Real Options Valuation: a Monte Carlo Simulation Approach[1]

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Abstract

This paper provides a valuation algorithm based on Monte Carlo simulation for valuing a wide set of capital budgeting problems with many embedded real options dependent on many state variables.
Along the lines of Gamba and Trigeorgis (2002b), we decompose a complex real option problem with many options into a set of simple options, properly taking into account deviations from value additivity due to interaction and strategical interdependence of the embedded real options, as noted by Trigeorgis (1993). The valuation approach presented in this paper is alternative to the general switching approach for valuing complex option problems (see Kulatilaka and Trigeorgis (1994) and Kulatilaka (1995)).
The numerical algorithm presented in this paper is based on simulation, and extends the LSM approach presented in Longstaff and Schwartz (2001) to a multi-options setting in order to implement the modular valuation approach introduced in Gamba and Trigeorgis (2002).
We provide also an array of numerical results to show the convergence of the algorithm and a few real life capital budgeting problems, including the extension of Schwartz and Moon (2000,2001) for valuing growth companies, to see how they can be tackled using our approach.

JEL Classification: C15, C63, G13, G31.

1 Introduction

Traditional Monte Carlo simulation has been considered a powerful and flexible tool for capital budgeting for a very long time. It is a recommended methodology for capital budgeting decisions in many Corporate Finance textbooks. Actually, it permits to include a wide set of value drivers, it is flexible enough to cope with many real life situations and it does not suffer the "curse of dimensionality" affecting other numerical methods. Yet, as pointed out by many authors,[1] it seems not so suited to tackle capital budgeting problems with (potentially) many real options.

Mason and Merton [36] first described a capital budgeting problem as a collection of real options, i.e. a set of opportunities that managers (usually) have to deviate from a previously decided course of actions. Real options are capital budgeting decisions contingent on some relevant and well specified state variables affecting the value of an investment project. Projects involving individual options have been studied since the early stage of development of the real options theory (see e.g. Majd and Myers [37] and McDonald and Siegel [40, 41]).[2] Generally speaking, the numerical techniques for financial options can be successfully employed to evaluate single real options: as far as the mathematics of real option valuation is concerned, there would be no need of a theory specifically devoted to individual real options. An exception is represented by Brennan and Schwartz [11] who evaluate the investment in a mine considering the compound effect of the flexibility to temporarily shut down and restart operations and to abandon the project.[3] A widely accepted classification of simple real options is the one presented in Mason and Merton [36] (see also Amram and Kulatilaka [1] and Trigeorgis [51] for more details and references therein) and includes: the

[1] See for instance Trigeorgis [51, pp. 54-57].

[2] For a comprehensive bibliography on the subject, see Dixit and Pindyck [20] and Trigeorgis [51].

[3] This line of research, involving the option to switch from one operating mode to the others and with the possibility to reverse the action at some cost, has been followed up by other authors. Dixit [19] studied an investment problem with the flexibility to start and close operations over time. Hodder and Triantis [25] present a general impulse control framework for optimal switching problems. Kulatilaka [27, 28] introduced a model to evaluate an investment project in a industrial plant firing two different types of fuel, endowed with the flexibility to switch from one fuel to the other according to the relative movements of their market prices. Kulatilaka and Trigeorgis [31] and Kulatilaka [29] (see also Trigeorgis [51, Ch. 5, pp. 171-201]) proposed a general model of managerial flexibility based on the option to switch among properly defined operating modes. In this work we propose a different alternative approach to model the general flexibility embedded in a capital budgeting problem.

option to defer an investment decision, the option to partially or completely abandon operations, the option to alter the scale of current operations, the options to switch the existing assets to an alternative use and many others. The valuation of these options can be easily done by employing the same techniques used for financial option pricing.[4]

Unfortunately, real life investment decisions usually present many options at once or, following Trigeorgis [49], an investment decision can be seen as a portfolio of interacting opportunities. The interactions among the contingent decisions make valuation harder. As a rule, the value of a portfolio of interacting options deviates from additivity and in some cases the difference with respect to the sum of the values of the individual real options (i.e., considered in isolation) can be significant. Hence, the problem of decomposing a complex investment project into a set of individual options quite often does not have a straightforward solution. This fact prevents the use of valuation techniques devoted to individual options, well known in financial option theory and calls for a valuation approach specific for problems involving many real options.[5]

Kulatilaka and Trigeorgis [31] and Kulatilaka [29] (see also Trigeorgis [51, Ch. 5, pp. 171-201]) proposed a valuation approach for complex problems based on the general idea of switching among different "operating modes". In their approach, given an investment with many embedded options, at any time a decision can be made, there is an option to switch from the current "mode" to a different one. The switching cost of the decision is the "strike price" of the option. This valuation method is based on the analogy between machines with many operating modes (and related switching costs) and a capital budgeting problem: the operating modes are decision that the management can make in a dynamic fashion. For instance, the usual wait-to-invest option (a call option on the present value of the cash flows from operations of a given investment project) can be described as an (irreversible) option to switch from the mode "wait to invest" to the mode "invest." According to this approach, a flexible capital budgeting problem can be seen as a complex compound switch option among several and properly defined "modes." The technique based on the general switching flexibility, joint with some discrete-time approximation of the continuous-time dynamics of the state variable (either binomial lattices or Markov chains), is widely applied to capital budgeting problems (examples are in [30, 31, 49, 51]). Besides other problems, mainly related to the computational efficiency of a

[4] A good and comprehensive reference on this is Trigeorgis [51].

[5] A notable exception is Geske [24].

numerical valuation procedure based on this approach and which we will discuss later, the general option to switch has the following main drawback. As discussed in Brekke and Øksendal [7], an *optimal switching problem* is a special type of impulse control problem (see Bensoussan and Lions [4] for a reference). If we are to model the problem in a continuous-time setting, and we use some discrete-time numerical valuation approach to obtain a solution, first one has to prove the existence of a finite solution and next the convergence of the discrete-time (numerical) solution to the continuous-time one. (Of course, for a switching problem with a finite number of decision dates the solution always exists.) In Brekke and Øksendal [7] the proof of the existence of an optimal solution in a continuous-time setting is offered for a class of switching problems.[6] The same cannot be said for a general optimal switching problem. This means that, although the approach based on the general option to switch is flexible, one has to be very careful to apply this approach in a continuous-time setting, since a solution might not exist.

Gamba and Trigeorgis [22] propose an alternative approach to map a complex real options problem into a set of simple options and way to comply with the hierarchical structure of the options. This approach always provides well defined problems with a finite solution also in a continuous-time setting, provided that each individual embedded option (or building block) has a finite solution. Lastly, even if the approach based on the general option to switch proves to be fruitful in a low-dimensional setting, it becomes computationally intractable if there are many (i.e., more than two) state variables. Since simulation methods requires a computational effort which is linear with respect to the dimension of the state space, in this paper we propose an alternative approach for valuing multi-options and multi-assets problems based on the simulation approach developed by Longstaff and Schwartz [34] and extendid this methodology to the class of problems described in Gamba and Trigeorgis [22].

Usually, real options embedded in a capital budgeting problem are American-type claims. This means that closed-form solutions are rarely available and some numerical methods must be employed. Many methods have been proposed for real option pricing purposes. Most of them are plain extensions of well known algorithms used to price financial options. Roughly, they can be divided into three main classes: finite difference methods and other approaches dealing directly with PDE's (first introduced by Brennan and Schwartz [10]), Monte Carlo simulation methods (introduced by Boyle [6])

[6]Basically, these are the same kind of problems described in Brennan and Schwartz [11].

and lattice methods first proposed by Cox, Ross and Rubinstein [18]. All these approaches have some flaws when applied to real options valuation. Finite difference are quite hard to implement if the project has many interacting options, because it may be difficult even to obtain the relevant PDE. Bi- or trinomial lattices,[7] although very flexible for valuing projects with many embedded options (see Trigeorgis [50] and [51, Ch. 10]), suffers the "curse of dimensionality." Yet, real life capital budgeting problems usually involve multiple state variables. This feature, assuming that the stochastic model of these variables are known,[8] makes real options even more difficult to evaluate. Multi-factor real options problems have been studied for instance by Triantis and Hodder [48], Cortazar and Schwartz [15], Geltner, Riddiuogh and Stojanovic [23], Cortazar, Schwartz and Salinas [16], Martzoukos and Trigeorgis [38], Brekke and Schieldrop [8] and many others.

For all the above mentioned reasons, simulation seems to be the most suited numerical technique for real options. Unfortunately traditional Monte Carlo simulation (as introduced by Boyle [6] for plain vanilla options) is a forward-looking technique, whereas dynamic programming (to evaluate American options) implies backward recursion. Many approaches have been proposed to match simulation and dynamic programming: Bossaerts [5] proposes two moment estimators of optimal stopping time; Tilley [47] provides an algorithm in which simulated paths are bundled to estimate probability weights of the state space; Barraquand and Martineau [2] give a stratification method for pricing high-dimensional options, in the same spirit of Tilley's approach; Broadie and Glasserman [12] proposed an algorithm based on simulated trees with a small number of dates where early exercise is allowed.

A very promising approach has been presented by Longstaff and Schwartz [34]. This numerical method is based on Monte Carlo simulation and uses least squares linear regression to determine the optimal stopping time of the problem. This method, called Least Squares Monte Carlo (LSM) approach, has the additional feature of being a very intuitive, pedagogically clear and flexible tool. We will provide an extension of this algorithm to evaluate

[7] In this class of algorithms we include also discrete-time and discrete state Markov chains. See for instance, Kulatilaka [29] for an application of Markov chains to real options valuation.

