Stochastic Process

- Their value changes over time in an uncertain manner.
- "discrete-time" vs. "continuous-time"
  (random walk) (Brownian motion)
- Markov process (a special case)
  *memoryless
  *weak form of the efficient-market hypothesis

  *Wiener process or Brownian motion $w(t)$

  - Δw over small Δt are independent.
  - Δw ~ normally distributed
    E (Δw) = 0
    Var (Δw) = Δt

    $\Delta w = \sqrt{\Delta t}\ \tilde{\varepsilon}_t$ where $\tilde{\varepsilon}_t \sim$ N(0,1)

〔Note that stock prices are closer to being log-normally distributed.〕

Ito process

- $dS = \alpha(S,t)dt + \sigma(S,t)dw$ (**)

- $E(dS) = \alpha(S,t)dt$

  $Var(dS) = \sigma^2(S,t)dt$
- Special case: geometric Brownian motion with drift
  (or standard diffusion Wiener process)

  where $\alpha(S,t) = \alpha S$ and $\sigma^2(S,t) = \sigma^2 S^2$

  α and σ are constant.

Then

$$\begin{cases} dS = \alpha S\ dt + \sigma S\ dw \\ \dfrac{dS}{S} = \alpha\ dt + \sigma\ dw \end{cases}$$

Thus $E(dS) = \alpha S\ dt$ and $Var(dS) = \sigma^2 S^2\ dt$

Ito's Lemma

Ito's lemma is easier to understand as a Taylor-series expansion:

$$C(S+\Delta S, t+\Delta t) = C(S,t) + \frac{\partial C}{\partial t}\Delta t + \frac{\partial C}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\Delta S)^2 + \ldots,$$

or

$$\Delta C = C(S+\Delta S, t+\Delta t) - C(S,t)$$

$$= \frac{\partial C}{\partial t}\Delta t + \frac{\partial C}{\partial S}\Delta S + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\Delta S)^2 + \ldots,$$

In the limit, as higher-order terms disappear,

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(dS)^2.$$

If *S* follows the standard diffusion (Ito) process in equation (＊＊), $(dS)^2$

behaves like $\sigma^2 S^2 dt$, so that Ito's lemma becomes

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{1}{2}\frac{\partial^2 C}{\partial S^2}(\sigma^2 S^2 dt).$$

Consider an option or a contingent claim *C(S,t)*

Ito's Lemma

Let $C(S_t, t)$ be a twice-differentiable function of t and of the random process S_t

$$dS_t = \alpha_t dt + \sigma_t dW_t, \qquad t \geq 0, \tag{1}$$

with well-behaved drift and diffusion parameters, a_t, σ_t. Then we have

$$dC_t = \frac{\partial C}{\partial S_t}dS_t + \frac{\partial C}{\partial t}dt + \frac{1}{2}\frac{\partial^2 C}{\partial S_t^2}\sigma_t^2 dt, \tag{2}$$

or, after substituting for dS_t using the relevant SDE,

$$dC_t = \left[\frac{\partial C}{\partial S_t}\alpha_t + \frac{\partial C}{\partial t} + \frac{1}{2}\frac{\partial^2 C}{\partial S_t^2}\sigma_t^2\right]dt + \frac{\partial C}{\partial S_t}\sigma_t dW_t, \tag{3}$$

where the equality holds in the mean square sense.

The Black-Scholes Formulae

* Standard Assumptions

1. Frictionless markets (for stocks, and options). This means that
   (a) there are no transactions costs or (differential) taxes;
   (b) there are no restrictions on short sales (such as margin requirements), and full use of proceeds is allowed;
   (c) all shares of all securities are infinitely divisible; and
   (d) borrowing and lending (at the same rate) are unrestricted.

   These assumptions allow continuous trading.

2. The risk-free (short-term) interest rate is constant over the life of the option (or known over time).

3. The underlying asset (stock) pays no dividends over the life of the option.

   This assumption is later removed, with appropriate dividend adjustments.

