suppose E(Δx)=0，Var(Δx)>0，then E[Δx^2]>0

- The result shows the differences between standard calculus and stochastic calculus

2. Motivation

- As the result, we can get the approximation

$$f(x_0 + \Delta x) - f(x_0) \cong f_x(\Delta x) + \frac{1}{2} f_{xx} E[(\Delta x)]^2$$

- Ch16 will examine how can we use E[(Δx)^2] to replace Δx^2

2. Motivation

- 當x爲deterministic

  $f(x_0+\Delta x)-f(x_0) \sim f_x \Delta x$ (1)

- 當x爲random variable

  $f(x_0+\Delta x)-f(x_0) \sim f_x \Delta x+1/2*f_{xx}E[(\Delta x)^2]$ (2)

2. Motivation

From(1)，divide both sides by Δx

$$\longrightarrow \frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \sim f_x$$

From(2)， divide both sides by Δx

$$\longrightarrow \lim_{\Delta x \to 0}\frac{f(x_0+\Delta x)-f(x_0)}{\Delta x} \sim f_x + \frac{1}{2} f_{xx} \lim_{\Delta x \to 0}\frac{(\Delta x)^2}{\Delta x}$$

$\longrightarrow$ it is not clear whether we can let $\Delta x \longrightarrow 0$

3. A Framework for Discussing Differentiation

- In applications to financial markets, what is of interests are the changes in asset prices over incremental time periods
- In stochastic calculus, the concept of derivative has to use some type of probabilistic convergence
- Next, we construct SDE from discrete time to continuous time to understand the different ways

Define：

$t \in [0,T]$

$0 = t_0 < t_1 < \ldots < t_k < \ldots t_n = T$

$t_k = kh$ 故 $n = T/h$

$S_k = S(kh)$

$\Delta S_k = S(kh) - S((k-1)h)$

∵ the corresponding expectations exist

∴ for any k under I_{k-1}

$\Delta W_k = [S_k - S_{k-1}] - E_{k-1}[S_k - S_{k-1}]$ (innovation term)

3. A Framework for Discussing Differentiation

Properties of innovation term:

- ΔW_k is unknown at the end of the interval (k-1)
- $E_{k-1}[\Delta W_k] = 0$ for all k
- They are know given I_k：
  $E_k[\Delta W_k] = \Delta W_k$
- ΔW_k is martingale difference*

*

The accumulated error process W_k will be given by

$W_k = \Delta W_1 + \ldots + \Delta W_k$

$= \sum_{i=1}^{k} \Delta W_i$ where $W_0 = 0$

We can show W_k is a martingale:

$E_{k-1} W_k = E_{k-1}[\Delta W_1 + \ldots + \Delta W_k]$

$E_{k-1} W_k = [\Delta W_1 + \ldots + \Delta W_{k-1}] = W_{k-1}$ under I_{k-1}

- What is the importance of ΔW_k?

1. For a decision maker, the important information contained in asset prices is indeed ΔW_k
2. These unpredictable "news" occur continuously and can be observed "on line" in all major networks such as Reuters or Bloomberg ΔW_k

These imply that in order to discuss differentiation in stochastic environments, one needs to study the properties of ΔW_k.

Next, we discuss that under some fairly acceptable assumptions, ΔW_k^2(ordW_t^2) cannot be considered as "negligible" in Talylor-style approximations.

4. The “Size” of Incremental Errors

- We use Merton’s approach to deal with the variation in second-order terms

  → Merton’s approach helps understand of the economics behind the assumption
- Merton’s approach is to study the characteristics of the information flow in financial markets and try to model this information flow in some precise way

- Define notations:

$Var(\Delta W_k)=V_k$

$V_k=E_0[\Delta W_k^2]$

The variance of cumulative errors is define as：

$$V=E_0[\sum_{k=1}^{n}\Delta W_k]^2 = \sum_{k=1}^{n} V_k$$

where the property that ΔW_k are uncorrelated across k is used and the expectation of cross terms are set equal to zero.

4. The “Size” of Incremental Errors

- Follow Merton (1990)，there are three assumptions:

1. $V>A_1>0$, where A_1 is independent of n (lower bound)
2. $V<A_2<\infty$, where A_2 is independent of n (upper bound)

   For the third assumption, define

   $$V_{max}=max[V_k, k=1,\ldots,n].$$
3. $\frac{V_k}{V_{max}}>A_3$, $0<A_3<1$, with A_3 independent of n (not concentrated)

We discuss some important properties of$(\Delta W_k)^2$ below

4. The “Size” of Incremental Errors

- The following proposition is at the center of stochastic calculus

  ⇨ Under assumptions 1,2,and 3, the variance of ΔW_k is proportional to h

i.e. $E[\Delta W_k]^2=\sigma_k^2 h$, where σ_k is a finite constant that does not depend on h. It may depend on the information at time k-1