[8] One of the major problems in real options applications is the specification of the stochastic model. Traditionally, some simple models are used (e.g. geometric Brownian motion or mean-reverting processes) which can be suited to describe the price of traded assets. We will rely upon the usual assumptions, leaving the issue of the underlying process specification as a subject for future research. We just want to stress, at this point, that even in this respect simulation is likely to be suiter than the other numerical methods, since it can be employed with a larger family of stochastic processes.

complex investment projects with many interacting options and many state variables, along the lines of the decomposition approach proposed in Gamba and Trigeorgis [22].

The paper is organized as follows. Section 2 introduces the suited environment for capital budgeting purposes. Instead of the usual CAPM economy, we develop our model in the equilibrium multi-factor economy proposed by Cox, Ingersoll and Ross [17]. This permits to point out the risk factors and the related premia, if the project depends on many (not necessary traded) factors, instead of a unique market factor as in CAPM. Section 3 shortly illustrates, as per Gamba and Trigeorgis [22], the way to describe a wide class of capital budgeting problems using a small set of well defined building blocks, which are specifically designed both to comprise the largest possible number of actual options problems and to be easily solved by the related simulation algorithm. These small problems will be the building blocks of our valuation algorithm. This valuation approach permits to decompose a complex real options problems into a sequence of simple options taking into account their interdependencies. Section 4 presents the proposed extension of the LSM algorithm to numerically evaluate the building blocks of our algorithm and hence extends Longstaff and Schwartz [34] approach to multi-options problems. Section 5 provides a set of numerical examples to see how to apply our approach to real life capital budgeting problems and to show the efficiency of the numerical algorithm based on simulation.

2 The economy

2.1 State variables

Let there be given a Cox-Ingersoll-Ross economy with financial market and a representative agent (see [17]).

There are n state variables $X_1, X_2, \ldots, X_n$, for short denoted $X' = (X_1, X_2, \ldots, X_n)$.[9] The values of these variables is the only relevant information to make the capital budgeting decisions. These variables can be either prices of traded securities or observable values of non-traded assets (factors). In particular, these variables can be thought of as the outputs of production processes. The dynamics of the state variables, with respect to the objective probability measure follow the Markov processes

$$dX_i(t) = a_i(t, X(t))dt + b_i(t, X(t))dB(t) \quad \text{with } X_i(0) = x_i, \quad i = 1, \ldots, n$$

[9] A prime denotes transposition.

where $a_i : \mathbb{R}^n \mapsto \mathbb{R}$ and $b_i : \mathbb{R}^n \mapsto \mathbb{R}^n$ are such that the solution of the stochastic differential equations above exists and $dB(t)$ is the increment of a standard n-dimensional Brownian motion, with $\mathbb{E}[dB_i(t)dB_j(t)] = 0$. With matrix notation, the process is

$$dX(t) = a(t, X)dt + b(t, X)dB(t) \qquad \text{with } X(0) = x$$

where $a' = (a_1, \ldots, a_n)$ and

$$b(t, x) = \begin{pmatrix} b_1(t, x) \\ \vdots \\ b_n(t, x) \end{pmatrix}$$

is a positive definite $n \times n$ matrix with full rank for all t.

2.2 Financial market

There is a financial market where n non-redundant financial assets[10] are traded, i.e the financial market is *complete*. The prices of these assets, given by the processes $\{P^j(t, X_t)\}$, $j = 1, .., n, \ldots$, are contingent on the n state variables (if they are not the state variables themselves). With no loss of generality, we can think of the j-th contingent claim as the security issued by the firm whose production depends on X_j. Since there is no confusion, in the rest of the paper the dependence of the financial asset prices on the state variables will be often dropped. The dynamics of the asset prices are

$$\frac{dP^j(t)}{P^j(t)} = (\mu_j(t) - \delta_j(t))\, dt + \sigma_j(t)dB(t)$$

where δ_j is the payout rate,[11] μ_j is the total expected instantaneous rate of return and σ_j is a n-dimensional vector valued function.

An instantaneously riskless asset is available with instantaneous rate of return r contingent on the state variables. We assume that the dimension of set of financial investments opportunities remains unchanged within the relevant time horizon.

The market is assumed to be in equilibrium and the related asset pricing relation is

$$\mu_j(t) = r(t) + \sum_{i=1}^{n} \Psi_i(t, X) \frac{P^j_{X_i}(t)}{P^j(t)} \quad j = 1, \ldots, n \tag{2.1}$$

[10]We will use the definition "financial asset" in a broad meaning. Actually, also traded commodities are included in this set.

[11]If the j-th financial asset is a traded commodity, δ_j is the related convenience yield (see Brennan [9]).

where $\Psi' = (\Psi_1, \ldots, \Psi_n)$ are the (factor) risk premia, $P^j_{X_i}$ is the derivative of asset j with respect to state variable X_i, and $r(t)$ is the equilibrium instantaneous riskless rate.

Accordingly, since the financial market is complete, there is a unique (equilibrium) risk-neutral probability measure. With respect to this probability measure, the dynamic of the state variables is

$$dX(t) = \hat{a}(t, X)dt + b(t, X)dB^*(t) \qquad \text{with } X(0) = x \tag{2.2}$$

where $\hat{a} = (a - \Psi)$ is the risk-adjusted drift and $\{B^*(t)\}$ is the Brownian motion under the equilibrium martingale measure.

To simplify our arguments, we will assume from now on that the riskless rate is non-stochastic and constant. The analysis would be the same, at the cost of more cumbersome formulas, if we assumed a stochastic riskless rate.

2.3 Contingent claim valuation

If a (necessarily) redundant contingent claim, for instance an option on a traded asset (or on a factor), is given, with maturity T and payoff $\Pi(T, X_T)$, where Π is a known function, we can evaluate the contingent claim with respect to the prices in the financial market. We assume that Π has finite expectation and variance with respect to the relevant probability measure can be properly defined.[12]

Let $F(t, X_t)$ be the value of the claim at $t \leq T$, with $F(T, X_T) = \Pi(T, X_T)$. If the claim is European, i.e. it can be exercised only at T, the price at any time $t < T$ is

$$F(t, X_t) = e^{-r(T-t)}\mathbb{E}^*_t[\Pi(T, X_T)], \tag{2.3}$$

and if the claim is American, i.e. it can be exercised at any time before T, and is still available at t,

$$F(t, X_t) = \max_{\tau \in \mathcal{T}(t,T)} \left\{ e^{-r(\tau-t)}\mathbb{E}^*_t[\Pi(\tau, X_\tau)] \right\}, \tag{2.4}$$

where $\mathcal{T}(t, T)$ is the set of stopping times in $[t, T]$ with respect to the information generated by the state variables X and $\mathbb{E}^*_t[\cdot]$ is the expectation with

[12]Formally, $\Pi \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{Q})$, the space of square-integrable functions with respect to $\mathbb{Q}$, where Ω denotes the space of all possible states of the economy, $\mathcal{F}$ is the filtration generated by the state variables and $\mathbb{Q}$ is the equilibrium risk-neutral probability measure on $\mathcal{F}$.

respect to the unique risk neutral probability, conditional on the information available at t.[13] See Bensoussan [3] and Karatzas [26].

Interesting enough, equations (2.3) and (2.4) are simply net present values, with respect to the riskless rate, of certainty equivalent payoffs. This valuation paradigm will be used also for valuing capital budgeting projects (and the related real options).

3 A general valuation approach for capital budgeting

In this section we shortly recollect from Gamba and Trigeorgis [22] the main points of the valuation approach based on decomposition of a complex project with many real options into a set of simple problems.

Real options are contingent claims on real investment projects. They are contingent on the state variables X of the reference economy. Since the financial market is complete, Fisher separation theorem holds and the valuation principle based on discounting certainty equivalent cash flows applies.

Assuming that simple (individual) options are well defined, we introduce a set of possible ways of interaction (or stylized problems) to properly capture the interdependencies among individual options. This methodology can be successfully applied to a large family of capital budgeting problems: basically it can be used for all projects which do not give the management the opportunity to reverse a previously made decision.[14]

The idea underlying the approach proposed in Gamba and Trigeorgis [22] is very simple: *a capital budgeting problem can be decomposed into a hierarchical set of simple options.* Hierarchy among individual (real) options coply with option interaction and interdependency. Some interdependencies are as simple as compoundedness.

Besides the compound option case, described above, we will present the following two additional possible way to interact: sum of independent options and mutually exclusive options. In Gamba and Trigeorgis [22] other ways of inteaction are presented.

The first case is the one with many independent options: the value of the portfolio of options is the sum of the values of the simple options. This is the case, described in corporate finance textbooks, of value additivity of investment projects. We stress that "independent" in this case means

[13] We will denote conditional expectation also with $\mathbb{E}_t^*[\cdot] = \mathbb{E}^*[\cdot \mid \mathcal{F}_t]$.

[14] The extension of the simulation technique to these problems is the subject of future research.

“strategically independent” and not “stochastically independent:” actually, the assets underlying the independent options may not be (and usually are not) stochastically independent.

Let there be given H options, with maturities T_h, payoffs $\Pi_h(t, X_t)$ and values $F_h(t, X_t)$, $h = 1, \ldots, H$. In what follows, since our approach is (backward) recursive, we will phrase the argument by choosing a generic step in the valuation procedure and assuming that the results of the previous steps are known. Hence, $F_h(t, X_t)$, $h = 1, \ldots, H$ have been already determined (together with the related optimal stopping times) in the dates following t.