4. (Distributional assumption concerning the stock-price process) Stock prices follow a stochastic diffusion Wiener process of the form:

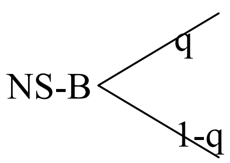
$$\frac{dS}{S} = \alpha \; dt + \sigma \; dz$$

* No-Arbitrage Approach (Dynamic Portfolio Replication Strategy)

[Idea]: Construct a portfolio consisting of buying **N** shares of underlying stock at its current price, **S**, and borrowing $**B** against them at riskless rate, **r**, (e.g., selling short Treasury bills).

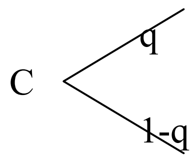
Net Cost: NS-B

NS^{+} -(1+r) B ———— (4)



NS^{-} -(1+r) B ———— (5)

$C^{+} = \max(S^{+}-E, 0)$ —————— (6)



$C^{-} = \max(S^{-}-E, 0)$ —————— (7)

If (4) = (5) and (6) = (7), then

$$\begin{cases} C = NS - B \\ NS - C = B \end{cases}$$

or [Risk-Neutral Valuation]

Continuous Time Framework:

$$\begin{cases} N = \dfrac{\partial C}{\partial S} \\ dC = N\, dS - dB \qquad \textit{with} \quad N = \dfrac{\partial C}{\partial S} \end{cases}$$

Substitute

$$\begin{cases} \sigma_t = \sigma S \\ \alpha_t = \alpha S \end{cases}$$ in (3). Then we have

$$dC = [\alpha S\ C_s + C_t + \frac{1}{2}\sigma^2 S^2 C_{ss}]dt + \alpha S\ C_s dW_t$$

So $$dB = N\ dS - dC$$

$$= Cs[\alpha S\ dt + \sigma S\ dW_t] - dC$$

$$= [\alpha S\ C_s]dt + [\sigma S\ C_s]dW_t - dC$$

$$= -[C_t + \frac{1}{2}\sigma^2 S^2 C_{ss}]dt$$

From no-arbitrage argument, we have

$$rBdt = -[C_t + \frac{1}{2}\sigma^2 S^2 C_{ss}]dt \qquad (8)$$

Substituting B= NS – C and N=C_s into (9)
And dividing throughout by dt , we get

$$C_t + \frac{1}{2}\sigma^2 S^2 C_{ss} + rSC_s - rC = 0$$

This is the Black-Scholes partial differential equation.
or

$$\frac{\partial C}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C}{\partial S^2}+rS\frac{\partial C}{\partial S}-rS=0,$$

with

$$C(0,t) = 0, \qquad C(S,t) \sim S \quad as\ S \rightarrow \infty$$

and

$$C(S,T) = \max(S-E,0).$$

[Trick]: Try to get rid of the 2nd and 3rd terms with S. We set

$$S=Ee^x, \qquad t=T-\tau/\frac{1}{2}\sigma^2, \qquad C=Ev(x,\tau).$$

$$\frac{\partial v}{\partial \tau}=\frac{\partial^2 v}{\partial x^2}+(k-1)\frac{\partial v}{\partial x}-kv$$

where $k = r/\frac{1}{2}\sigma^2$. The initial condition becomes

$$v(x,o) = \max(e^x-1,0).$$

$$v=e^{\alpha x+\beta\tau}u(x,\tau),$$

$$\beta u+\frac{\partial u}{\partial \tau}=\alpha^2 u+2\alpha\frac{\partial u}{\partial x}+\frac{\partial^2 u}{\partial x^2}+(k-1)(\alpha u+\frac{\partial u}{\partial x})-ku.$$

$$\beta=\alpha^2+(k-1)\alpha-k,$$

$$0=2\alpha+(k-1)$$

$$\alpha=-\frac{1}{2}(k-1), \qquad \beta=-\frac{1}{4}(k+1)^2.$$

$$v=e^{-\frac{1}{2}(k-1)x-\frac{1}{4}(k+1)^2\tau}u(x,\tau),$$

$$\frac{\partial u}{\partial \tau}=\frac{\partial^2 u}{\partial x^2} \qquad for \qquad -\infty<x<\infty, \qquad \tau>0,$$