- Proof：

Use assumption 3：

$$V_k > A_3 V_{max.}$$

Sum both sides over all intervals：

$$\sum_{k=1}^{n} V_k > nA_3V_{max}$$

Under assumption 2：

$$A_2 > \sum_{k=1}^{n}(V_k) > nA_3V_{max}$$

Divide both side by nA_3：

$$\frac{1}{n}\frac{A_2}{A_3} > V_{max}$$

Note that $n=\frac{T}{h}$. Then,

$$\frac{1}{n}\frac{A_2}{A_3} > V_{max} > V_k$$

$$\frac{h}{T}\frac{A_2}{A_3} > V_k$$

This gives an upper bound on V_k that depends only on h. We now obtain a lower that depends only on h also.

We know that $\sum_{k=1}^{n} V_k > A_1$ is true. Then,

$$nV_{max} > \sum_{k=1}^{n} V_k > A_1 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(*)$$

Use assumption 3：

$$V_k>A_3V_{max}$$

Divide (*) by n：

$$V_{max} > \frac{A_1}{n}$$

Then,

$$V_{max} > \frac{A_1}{T} h$$

$$V_k > A_3 V_{max} > \frac{A_3 A_1}{T} h$$

This means that

$$V_k > \frac{A_3 A_1}{T} h$$

Therefore,

$$\frac{h}{T}\frac{A_2}{A_3} > V_k > \frac{A_3 A_1}{T} h$$

V_k has upper and lower bounds that are linear functions of h, regardless of what n is. This means that we should be able to find a constant σ_k depending on k, s.t. $V_k=E[\Delta W_k]^2=\sigma_k^2 h$

5. One Implication

- According to the proposition, if corresponding expectation exist, one can always write

$S_k-S_{k-1}=E_{k-1}[S_k-S_{k-1}]+\sigma_k \Delta W_k$, where $Var(\Delta W_k)=h$

After dividing both sides by h:

$$\frac{S_k - S_{k-1}}{h} = \frac{E_{k-1}[S_k - S_{k-1}]}{h} + \frac{\sigma_k \Delta W_k}{h}$$

But according to proposition:

$$E[\Delta W_k^2]=h$$

Suppose we use this to justify the approximation:

$$\Delta W_k^2 \cong h \qquad \text{(chapter 9)}$$

Suppose we do the same here and pretend we can take the "limit" of the random variable:

$$\lim_{h\to 0}\frac{W_{(k-1)h+h} - W_{(k-1)h}}{h}$$

The approximation indicates that the derivatives may not be well defined :

$$\lim_{h\to 0}\frac{\left|W_{(k-1)h+h} - W_{(k-1)h}\right|}{h} \to \infty$$

Figure 2 shows $f(h)=\frac{h^{\frac{1}{2}}}{h}$ this graphically

5. One Implication

- In general，we cannot use traditional calculus to deal with random variables
- Under stochastic environment, we should construct stochastic calculus to solve problems

6. Putting the Result Together

- Up to this point, we have accomplished two things:
  1. $S_k-S_{k-1}=E_{k-1}[S_k-S_{k-1}]+\sigma_k \Delta W_k$
  2. $Var(\Delta W_k)=h$
- In order to obtain a stochastic difference equation, let

$$E_{k-1}[S_k-S_{k-1}]=A(I_{k-1},h)$$

If A(.) is a smooth function of h, then the Taylor series expansion around h=0

$$A(I_{k-1},h)=A(I_{k-1},0)+a(I_{k-1})h+R(I_{k-1},h)$$

If h=0

$A(I_{k-1},0)=0$

In the literature dealing with ordinary stochastic differential equation is that any deterministic terms having power>1 are small enough to be ignored.

Thus, we can let

$R(I_{k-1},h) \cong 0$

and obtain the first-order Taylor series approximation:

$$E_{k-1}[S_k-S_{k-1}] \cong a(I_{k-1},kh)h$$

6. Putting the Result Together

- Rewrite the result, we can get
  $S_{kh}-S_{(k-1)h}=a(I_{k-1},kh)h+\sigma_k[W_{kh}-W_{(k-1)h}]$

- In later chapter we let h ⟶ 0 and obtain the SDE:

$$dS(t)=a(I_t,t)dt+\sigma_t dW(t)$$

$a(I_t,t)dt$: drift term σ_t: diffusion term

6.1 Stochastic Differentials

- Question:
  How can these terms be made more explicit?
  ➡ We need to define the fundamental concept of the Ito integral. (chapter 9)

7. Conclusions

- Differential in standard calculus cannot be extend in straightforward fashion to stochastic derivatives.
- We can constructed a SDE by decomposing in a stochastic process into a predictable and an unpredictable part, and then making some assumptions about the smoothness of the predictable part.