The possibility to exercise all the H options independently is itself an option. Since the H subsequent options are independent of each others, the value of the option to exercise them, denoted $G(t, X_t)$, is the sum of their values:

$$G(t, X_t) = \sum_{h=1}^{H} F_h(t, X_t). \tag{3.1}$$

This approach can encompass also the presence of technical uncertainty affecting the future decisions on such project. More specifically, the availability of the H options above may depend on some event with H possible outcomes. We consider the technical uncertainty to be stochastically independent of X. Usually, the probability of the technical event affecting the project is (assumed to be) known. If the event has H possible outcomes, $p_h > 0$ is the probability of the h-th one, $\sum_h p_h = 1$, and assuming that the technical uncertainty dissipates at $T' < T_h$, $h = 1, \ldots, H$, the option value is

$$G(t, X_t) = e^{-r(T'-t)} \sum_{h=1}^{H} p_h \mathbb{E}_t^* \left[F_h(T', X_{T'}) \right], \tag{3.2}$$

where the h-th subsequent option can be exercised, if American, in the interval $[T', T_h]$ and, if European, at T_h. This model can be easily generalized to many sources of technical uncertainty and to the case the event can happen in a given time interval according to a continuous-time distribution (eg. a Poisson process) (see Gamba and Trigeorgis [22]).

The second stylized problem is the one involving compound options: a real option can offer, when exercised, more opportunities. This happens in many staged investments in which each installment is an option on the subsequent stages. If this is the case, then the value of the previous claim depends also on the value of the subsequent one.

Let there be given H compounded real options, that is, the h-th option, besides its own payoff, gives the “right” to exercise the $(h+1)$-th option, $h =$

$1, \ldots, H-1$. The payoffs of option h is denoted $\Pi_h(t, X_t)$. For definiteness, let the maturities be $T_1 \leq T_2 \leq \ldots \leq T_H$. We will denote F_h the value of the h-th option and, given the (backward) recursive nature of the algorithm, we assume that its value is already known. In what follows, if t is greater than T_h, then the value of the h-th option is zero, because that option is not available any longer: $F_h(t, X_t) \equiv 0$ if $t > T_h$. The value F_h depends also on F_{h+1}. If the h-th option is American, then its value at $t \leq T_h$ is

$$F_h(t, X_t) = \max_{\tau \in \mathcal{T}(t,T_h)} \left\{ e^{-r(\tau - t)} \mathbb{E}_t^* [\Pi_h(\tau, X_\tau) + F_{h+1}(\tau, X_\tau)] \right\}. \tag{3.3}$$

If the h-th option is European,

$$F_h(t, X_t) = e^{-r(T_h - t)} \mathbb{E}_t^* [\Pi_h(T_h, X_{T_h}) + F_{h+1}(T_h, X_{T_h})], \tag{3.4}$$

The above is true for $h = 1, \ldots, H-1$.[15]

The third stylized problem is the one involving the choice of many alternative opportunities. Let there be given H mutually exclusive real options. For the sake of definiteness, we may think of two opposite decisions regarding the same real asset (abandon/expansion, lease/sell, ...): once the decision is made, the other (alternative) competing options expire. These options have payoffs Π_h, $h = 1, 2, \ldots, H$, and maturities T_h, $h = 1, 2, \ldots, H$. With no loss of generality, we assume $T_1 \leq T_2 \leq \ldots \leq T_H$. As usual, let $F_h(t, X_t)$ be the value of the h-th real option. The management is asked to decide, within the time horizon T_H, for the best alternative. We assume that the decision, once taken, is irreversible. In this sense, there is a timing option also in the opportunity to choose the best (out of H) option. Actually, since the decision is irreversible, the management may be interested in delaying the choice of the option to be exercised (and keeping the options open); i.e., there is a timing option also in the choice of the best opportunity.

[15] The above expressions encompasses well know cases. If, $H = 2$, $\Pi_1 \equiv -K_1$, and Π_2 is either $\max\{P_t - K_2, 0\}$ or $\max\{K_2 - P, 0\}$, i.e. the subsequent option is an European call or put on a non-dividend paying asset whose price, $P_t = P(t, X_t)$, evolves according to a geometric Brownian motion, and strike K_2, and the previous option is a call on the second option, then closed form solutions are available from Geske [24]. Again, if $H = 2$, the first option is an European call or put and the second is an option to exchange one asset for another (i.e., $\Pi_2 = \max\left\{P^1 - P^2, 0\right\}$), the assets pay no dividend and their prices are geometric Brownian motions, then closed form formulas are available in Carr [13] (extending the results in Margrabe [35]). If the assets pay a continuous dividend, Martzoukos e Trigeorgis [38] provide extensions of the closed-form formulae in Geske [24] and Carr [13]. In all the other cases, closed-form formulas are not available. We will use close form solutions as a benchmark for our numerical evaluation approach (see Table 6).

Let $G(t, X_t)$ be the value of the opportunity to choose the best one out of H available options. To avoid trivial situations, we assume that at least one of the H options is an American-style claim. We define the control as a couple (τ, ζ), where τ is a stopping time in $\mathcal{T}(t, T_H)$ and ζ takes value in the set $\{1, 2, \ldots, H\}$. The value of the opportunity to select the best option is[16]

$$G(t, X_t) = \max_{(\tau,\zeta)} \left\{ e^{-r(\tau-t)} \mathbb{E}_t^*[F_\zeta(\tau, X_\tau)] \right\}. \tag{3.5}$$

Since the decision about the option is irreversible, although the opportunity to select the best option seems to depend on the *values* of the subsequent options, F_h, $h = 1, \ldots, H$, the choice is not made until the time to exercise the most favorable option has come.[17]

At any steps of the above described procedure, F_{h+1} can be interpreted either as the value of an option, but also as the value of many independent options available at the same time (G in Equation (3.1)), or the expected value of the options that will be available as soon as some technical uncertainty resolves, (G in Equation (3.2)), or the best out of a given set of options (G in (3.5)).

4 A generalization of the Least Square Monte-Carlo approach

Longstaff and Schwartz [34] provide a valuation algorithm, called Least Squares Monte Carlo (LSM) based on simulation that implement backward dynamic programming. Their algorithm provides a way to determine the optimal stopping time of an American-like claim and then, by applying Equation (2.4), to find the estimate of the claim. In what remains of this section, we first describe shortly the LSM and next we extend it in order

[16] If the options are all European-style claims, that is, the maturity T_h is the only date when the h-th option can be exercised, for all h, then the problem is still (3.5), but the stopping time is restricted to the set $\{T_1, \ldots, T_H\}$.

[17] This framework encompasses some known results. If we restrict the maturities of the options, so that $T_1 = T_2 = \ldots = T_H$, assume that these options are European, and put $\Pi_h(t, X_t) = \max\{P^h(X_t) - K, 0\}$, where P^h follows a geometric Brownian motion for all h, then the problem in (3.5) reduces to the well known option on the maximum on H assets with prices $P_h(X_t)$ dependent on the state variables X_t and strike K (see Stulz [46], Johnson [32], and Martzoukos e Trigeorgis [38] if the underlying assets pay a continuous dividend yield). As above, in all the other cases we have to resort to numerical evaluation. To carry out numerical experiments, we will benchmark numerical results against closed-form solutions, if these are available (see Table 6).

to solve the basic (building block) option problems needed to apply the decomposition approach presented in Gamba and Trigeorgis [22].

4.1 Longstaff-Schwartz approach for simple options

Given the valuation problem in (2.4) for an American claim contingent on X and expiring at T, an approximation of the value is obtained by choosing an integer N so that the time span $[0, T]$ is divided into N intervals whose lenght is $\Delta t = T/N$. Next, the dynamics of the state variables is simulated by generating K paths of the stochastic process $\{X_t\}$. We will denote $X_t(\omega)$ the value of the process at time t along the ω-th simulated path and $\tau(\omega)$ the path-wise stopping time with respect to the information generated by $\{X_t\}$ in the discrete set of dates where the state variables dynamics are generated.

The goal of the algorithm is to find the optimal exercise time restricted to the set of dates

$$\{t_0 = 0, t_1 = \Delta t, \ldots, t_N = N\Delta t\}.$$

As usual, the optimal policy is obtained by backward dynamic programming: if at time t_n, along the path ω, the claim has not been exercised yet (i.e., the stopping time along the ω-th path, as determined in previous time steps of the algorithm, is greater that t_n), the optimal decision is made by comparing the payoff $\Pi(t_n, X_t(\omega))$ with $F(t, X_t(\omega))$, the (optimal) value function of problem (2.4). If $F(t, X_t(\omega)) = \Pi(t_n, X_t(\omega))$ then $\tau(\omega) = t_n$ (the optimal stopping time along the ω-th path is updated). The intuition behind this recursive procedure is that the stopping time satisfies the following condition:

$$\tau = \inf\{t \mid F(t, X_t) = \Pi(t, X_t)\} :$$

it is the first time (in a path-wise sense) the value of the option is equal to the payoff from exercise.