$$u(x,0)=u_0(x)=\max(e^{\frac{1}{2}(k+1)x}-e^{\frac{1}{2}(k-1)x},0).$$

$$u(x,\tau)=\frac{1}{2\sqrt{\pi\tau}}\int_{-\infty}^{\infty}u_0(s)e^{-(x-s)^2/4\tau}ds$$

$$x' = (s - x)/\sqrt{2\tau}$$

$$u(x,\tau) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{\infty} u_0(x'\sqrt{2\tau} + x)e^{-\frac{1}{2}x'^2}\,dx'$$

$$= \frac{1}{2\sqrt{\pi}}\int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k+1)(x+x'\sqrt{2\tau})} e^{-\frac{1}{2}x'^2}\,dx'$$

$$-\frac{1}{2\sqrt{\pi}}\int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k-1)(x+x'\sqrt{2\tau})} e^{-\frac{1}{2}x'^2}\,dx'$$

$$= I_1 - I_2,$$

$$I_1 = \frac{1}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{2}(k+1)(x+x'\sqrt{2\tau})} e^{-\frac{1}{2}x'^2}\,dx'$$

$$= \frac{e^{\frac{1}{2}(k+1)x}}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}}^{\infty} e^{\frac{1}{4}(k+1)^2\tau} e^{-\frac{1}{2}(x'-\frac{1}{2}(k+1)\sqrt{2\tau})^2}\,dx'$$

$$= \frac{e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau}}{\sqrt{2\pi}}\int_{-x/\sqrt{2\tau}-\frac{1}{2}(k+1)\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}\rho^2}\,d\rho$$

$$= e^{\frac{1}{2}(k+1)x+\frac{1}{4}(k+1)^2\tau} N(d_1),$$

where

$$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}$$

and

$$N(d_1) = \frac{1}{2\sqrt{\pi}}\int_{-\infty}^{d_1} e^{-\frac{1}{2}s^2}\,ds$$

$$v(x,\tau) = e^{-\frac{1}{2}(k-1)x-\frac{1}{4}(k+1)^2\tau} u(x,\tau)$$

and then putting $x = \log(S/E)$, $\tau = \frac{1}{2}\sigma^2(T-t)$ and $C = Ev(x, \tau)$, to recover

$$C(S,t) = SN(d_1) - Ee^{-\gamma(T-t)}N(d_2),$$

where

$$d_1 = \frac{\log(S/E) + (\gamma + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}},$$

$$d_2 = \frac{\log(S/E) + (\gamma - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}.$$

$$u(x,0) = \max(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0),$$

put-call parity formula

$$C - P = S - Ee^{-\gamma(T-t)}$$
$$P(S,t) = Ee^{-\gamma(T-t)}N(-d_2) - SN(-d_1),$$

Where we have used the identity $N(d)+N(-d)=1$.

$$\Delta = \frac{\partial C}{\partial S}$$
$$= N(d_1) + S\frac{\partial}{\partial S}N(d_1) - Ee^{-\gamma(T-t)}\frac{\partial}{\partial S}N(d_2)$$
$$= N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - Ee^{-\gamma(T-t)}N'(d_2)\frac{\partial d_2}{\partial S}$$
$$= N(d_1) + SN'(d_1) - Ee^{-\gamma(T-t)}N'(d_2)/S\sigma\sqrt{T-t}$$
$$= N(d_1),$$

$$SN'(d_1) = Ee^{-\gamma(T-t)}N'(d_2)$$

(divide both sides by $N'(d_2) = (1/\sqrt{2\pi})e^{-\frac{1}{2}d_2^2}$ *first*).

Delta for the put is

$$\Delta = \frac{\partial P}{\partial S}$$
$$= N(d_1) - 1.$$