Unfortunately, $F(t, X_t)$ is not available at this step of the procedure. A way around this difficulty is offered by the Bellman equation of the optimal stopping problem in discrete time:

$$F(t_n, X_{t_n}) = \max\left\{\Pi(t_n, X_{t_n}), e^{-r(t_{n+1}-t_n)}\mathbb{E}^*_{t_n}\left[F(t_{n+1}, X_{t_{n+1}})\right]\right\}.$$

By this equation, we can determine the path-wise optimal policy, restricted to the given dates, by comparing the continuation value,

$$\Phi(t_n, X_{t_n}) = e^{-r(t_{n+1}-t_n)}\mathbb{E}^*\left[F(t_{n+1}, X_{t_{n+1}}) \mid \mathcal{F}_{t_n}\right] \tag{4.1}$$

with the payoff, $\Pi(t_n, X_{t_n})$. So, the decision rule at time step t_n along the ω-th path is:

$$\text{if} \qquad \Phi(t_n, X_{t_n}(\omega)) \leq \Pi(t_n, X_{t_n}(\omega)) \quad \text{then} \quad \tau(\omega) = t_n. \tag{4.2}$$

At $t_n = T$, since the claim is expiring, $\Phi(t_n, X_{t_n}) = 0$, and the rule is to exercise the claim if the payoff is positive. At any t_n, the optimal stopping time is found by recursively applying the decision rule in (4.2), from $t_n = T$ back to t_n. If we have determined, at some previous step of this procedure, $\tau(\omega) > t_n$, and condition (4.2) holds at the current step, then the stopping time along path ω is updated: $\tau(\omega) = t_n$. At $t_n = 0$, when the optimal stopping times along all paths are determined, the value of the American contingent claim is estimated by averaging the path-wise values:

$$F(0, x) = \frac{1}{K} \sum_{\omega=1}^{K} e^{-r\tau(\omega)} \Pi(\tau(\omega), X_{\tau(\omega)}(\omega)).$$

The valuation problem above boils down to one of finding the continuation value at (t, X_t), in order to apply the decision rule in (4.2). This is the point where LSM differs from all other approaches proposed to evaluate American-type contingent claim with simulation. The intuition behind LSM is the following: if at t the claim is still available, the continuation value is the expectation, conditional on the information available at that date, of future optimal payoffs from the contingent claim. To clarify the next steps, we slightly modify the previously introduced notation: let $\Pi(t, s, \tau, \omega)$ be the (non-necessarily positive) cash flow from the contingent claim optimally exercised at time s (with respect to the stopping time $\tau(\omega)$), conditional on not being exercised at $t < s$, along the ω-th path. Hence,

$$\Pi(t, s, \tau, \omega) = \begin{cases} \Pi(s, X_s(\omega)) & \text{if } s = \tau(\omega) \\ 0 & \text{if } s \neq \tau(\omega). \end{cases}$$

The dependence of this cash flow on t is due to the fact that, when we apply recursively the decision rule in (4.2), the stopping time along the ω-th path can change step by step.

The continuation value at t_n is the present value (with respect to the equilibrium risk neutral probability) of all future expected cash flows from the contingent claim

$$\Phi(t_n, X_{t_n}) = \mathbb{E}^*_{t_n} \left[\sum_{i=n+1}^{N} e^{-r(t_i - t_n)} \Pi(t_n, t_i, \tau, \cdot) \right]. \tag{4.3}$$

Since Φ is an element of a linear vector space,[18] then we can represent the continuation value as

$$\Phi(t, X_t) = \sum_{j=1}^{\infty} \phi_j(t) L_j(t, X_t)$$

where L_j is the j-th element in the orthonormal basis. In Longstaff and Schwartz [34] $L_j(t, X_t)$ are either Hermite, or Laguerre polynomials or also powers of X_t. If only $J < \infty$ elements in the basis are used to determine Φ, we obtain an approximation of the continuation value. Following Longstaff and Schwartz,

$$\Phi^J(t, X_t) = \sum_{j=1}^{J} \phi_j(t) L_j(t, X_t).$$

Now, $\phi_j(t)$ can be estimated by a linear least squares regression of $\Phi^J(t, X_t)$ onto the basis $\{L_j(t, X_t)\}$:[19]

$$\left\{\hat{\phi}_j(t_n)\right\}_{j=1}^{J} = \\ = \arg \min_{\{\phi_j\}_{j=1}^{J}} \left\| \sum_{j=1}^{J} \phi_j(t_n) L_j(t_n, X_{t_n}) - \sum_{i=n+1}^{N} e^{-r(t_i - t_n)} \Pi(t, t_i, \tau, \cdot) \right\|^2 .$$

The estimated continuation value,

$$\hat{\Phi}^J(t_n, X_{t_n}) = \sum_{j=1}^{J} \hat{\phi}_j(t_n) L_j(t, X_{t_n})$$

is then used to apply recursively the decision rule in (4.2).

Accuracy of the estimates of the value of the American contingent claim can be increased by increasing the number of time steps, N, the number of simulated paths, K, and the number of basis function, J. Actually, given N, the algorithm has been proved to converge to the actual value of the (corresponding Bermudan with N dates) claim if $J \to \infty$ and if $K \to \infty$ and the estimation errors are asymptotically normally distributed (see Clément, Lamberton and Protter [14]).

[18] Φ belongs to the Hilbert space $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{Q})$ and any Hilbert space has a countable orthonormal basis.

[19] We denote by $\|\cdot\|$ the norm in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{Q})$.

Moreno and Navas [43] have checked the robustness of LSM simulation approach for different choices of the basis functions, and show that, as far as the simple American option on one underlying assets is concerned, few basis functions are enough for a fairly accurate estimate of the option price and different alternative basis functions give the same results.

4.2 An extension to multi-option problems

Since we are interested in valuing capital budgeting projects with many embedded (American) options, we have to extend the LSM algorithm presented in Section 4.1 to the framework introduced in Gamba and Trigeorgis [22] (see Section 3).

As far as the case with H independent options is concerned, the value of the option to exercise them, according to Equation (3.1), is simply the sum of their values obtained with the LSM algorithm. If there is a project-specific source of uncertainty, which is stochastically independent on the state variables, and which resolves at T', $0 < T' < T_h$, for $h = 1, \dots, H$, according to our notation, the relevant equation is (3.2). In this case, the valuation approach is slightly different from the one described above. At this step of the algorithm we have already found the values (and the related stopping times) of the subsequent options, F_h. Actually, since the subsequent options cannot be exercised in the interval $[0, T']$, their value is

$$\begin{aligned} F_h(0,x) &= e^{-rT'}\mathbb{E}^*\left[F_h(T', X_{T'})\right] \\ &= \mathbb{E}_0^*\left[\sum_{i=1}^{N} e^{-rt_i}\Pi_h(0, t_i, \tau, \cdot)\right] \end{aligned}$$

where

$$\Pi_h(t,s,\tau,\omega) = \begin{cases} \Pi_h(s, X_s(\omega)) & \text{if } s = \tau_h(\omega) \\ 0 & \text{if } s \neq \tau_h(\omega), \end{cases}$$

where τ_h denotes the stopping time for option h, $h = 1, \dots, H$. Note that $T' \leq \tau_h \leq T_h$ and that $\Pi_h(0, t_i, \tau, \omega)$ has been already found applying the LSM approach to the h-th option. To find G we just need Equation (3.2):

$$G(0,x) = \sum_{h=1}^{H} p_h F_h(0,x).$$

For the compound option case, the algorithm is the following. According to the recursive nature of the valuation problem, we assume that the pathwise stopping time for the (subsequent) $(h+1)$-th option (and the ones embedded in that options) has been already determined. We are to compute the path-wise stopping time for the h-th option. The Bellman equation for problem (3.3) is

$$F_h(t_n, X_{t_n}) =$$
$$\max\left\{\Pi_h(t_n, X_{t_n}) + F_{h+1}(t_n, X_{t_n}), e^{-r(t_{n+1}-t_n)}\mathbb{E}^*_{t_n}\left[F_h(t_{n+1}, X_{t_{n+1}})\right]\right\}.$$

Hence, to find out the stopping time for option h, denoted $\tau_h(\omega)$, at t_n on the ω-th path, the decision rule is the following:

$$\text{if } \Phi_h(t_n, X_{t_n}(\omega)) \leq \Pi_h(t_n, X_{t_n}(\omega)) + F_{h+1}(t_n, X_{t_n}(\omega)) \text{ then } \tau_h(\omega) = t_n \tag{4.4}$$

where Φ_h is the continuation value from the Bellman equation (see Equation (4.1)), Π_h is the payoff of the h-th option, F_{h+1} is the value of the $(h+1)$-th option and τ_h is the stopping time for option h. This decision rule replaces the one in (4.2), as long as compound options are considered.

To apply this rule we have to estimate the continuation value Φ_h and the value of the subsequent option, F_{h+1}. The former is found by extending the Longstaff and Schwartz idea. Note that

$$\Phi_h(t_n, X_{t_n}(\omega)) = \mathbb{E}^*_{t_n}\left[\sum_{i=n+1}^{N} e^{-r(t_i-t_n)} \sum_{\ell=h}^{H} \Pi_\ell(t_n, t_i, \tau, \cdot)\right]$$

On the other hand,

$$F_{h+1}(t_n, X_{t_n}(\omega)) = \mathbb{E}^*_{t_n}\left[\sum_{i=n}^{N} e^{-r(t_i-t_n)} \sum_{\ell=h+1}^{H} \Pi_\ell(t_n, t_i, \tau, \cdot)\right],$$

i.e., according to Equation (4.3), the value of the $(h+1)$-th option is the present value of expected cash flow obtained from optimally exercising that option and all subsequent options, starting from the current date. It should be noted that, at this step, $\Pi_\ell(t_n, t_n, \tau, \omega)$ is known, for $\ell = h+1, \ldots, H$.

In order to apply this rule, since the conditions in Longstaff and Schwartz [34] still apply,[20] Φ_h is approximated by Φ_h^J and this can be estimated by least squares regression of the discounted conditional cash flows from option h onto the basis $\{L_j, j = 1, \ldots, L\}$.

It should be noted that the above procedure encompasses also the case in which some of the real options are European. Actually, if option $(h+1)$

[20] Because Φ_h is in $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{Q})$.

is European with maturity T_{h+1}, at $t_n < T_{h+1}$, $\Pi_{h+1}(t_n, t_i, \tau, \cdot) \equiv 0$ for all $t_i \neq T_h$.

As far as the case with H mutually exclusively options is considered, at any time step we have to find the optimal control (τ, ζ), according to Equation (3.5). The Bellman equation at time t_n is

$$G(t_n, X_{t_n}) =$$
$$\max\left\{F_1(t_n, X_{t_n}), \ldots, F_H(t_n, X_{t_n}), e^{-r(t_{n+1}-t_n)}\mathbb{E}^*_{t_n}\left[G(t_{n+1}, X_{t_{n+1}})\right]\right\}.$$

Hence, the decision rule, along the ω-th path is:

$$\text{if } \Phi(t_n, X_{t_n}(\omega)) \leq \max_h \{F_h(t_n, X_{t_n}(\omega))\} \text{ then } (\tau, \zeta)(\omega) = (t_n, \bar{h}) \tag{4.5}$$

where Φ is the continuation value according to the Bellman equation,

$$\bar{h} = \arg\max_h \{F_h(t_n, X_{t_n}(\omega))\},$$

and $(\tau, \zeta)(\omega) = (\tau(\omega), \zeta(\omega))$. In order to apply the decision rule in (4.5) we have to estimate $\Phi(t_n, X_{t_n})$ and $F_h(t_n, X_{t_n})$. To this aim, let

$$\Pi(t, s, \tau, \zeta, \omega) = \begin{cases} \Pi_h(t, s, \tau, \omega) & \text{if } h = \zeta(\omega) \\ 0 & \text{otherwise} \end{cases}$$

Since the continuation value of the option to select the best option out of H available options is the present value of the expected cash flows conditional on following the optimal exercise strategy, then

$$\Phi(t_n, X_{t_n}) = \mathbb{E}^*_{t_n}\left[\sum_{i=n+1}^{N} e^{-r(t_i-t_n)}\Pi(t_n, t_i, \tau, \zeta, \cdot)\right].$$

This can be approximated by Φ^J according to Longstaff and Schwartz [34], and Φ^J can be estimated by Least Squares regression of the discounted cash flows $\Pi(t_n, t_i, \tau, \zeta, \omega)$ onto the basis $\{L_j, j = 1, \ldots, J\}$. To apply the decision rule in (4.5), also F_h need to be estimated. Yet, since at this step, $\Pi_h(t_n, t_n, \tau, \omega)$ is known, $h = 1, \ldots, H$, and

$$F_h(t_n, X_{t_n}(\omega)) = \mathbb{E}^*_{t_n}\left[\sum_{i=n}^{N} e^{-r(t_i-t_n)}\Pi_h(t_n, t_i, \tau, \cdot)\right]$$

hence, we can apply (4.5) to find out the control $(\omega, \zeta)(\omega)$ at t_n.

All the above cases are plain extensions of LSM. Hence, the convergence results in Clément, Lamberton and Protter [14] still apply.

5 Applications

In this section we provide several numerical experiments to show how the approach presented in Section 4 can be used to model and evaluate complex capital budgeting problems with many underlying assets and many interacting real options. To illustrate the efficiency of the extended LSM approach, assuming that the underlying factors are geometric Brownian motions, we benchmark the estimates of the values against the results obtained by applying the extended Log-Transformed binomial lattice approach (see Gamba and Trigeorgis [21]) and, if closed-form formula are available, against exact solutions.

There are two families of examples. The first set of numerical experiments are abstract situations whose main purpose is to illustrate the efficiency of the valuation approach when applied to the building blocks of the decomposition approach. We will present examples for the independent options case, the compound option case and the case with mutually exclusively options. These examples permit to see the influence of various parameter values on the accuracy of the numerical method proposed in this work. This includes, as a numerical experiment, the valuation of a complex real option presented in Lint and Pennings [33] and Martzoukos and Trigeorgis [38] involving four sources of uncertainty.

In the second set of numerical experiments, we apply the methodology in Section 4 to Schwartz and Moon's [44, 45] model for valuing growth companies. In particular, starting from the analysis presented in those papers, we introduce (and evaluate) a (contingent) expansion strategy of a growth firm, always including the possibility to default.

We will see how different problems specifications can be easily modelled in our approach.

5.1 Warm up applications

Example 1. Let there be given an investment project which, after a first outlay, K_0, (for instance, R&D expenditure, infrastructure outlay, etc.) can have two possible outcomes at time T'. Each outcome can give rise to a potential business. These business can be obtained with some additional capital expenditure.[21] The values of these business are the present values

[21] For the sake of definiteness, one can think of an investment project in an undeveloped land with potential oil or gas reserves. The first outlay is given by the exploration costs. We assume that the geological tests can show, after a given time-period, that alternatively either gas or oil can be extracted. The outcomes of exploration are uncertain. The

of the respective cash flows from operations. We assume that the values of the business are, under the risk neutral equilibrium probability, GBM's with dynamics

$$\frac{dV_i(t)}{V_i(t)} = \alpha_i dt + \sigma_i dB_i^*(t) \qquad V_i(0) = V_i$$

$i = 1, 2$, where $\alpha_i = r - \delta_i$, δ_i is an equilibrium shortfall rate of return[22] or a convenience yield, r is the annualized continuously compounded risk-free interest rate, α_i and σ_i are given on an annual basis, and $\mathbb{E}[dB_1^* dB_2^*] = \rho dt$. The decision on the business to be developed can be deferred until the technical uncertainty and the market uncertainty are dissipated. Hence, the following real options are embedded in the case at hand:

option to develop business V_1: by paying K_1 within $T_1 = 5$ years. The payoff of this option is $\Pi_1(t, V_1) = \max\{V_1 - K_1, 0\}$ and the value will be denoted F_1;

option to develop business V_2: the additional costs is K_2 and maturity is $T_2 = T_1 = 5$ years. The related payoff is $\Pi_2(t, V_2) = \max\{V_2 - K_2, 0\}$ and the value is F_2.

For simplicity, we assume that the decision to spend the capital outlays $K_0 = 1$ is committed (i.e., not an option). We can see K_0 as the cost needed to obtain information about the feasibility of the project. The probabilities of the possible outcomes are p_i. The options to develop the business can be exercised in the interval $[T', T_i]$.

The base case parameters are

i	1	2
δ_i	δ	0.05
σ_i	σ	0.15
V_i	S	80
K_i	100	80
T_i	T	
ρ	0	
K_0	8	
T'	1	
$p_1 = p_2$	0.5	
r	0.05	

additional capital outlays are needed to build the facilities for extraction.

[22] For more details on the equilibrium shortfall rate of return for non traded real assets, see McDonald and Siegel [39].

The values of the investment project for several choices of the parameters of the stochastic process of the first business (S, σ and δ) and for different maturities are presented in Table 2.[23] By inspection, we can see that the overall accuracy (as measured by Root Mean Square Error of the numerical estimate[24]) is fair. Keeping the parameters of one of the two businesses constant, we can see that the value of the project is increasing in the current value of one of the two underlyings, increasing with respect to volatility and to maturity, whereas it is decreasing in the convenience yield.

The second and the third examples can be considered two strategic alternatives of the same capital budgeting problem.

Example 2. Let there be given a real asset (a business) whose value V_t follows a GBM

$$\frac{dV(t)}{V(t)} = \alpha dt + \sigma dB(t), \qquad V(0) = V$$

under the equilibrium risk-neutral probability, where $\alpha = r - \delta$, δ is an equilibrium shortfall rate of return and r is the (continuously compounded) annual riskless rate. The following options are available:

option to defer the investment: the payoff as if the option was in isolation is $\Pi_1(t, V) = \max\{e_1 V - I_1, 0\}$, that is, with a cost outlay I_1 we can get a percentage e_1, $0 < e_1 < 1$, of the whole business. The maturity of the option is T_1 (years). As usual F_1 will denote the option value;

option to expand: the payoff is $\Pi_2(t, V) = \max\{e_2 V - I_2, 0\}$ with $e_2 = 1 - e_1$; i.e., with a capital expenditure I_2 we can complete the investment in the business. The maturity is T_2 years, the value F_2.

The payoffs above are not the true ones for the problem at hand. Actually, since the expansion option is available only after the investment option has been exercised, then the payoffs of the investment option is

[23] We point out that, since the options to develop the two business are strategically independent, then the value of the project is not dependent on the correlation ρ between the two business.

[24] Root Mean Square Error (RMSE) is defined as

$$RMSE = \sqrt{\frac{1}{m} \sum_{i=1}^{m} \left(\frac{F_i - \hat{F}_i}{F_i} \right)^2}, \tag{5.1}$$

where F_i is the (accurate) value of the i-th project value, $\hat{F}_i$ is the related estimate obtained by simulation and m is the number of cases (in the table).

$\max\{e_1 V - I_1 + F_2(t, V), 0\}$. We stress that, although the second option can be exercised in the interval $[0, T_2]$, the actual time interval for the second option is from the (stopping) time the first option is exercised to T_2. The base case parameters are

$I_1 = I_2$	80
$e_1 = e_2$	0.5
k	0.5
X	80
V_0	S
T_2	T
T_1	$T - 2$
r	0.05

The numerical results for a set of value of the parameters of the project are in Table 3. As far as accuracy and the dependence of the project value on parameters is concerned, what we said for Table 2 still holds.

Example 3. Given the same real asset in Example 2, we are going to evaluate a different strategic alternative

option to defer the investment: by paying the outlay $I = I_1 + I_2$ we can get the whole business whose value is V and the opportunity is available until T_1 (years). Hence, the payoff of this option is $\Pi_1(t, V) = \max\{V - I, 0\}$ and its value is F_1;

option to contract the scale of the project: we can save part of the initial outlay, $X = I_2$, by reducing the scale of the business by k percent. This option is available, after the option to invest has been exercised, until T_2. Hence, the payoff is $\Pi_2(t, V) = \max\{X - kV, 0\}$ and the value is F_2.

As in Example 2, since the option to defer gives rise, when exercised, to the option to reduce the scale of the project, then the actual payoff of the first option is $\max\{V - I + F_2(t, V), 0\}$.

From a strategic viewpoint, this example offers an alternative approach to the same investment opportunity showed in Example 2. Actually, in that case the approach was more conservative, because the second stage takes place only if the first step is successful. In this example, on the other hand, we can obtain the same real asset, but we can recover part of the sunk cost if the business turn to be less favorable than expected. Although at a first sight the two alternatives might seem basically the same, Example 3 has a larger "operating leverage" given the higher level of fixed costs.

The values of the project, for a set of parameter, are in Table 4. The usual comments on accuracy we did for the previous table still apply.

Example 4. On the same asset as is Example 2, let there be given the following opportunities:

option to defer the investment: by paying I_1 we can acquire e_1 percent of the asset. This option can be exercised within T_1 (years). The payoff is $\Pi_1(t, V) = \max\{e_1 V - I_1 + G, 0\}$, where G is the value of the option to choose the best of two subsequent opportunities, and its value is F_1.

The exercise of this option give the opportunity to choose the best of two alternatives. Either

option to expand: we can get the remaining part ($e_2 = 1 - e_1$) of the business with an additional capital expenditure I_2 by time T_2. The related payoff is $\Pi_2(t, V) = \max\{e_2 V - I_2, 0\}$ and the value is F_2. Or

option to abandon: by year T_3, as an alternative to the option to expand, we can abandon the business ($k = e_1$), after the first investment, saving $X < I_1$. The payoff of this option is $\Pi_3(t, V) = \max\{X - kV, 0\}$ and the value is F_3.

Since the option to expand and the option to abandon are mutually exclusive, only one of them can be exercised. The base case parameters are

$I_1 = I_2$	50
X	30
$e_1 = e_2$	0.5
k	0.5
V_0	S
T_1	$T - 2$
T_2	T
T_3	$T - 0.5$
r	0.05

The results for a wide set of parameter values, are in Table 5.

As an application of the stylized option problem in Example 4, we apply our numerical methodology to an actual case study drawn from Lint and Pennings [33] and Martzoukos and Trigeorgis [38].

The case can be described as follow: there is a firm which is considering the development of two product standards in consumer electronic industry in a given time horizon. There is a cost outlay to be paid upfront to obtain the options to invest in the two product standards. The standard that finally will prevail is uncertain at the date of the first outlay. If the firm invests in both technologies, it acquires an option on the best of two assets (product standards). Each underlying asset of this option is the market (present) value of the resulting cash flows if that standard prevails. Moreover, the underlying assets are correlated. The cost of introducing each standard is the strike price of the option. Also the strike prices for the two technologies are stochastic and correlated with the other state variables.

The underlying assets are V_i, the market value of i-th business (i.e., the value of cash flows obtained by product standard i), and C_i, the cost to introduce the standard i, $i = 1, 2$. These variables are assumed to follow correlated geometric Brownian motions (under the equilibrium martingale measure) with equilibrium rate of return shortfall and volatilities respectively δ_{V_i}, σ_{V_i}, δ_{C_i} and σ_{C_i}, $i = 1, 2$. Correlations are ρ_{ij}, $i \neq j$, $i, j = 1, \ldots, 4$.

Hence, the investment project has the following embedded options:

option to defer investment: by paying I we can acquire the option to choose the best of the two standards later on. This option can be exercised within T_0 (years). The payoff is $\Pi_0 = \max\{G - I, 0\}$, where G is defined below, and its value is F_0;

options to defer investment: with an additional capital expenditure C_h, we can get the value of the related product standard, V_h. The maturity of this option is T_h. The related payoff is $\Pi_h = \max\{V_h - C_h, 0\}$ and the value is F_h, $h = 1, 2$.

As usual, since the option to invest in both standards provides the opportunity to choose for the best, then the actual payoff of the first option is

$$\max\{G(t, V_1, V_2, C_1, C_2) - I, 0\}.$$

The base case parameters are

$V_i(0)$	100	$i = 1, 2$
$C_i(0)$	100	$i = 1, 2$
r	0.07	$i = 1, \ldots, 4$
δ_i	0.1	$i = 1, \ldots, 4$
σ_i	0.2	$i = 1, \ldots, 4$
ρ_{ij}	0.5	$i \neq j, i, j = 1, \ldots, 4$
T_0	0	
I	0	
T_i	2	$i = 1, 2$

Besides the above described case, we have evaluated different versions of the investment problem by considering several features of the set of opportunities. In particular, we have evaluated also the impact of higher volatility, lower correlation, longer maturity and different investment scale on the option value.

To compare numerical results with exact solutions, we consider also the case of non-stochastic development costs for both technologies and $C_1 = C = C_2$. With this choice of parameters the problem has an analytic solution: if both the dividend yields are zero ($\delta_{V_1} = 0 = \delta_{V_2}$), then the model reduces to the European[25] option on the maximum of two risky assets and the solution formula has been provided by Stulz [46]; if at least one of the dividend yields is not zero, the extension of Stulz' formula for the European option on the maximum of two assets is in Martzoukos and Trigeorgis [38]. Moreover, if $V_2(0) = C_2(0) = 0$ (i.e., only one of the two standards is valuable), the options are European and $I = 0.1C_1(0)$, then the problem reduces to the compound-exchange option studied by Carr [13] and a closed-form valuation formula is available. Again, if at least one of the dividend yield is not zero, the extension of Carr's formula is in Martzoukos and Trigeorgis [38]. Numerical results are presented in Table 6. As for the other examples, accuracy is fair in most of the cases.

5.2 Rational pricing of a growth company

In this section we apply the simulation approach described above to value a growth company according to Schwartz and Moon [44].[26] Since in Schwartz and Moon [44] no real option (but the option to default) has been foreseen for such a company, we extend their analysis by explicitely introducing some growth options in the strategy of a growth company, while preserving the option to default.[27]

[25] Note that, if both the dividend yields are zero, the American option and the European option are the same.

[26] Although in Schwartz and Moon [44] the model has been applied to internet companies, it is suited to describe the dynamic of the value of any company with growth characteristics.

[27] A more advanced model has been proposed by Schwartz and Moon [45]. In that paper, besides some refinements of the previous model, a specific (path-dependent) pattern of growth of physical capital has been introduced. Since in this model, capital expenditure is a fixed proportion of revenues (i.e., not a contingent decision), the growth of physical capital is not the result of the exercise of expansion/growth options. For this reason, in what follow, we will explicitly include in the model expansion/growth options whose underlying is the value of cash flow from the business.

What remains of this section is organized as follows: first we describe the dynamics of the (present) value of the cash flows from the business with no option, and next we value some typical strategic (real) growth options on that business, as per the stylized examples seen in the first part of this section.

Schwartz and Moon's [44] model has two sources of uncertainty: revenues and growth rate of revenues. Let $\{R(t)\}$ denote the rate of revenues at t. It is a stochastic process

$$\frac{dR(t)}{R(t)} = \mu(t)dt + \sigma(t)dB_1(t) \quad \text{ with } R(0) = R_0$$

where μ is the actual expected growth rate of revenues. It is assumed that $\{\mu(t)\}$ is a stochastic process whose dynamic is

$$d\mu(t) = \kappa(\bar{\mu} - \mu(t))dt + \eta(t)dB_2(t) \quad \text{ with } \mu(0) = \mu_0$$

and $dB_1 dB_2 = \rho dt$. The volatilities of these two processes are assumed to follow a deterministic pattern towards a long term value:

$$\sigma(t) = \bar{\sigma} + e^{-\kappa t}(\sigma_0 - \bar{\sigma})$$

and

$$\eta(t) = \eta_0 e^{-kt}$$

Note that in this model, κ is a global rate of mean-reversion: it is related to the average time of normalization of the firm; i.e., the time needed for the firm to assume its long-term growth characteristics (as introduced in Schwartz and Moon [45]).

Costs have two components: a part which is (costant) proportion of revenues (γR), and a fixed part (committed investment) (F) per year. So, total cost are $C(t) = \gamma R(t) + F$. To simplify the analysis, we omit loss-carry-forward and the related tax savings.

Hence, the cash flow available to the firm is

$$Y(t) = [R(t) - C(t)]\,(1 - \tau)$$

where τ is the corporate tax rate and depreciation and interest tax shields are omitted for simplicity.

For valuation purposes, we introduce a risk-neutral probability measure by adjusting the drifts of the value drivers. In particolar, following Schwartz and Moon [45], we assume that only revenues has a "beta" risk premium.[28] Hence, given a market unit risk premium $\bar{\lambda}$, the risk premium on reveues is $\lambda(t) = \bar{\lambda}\sigma(t)$. According to the risk-neutral probability measure, the stochastic process of revenues is

$$\frac{dR(t)}{R(t)} = \left[\mu(t) - \bar{\lambda}\sigma(t)\right] dt + \sigma(t) dB_1^*(t).$$

Moreover, we assume that, over the valuation period, there is a non-stocastic risk-free rate r.

In this setting, the value of the firm is the expected value of discounted cash flow. As it is customary, we assume that, after a prespecified T, the company will be in its steady state. Hence, to obtain the value of the firm, we sum up the value of cash flow obtained within T to the terminal value, given by M times the EBITDA, $R(T) - C(T)$:

$$V(t) = \mathbb{E}_t^* \left[\int_t^T Y(s) e^{-r(s-t)} ds + M \left[R(T) - C(T)\right] e^{-r(T-t)} \right]$$

In what follows, we assume that $M = 1/r$.

Differently from Schwartz and Moon [44] and [45], we omit the possibility that the firm goes bankrupt when the accumulated cash flow level reaches a predetermined threshold Y^*. Instead, we will model bankruptcy as an option to abandon the current business (put option), with "strike price" S.[29] This option is exercised when the value, $V(t)$, of the firm reaches an endogenously determined level, $V^*(t)$.

In order to determine the value of shares of this company, $P = V/n$, where n is the number of outstanding shares, we will assume that the number of share is fixed over the time horizon and that the firm is all equity financed. The parameters of the model are in Table 1.

In order to apply our simulation approach, we take time steps of one month. Hence $N = 72$ and $K = 100\ 000$ paths.

Now, we can evaluate the business without options. By adopting the discrete-time approximation in Schwartz and Moon [45, Eq. (25)-(27)], the distribution of the $t = 0$ present value of cash flow is represented in Figure 1. The expected value is 17.266 and the standard deviation is 0.076.

Next, we add to the base case, the following real options:

[28] We point out that this is slightly in contrast with the assumption that the rate of growth of revenues can be correlated with the revenues rate, as assumed by Schwartz and Moon [45].

[29] In Merton's [42] framework, S is the level of debt.

Table 1: Firm valuation: parameters of the model

valuation horizon	T	6 y
initial value of revenues	R_0	500 m
initial volatility of revenues	σ_0	0.15
long term volatility of revenues	$\bar{\sigma}$	0.08
mean-reversion rate	κ	0.2
initial growth rate	μ	0.4
long term growth rate	$\bar{\mu}$	0.05
initial volatility of growth rate	η_0	0.3
correlation	ρ	0
annual fixed costs	F	35 m
operative costs (% of revenues)	γ	0.7
tax rate	τ	0.35
market unit risk premium	λ	1.25
risk-free rate	r	0.05
number of shares	n	100 m

1. an option to expand, with expansion rate e, investment cost I, and maturity T_1;

2. a staged expansion strategy, with maturities $T_2 = T_1/2$ and $T_3 \equiv T_1$, expansion rate respectively $e_1 = e_2 = e/2$, and investment costs $I_1 = I_2 = I/2$;

3. an option to abandon, with strike price S and maturity T_3;

4. the best of the following two alternatives: an expansion option according to (2) or an option to abandon as per (3).

For every strategic plan we will consider both the American and the European case. The parameters of the real option valuation problem are:

T_1	4 y
e	0.5
I	1000 m
S	1000 m

To solve the above real options problems, we take $V(t)$ obtained from the simulation of business as the underlying asset of the options; i.e. the simulation approach proposed in Section 4 for real option valuation uses

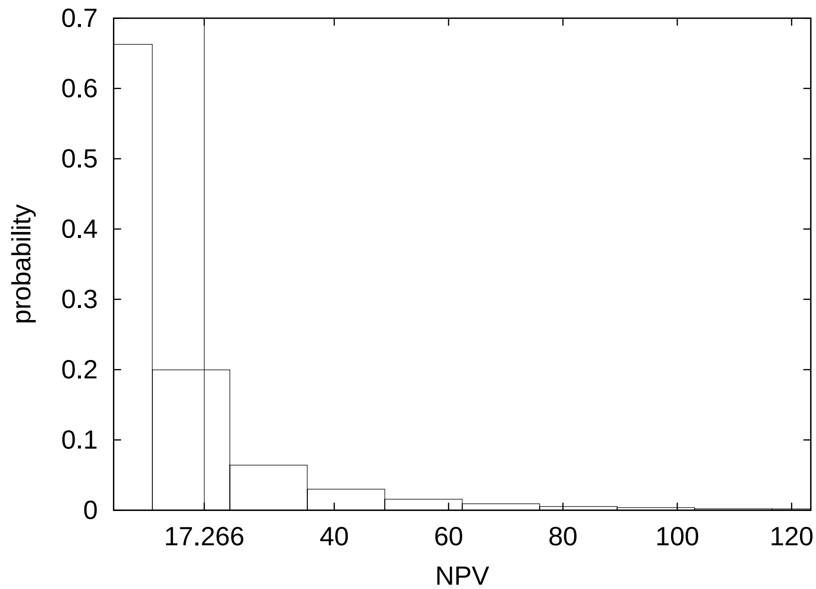

Figure 1: Rational pricing of a growth company: NPV distribution ($\mathbb{E}[NPV] = 17.266$, standard deviation = 0.076)

directly $\{V(t, \omega)\}$, for $t = 1, \ldots, N$ and $\omega = 1, \ldots, K$, as per the result of the above simulation.

The numerical results are presented in Table 7. The accuracy, measured by standard deviation of the numerical results, is fair. It can be seen that the value of the shares according to each and every flexible strategy is higher than the price of the share in the base case. Moreover, from Table 7, we can appreciate the incremental contribution of every option to the value of the strategic plan.

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Table 2: Example 1

				American				European			
S	σ (%)	δ (%)	T	Lattice	Sim	std. dev.	rel.err.	Lattice	Sim	std. dev.	rel.err.
100	20	1	4	7.302	7.224	0.118	- 0.011	7.051	7.020	0.102	- 0.004
100	20	1	5	9.202	9.116	0.135	- 0.009	8.844	8.828	0.110	-0.002
100	20	3	4	4.796	4.820	0.048	0.005	4.497	4.509	0.122	0.003
100	20	3	5	6.074	6.080	0.084	0.001	5.621	5.628	0.083	0.001
100	30	1	4	10.551	10.527	0.175	-0.002	10.300	10.306	0.153	0.001
100	30	1	5	12.656	12.570	0.122	-0.007	12.294	12.286	0.208	-0.001
100	30	3	4	8.157	8.171	0.109	0.002	7.746	7.781	0.161	0.004
100	30	3	5	9.701	9.749	0.155	0.005	9.072	9.098	0.160	0.003
90	20	1	4	4.036	3.985	0.087	-0.013	3.785	3.789	0.080	0.001
90	20	1	5	5.826	5.770	0.069	-0.010	5.468	5.499	0.079	0.006
90	20	3	4	2.086	2.072	0.049	-0.007	1.812	1.836	0.071	0.013
90	20	3	5	3.322	3.315	0.081	-0.002	2.914	2.926	0.076	0.004
90	30	1	4	7.268	7.213	0.151	-0.007	7.017	7.046	0.153	0.004
90	30	1	5	9.276	9.212	0.143	-0.007	8.916	8.873	0.201	-0.005
90	30	3	4	5.283	5.316	0.078	0.006	4.931	4.973	0.098	0.008
90	30	3	5	6.781	6.814	0.074	0.005	6.243	6.269	0.107	0.004
						RMSE	0.007			RMSE	0.005

"Lattice" is the value obtained by the Generalized Log-Transformed binomial lattice approach (see Gamba and Trigeorgis [21]) with $N = 600$ time steps.
"Sim" is the estimate obtained with the extended LSM approach proposed in this paper with $N = 50$ time steps, powers of the underlyings with $J = 8$ terms (and mixed terms up to the second power), and $K = 100\,000$ paths.
"std.dev" is the standard deviation of the LSM estimate. It is obtained by iterating 20 times the LSM and then calculating the standard deviation of the estimate.
"RMSE" is the root means square error; i.e., the square root of the sum of the squares of the relative errors (rel.err) with respect to the value obtained with the lattice approach (see (5.1)).

Table 3: Example 2

				American				European			
S	σ (%)	δ (%)	T	Lattice	Sim	std. dev.	rel.err.	Lattice	Sim	std. dev.	rel.err.
100	20	3	4	1.364	1.363	0.037	-0.001	1.363	1.353	0.059	-0.007
100	20	3	5	2.857	2.843	0.074	-0.005	2.850	2.862	0.056	0.004
100	20	5	4	0.899	0.908	0.025	0.010	0.871	0.873	0.027	0.003
100	20	5	5	1.858	1.854	0.050	-0.002	1.786	1.792	0.044	0.004
100	30	3	4	5.197	5.218	0.101	0.004	5.170	5.165	0.124	-0.001
100	30	3	5	8.311	8.282	0.165	-0.003	8.241	8.142	0.223	-0.012
100	30	5	4	4.116	4.160	0.101	0.011	3.974	3.960	0.104	0.004
100	30	5	5	6.489	6.491	0.111	-0.000	6.193	6.217	0.117	0.004
110	20	3	4	2.781	2.771	0.072	-0.004	2.777	2.786	0.056	0.003
110	20	3	5	4.917	4.879	0.091	-0.008	4.903	4.894	0.094	-0.002
110	20	5	4	1.920	1.915	0.052	-0.003	1.857	1.857	0.039	-0.000
110	20	5	5	3.341	3.363	0.060	0.007	3.198	3.202	0.088	0.001
110	30	3	4	7.942	7.916	0.233	-0.003	7.896	7.890	0.114	-0.001
110	30	3	5	11.659	11.690	0.139	0.003	11.550	11.539	0.177	-0.001
110	30	5	4	6.419	6.461	0.097	0.007	6.184	6.192	0.128	0.001
110	30	5	5	9.280	9.356	0.149	0.008	8.825	8.824	0.197	-0.000
						RMSE	0.006			RMSE	0.004

"Lattice" is the value obtained by the Generalized Log-Transformed binomial lattice approach (see Gamba and Trigeorgis [21]) with $N = 10^4$ time steps.

"Sim" is the estimate obtained with the extended LSM approach proposed in this paper with $N = 50$ time steps, powers of the underlyings with $J = 8$ terms, and $K = 100\ 000$ paths.

"std.dev" is the standard deviation of the LSM estimate. It is obtained by iterating 20 times the LSM and then calculating the standard deviation of the estimate.

"RMSE" is the root means square error; i.e., the square root of the sum of the squares of the relative errors (rel.err) with respect to the value obtained with the lattice approach (see (5.1)).

Table 4: Example 3

				American				European			
S	σ (%)	δ (%)	T	Lattice	Sim	std. dev.	rel.err.	Lattice	Sim	std. dev.	rel.err.
100	20	3	4	1.198	1.194	0.029	-0.004	1.175	1.162	0.066	-0.011
100	20	3	5	2.627	2.610	0.041	-0.006	2.595	2.594	0.060	-0.000
100	20	5	4	0.913	0.926	0.029	0.015	0.895	0.902	0.024	0.008
100	20	5	5	1.898	1.909	0.049	0.006	1.850	1.849	0.046	-0.001
100	30	3	4	4.894	4.874	0.087	-0.004	4.835	4.821	0.087	-0.003
100	30	3	5	8.005	7.982	0.180	-0.003	7.931	7.995	0.090	0.008
100	30	5	4	4.198	4.218	0.051	0.005	4.110	4.117	0.075	0.002
100	30	5	5	6.641	6.635	0.150	-0.001	6.440	6.468	0.110	0.004
110	20	3	4	2.492	2.483	0.040	-0.004	2.452	2.463	0.055	0.005
110	20	3	5	4.576	4.556	0.081	-0.004	4.530	4.518	0.067	-0.003
110	20	5	4	1.955	1.985	0.035	0.015	1.917	1.925	0.044	0.004
110	20	5	5	3.416	3.417	0.051	-0.000	3.322	3.304	0.066	-0.005
110	30	3	4	7.541	7.540	0.092	-0.000	7.462	7.459	0.136	-0.000
110	30	3	5	11.287	11.229	0.159	-0.005	11.186	11.229	0.164	0.004
110	30	5	4	6.555	6.557	0.114	-0.000	6.410	6.443	0.088	0.005
110	30	5	5	9.505	9.492	0.105	-0.001	9.189	9.177	0.129	-0.001
						RMSE	0.006			RMSE	0.005

"Lattice" is the value obtained by the Generalized Log-Transformed binomial lattice approach (see Gamba and Trigeorgis [21]) with $N = 10^4$ time steps.

"Sim" is the estimate obtained with the extended LSM approach proposed in this paper with $N = 50$ time steps, powers of the underlyings with $J = 8$ terms, and $K = 100\ 000$ paths.

"std.dev" is the standard deviation of the LSM estimate. It is obtained by iterating 20 times the LSM and then calculating the standard deviation of the estimate.

"RMSE" is the root means square error; i.e., the square root of the sum of the squares of the relative errors (rel.err) with respect to the value obtained with the lattice approach (see (5.1)).

Table 5: Example 4

				American				European			
S	σ (%)	δ (%)	T	Lattice	Sim	std. dev.	rel.err.	Lattice	Sim	std. dev.	rel.err.
100	20	10	3	6.533	6.519	0.017	-0.002	5.094	5.110	0.028	0.003
100	20	10	5	8.447	8.405	0.032	-0.005	5.414	5.411	0.051	0.000
100	20	5	3	9.131	9.163	0.031	0.004	8.822	8.873	0.031	0.006
100	20	5	5	13.120	13.110	0.030	-0.001	12.228	12.233	0.057	0.000
100	40	10	3	15.566	15.372	0.037	-0.012	13.475	13.504	0.083	0.002
100	40	10	5	20.459	20.131	0.060	-0.016	15.615	15.622	0.106	0.000
100	40	5	3	18.568	18.612	0.065	0.002	17.930	18.001	0.096	0.004
100	40	5	5	26.080	25.868	0.095	-0.008	24.253	24.284	0.181	0.001
80	20	10	3	0.904	0.918	0.010	0.015	0.732	0.742	0.017	0.013
80	20	10	5	2.311	2.292	0.020	-0.008	1.640	1.640	0.028	0.000
80	20	5	3	1.694	1.711	0.017	0.010	1.646	1.674	0.000	0.017
80	20	5	5	4.786	4.776	0.039	-0.002	4.545	4.553	0.032	0.002
80	40	10	3	6.573	6.483	0.045	-0.014	5.783	5.847	0.047	0.011
80	40	10	5	11.169	11.125	0.094	-0.004	8.847	8.813	0.101	-0.004
80	40	5	3	8.392	8.337	0.065	-0.007	8.138	8.217	0.069	0.010
80	40	5	5	15.294	15.225	0.089	-0.005	14.383	14.422	0.149	0.003
						RMSE	0.009			RMSE	0.007

"Lattice" is the value obtained by the Generalized Log-Transformed binomial lattice approach (see Gamba and Trigeorgis [21]) with $N = 10^4$ time steps.
"Sim" is the estimate obtained with the extended LSM approach proposed in this paper with $N = 50$ time steps, powers of the underlyings with $J = 9$ terms, and $K = 100\,000$ paths.
"std.dev" is the standard deviation of the LSM estimate. It is obtained by iterating 20 times the LSM and then calculating the standard deviation of the estimate.
"RMSE" is the root means square error; i.e., the square root of the sum of the squares of the relative errors (rel.err) with respect to the value obtained with the lattice approach(see (5.1)).

Table 6: Best of two product standards

	American				European			
	Lattice	Sim	std. dev.	rel.err.	Lattice	Sim	std. dev.	rel.err.
base case	16.427	16.605	0.070	0.011	15.707	15.753	0.071	0.003
$\rho_{ij} = 0$	23.286	23.356	0.123	0.003	22.261	22.258	0.086	0.000
$V_1 = C_1 = 90,\ V_2 = C_2 = 100$	15.616	15.799	0.069	0.012	14.931	14.973	0.054	0.003
$T_1 = 3,\ T_2 = 2$	17.280	16.957	0.059	-0.019	16.041	16.060	0.074	0.001
$T_1 = T_2 = 5$	21.567	21.747	0.105	0.008	18.402	18.437	0.079	0.002
$C_1 = 90,\ \sigma_{C_1} = .1,\ \sigma_{C_2} = .3,\ \rho_{ij} = 0$	24.876	24.306	0.113	-0.023	23.775	23.737	0.066	-0.002
$C_1 = 90,\ \sigma_{C_1} = .3,\ \sigma_{C_2} = .1,\ \rho_{ij} = 0$	25.575	25.168	0.088	-0.016	24.238	24.199	0.092	-0.002
$\sigma_i = .3$	24.627	25.007	0.149	0.015	23.545	23.578	0.087	0.001
$I = 10,\ T_0 = 2,\ T_1 = T_2 = 3$	11.432	11.049	0.072	-0.034	10.154	10.166	0.069	0.001
$\sigma_{C_i} = 0,\ \delta_i = 0$	26.593	26.897	0.157	0.011	26.593	26.604	0.051	0.000
$\sigma_{C_i} = 0$	12.567	12.550	0.080	-0.001	11.403	11.425	0.065	0.002
$I = .1C_1,\ V_2 = C_2 = 0,\ T_0 = 2,\ T_1 = 3$	6.668	6.551	0.029	-0.017	5.705	5.605	0.034	-0.018
			RMSE	0.016			RMSE	0.005

"Lattice" is the value obtained by the Generalized Log-Transformed binomial lattice approach (Gamba and Trigeorgis [21]) with $N = 500$ time steps for the cases with 2 underlyings (below the double line) and with $N = 50$ steps for the cases with 4 underlyings (above the double line).

"Sim" is the estimate provided with the extended LSM approach with $N = 50$ time steps, power series with $J = 9$ terms (and mixed terms up to the second power) and simulating $K = 100\ 000$ paths when the problem has 2 underlyings (below the double line) and $K = 50\ 000$ when the problem has 4 underlyings (above the double line).

"std.dev" is the standard deviation of the LSM estimate. It is obtained by iterating 10 times the LSM and then calculating the standard deviation of the estimate.

"RMSE" is the root means square error (see (5.1)).

Exact solutions are given in the last three cases, as far as European options are concerned. When $\sigma_{C_i} = 0,\ \delta_i = 0$ (Stulz [46]), $F(0, x) = 26.608$. If $\sigma_{C_i} = 0,\ \delta_i = 0$ (Martzoukos and Trigeorgis [38]) $F(0, x) = 11.411$. If $I = .1C_1,\ V_2 = C_2 = 0,\ T_0 = 2,\ T_1 = 3$ (Carr [13]) $F(0, x) = 5.709$.

Table 7: Rational pricing of a growth company

	American		European	
	Sim	std.dev.	Sim	std.dev.
Abandon	22.005	0.083	22.012	0.089
Expansion I	19.923	0.089	18.749	0.106
Expansion II	19.968	0.099	19.160	0.089
Best of two strategies	24.568	0.106	23.868	0.141

"Sim" is the estimate provided with the extended LSM approach with $N = 72$ time steps, power series with $J = 9$ terms and simulating $K = 100\ 000$ paths.
"std.dev" is the standard deviation of the LSM estimate. It is obtained by iterating 20 times the LSM and then calculating the standard deviation of the estimate.
Base case: $NPV = 17.266$ with standard deviation 0.076.