A SIMPLE EXPONENTIAL MODEL FOR DEPENDENT DEFAULTS

Kay Giesecke[*]

Cornell University

May 2, 2002; this version August 8, 2003

Abstract

A thorough understanding of the joint default behavior of credit-risky securities is essential for credit risk measurement as well as the valuation of multi-name credit derivatives and Collateralized Debt Obligations. In this paper we study a simple and tractable intensity-based model for correlated defaults, in which unpredictable default arrival times are jointly exponentially distributed. Since all relevant results are given in closed-form, the model can be easily implemented. The efficient simulation of dependent default times for pricing and risk management purposes is straightforward as well. Parameter calibration relies on market data as well as data and figures provided by rating agencies and credit risk management solutions.

Key words: correlated defaults; multivariate exponential model; simulation.

JEL Classification: G12; G13

1

1 Introduction

Credit risk refers to the risk of incurring losses due to unexpected changes in the credit quality of a counterparty or issuer. Based on the dependence of issuers on general economic factors or direct firm inter linkages, credit quality changes of several issuers are often correlated. Investors holding positions with numerous counterparties, such as financial institutions, are therefore exposed to the aggregated risk of losses due to correlated credit events arrivals.

The effective measurement and management of this aggregated credit risk is one of the core businesses of a financial institution. Credit risk measurement involves estimating the distribution of aggregated losses. Credit derivatives, which allow to isolate and trade credit risk by providing a payoff upon a credit event arrival with respect to a reference entity, allow the active management of credit risk. In the portfolio context of credit a significant role is played by multi-name credit derivatives, which have payoffs contingent on the credit quality of a number of reference entities. Collateralized Debt Obligations (CDO's) allow to restructure portfolios of credit-risky securities. They involve prioritized tranches whose cash flows are linked to the performance of a pool of debt instruments.

The estimation of aggregated loss distributions in credit risk measurement and the valuation of multi-name credit derivatives and CDO's requires a model for the joint default behavior of numerous credit-risky securities such as bonds or loans. In the class of intensity based models, where the stochastic structure of default is prescribed by an (exogenous) intensity or conditional default arrival rate, single-name approaches have been successfully extended to multi-firm settings. Duffie & Garleanu (2001) assume that an individual firm's default intensity is composed of some idiosyncratic component and some systematic component affecting all considered firms. Duffie & Singleton (1998) present several computationally efficient models relying on intensity processes which exhibit common and/or correlated jumps. Schönbucher & Schubert (2001) impose the assumed default dependence structure directly on a generic stochastic intensity model for individual firms. Giesecke (2002) underpins a multi-firm intensity model with a structural asset based model, where firms' assets and default thresholds are correlated.

In this paper we study a simple and tractable intensity-based model for correlated default times, which is in the spirit of the approach of Duffie & Garleanu (2001). The model is based on the idea that a firm's default is driven

by idiosyncratic as well as other regional, sectoral, industry, or economy-wide shocks, whose arrivals are modeled by independent Poisson processes. In this approach a default is governed again by a Poisson process and default times are jointly exponentially distributed. Various important statistics, such as moments, default correlations, conditional default probabilities, and joint default probabilities can be calculated explicitly. A closely related approach is studied in Lindskog & McNeil (2001), who consider default losses of different types.

The complete non-linear default dependence structure can be described by the copula of the default times. In the exponential model the *exponential copula* arises naturally. This explicitly available copula facilitates the calculation of non-linear default correlation measures and lends itself to flexible and efficient simulation algorithms for correlated default arrival times for pricing and risk management purposes. In particular, these algorithms can be used to generate default times with exponential dependence structure but arbitrary marginal default distribution. That means that one can combine any given single-name default probability model with the exponential correlation model discussed here.

The parameters of the model can be calibrated from single-name default swap or bond price data (for pricing purposes) as well as data provided by rating agencies (e.g. Moody's Diversity Score) and often implemented credit risk management solutions (e.g. KMV's CreditMonitor).

In Section 2, we consider the model first in the instructive two-firm case and extend then to the general multivariate case. The flexible copula-based simulation of correlated default arrival times is discussed in Section 3. Parameter estimation is considered in Section 4. Two applications of the model are presented in Section 5.

2   The Exponential Model

2.1   Bivariate Case

We begin by deriving the basic properties of the model in the instructive bivariate case with two firms labeled 1 and 2; in the following section we extend to the general case. The idea of the approach is to let defaults of firms be driven by firm-specific as well as economy-wide shock events. Suppose there

are Poisson processes[1] N_1, N_2, and N with respective intensities λ_1, λ_2, and λ. We interpret λ_i as the idiosyncratic shock intensity of firm i, while we think of λ as the intensity of a macro-economic or economy-wide shock affecting both firms simultaneously. We define the default time τ_i of firm i by

$$\tau_i = \inf\{t \geq 0 : N_i(t) + N(t) > 0\},$$

meaning that a default takes place completely unexpectedly if either an idiosyncratic or a systematic shock (or both) strikes the firm for the first time. Note that there is a positive probability of a simultaneous default of both firms. Thus firm i defaults with intensity $\lambda_i + \lambda$ and we have

$$s_i(t) = P[\tau_i > t] = P[N_i(t) + N(t) = 0] = e^{-(\lambda_i + \lambda)t}.$$

The expected default time and the default time variance are given by

$$E[\tau_i] = \frac{1}{\lambda_i + \lambda}, \qquad \mathrm{Var}[\tau_i] = \frac{1}{(\lambda_i + \lambda)^2}.$$

The joint survival probability is found to be

$$\begin{aligned} s(t, u) = P[\tau_1 > t, \tau_2 > u] &= P[N_1(t) = 0, N_2(u) = 0, N(t \vee u) = 0] \\ &= e^{-\lambda_1 t - \lambda_2 u - \lambda(t \vee u)} \\ &= e^{-(\lambda_1 + \lambda)t - (\lambda_2 + \lambda)u + \lambda(t \wedge u)} \\ &= s_1(t) s_2(u) \min(e^{\lambda t}, e^{\lambda u}). \end{aligned} \tag{1}$$

This *bivariate exponential distribution* is well-known in reliability modeling, cf. Marshall & Olkin (1967). s has an absolutely continuous and singular component, which can be calculated explicitly. The moment generating function of s is available analytically as well.

There exists a unique function $C^\tau : [0, 1]^2 \to [0, 1]$, called the *survival copula* of the default time vector (τ_1, τ_2), such that joint survival probabilities can be represented as

$$s(t, u) = C^\tau(s_1(t), s_2(u)),$$

---

[1]All random variables are defined on a fixed probability space $(\Omega, \mathcal{F}, P)$. Depending on the specific application, P is the physical probability (risk management setting) or some risk neutral probability (valuation setting).

cf. Nelsen (1999) for background reading. The copula C^τ describes the complete non-linear *default time dependence structure*. As C^τ couples exponential marginals s_i with the exponential joint survival function s, we will call C^τ the exponential copula (in the reliability literature this copula is also known as the Marshall-Olkin copula). Letting s_i^{-1} denote the inverse function of s_i and defining

$$\theta_i = \frac{\lambda}{\lambda_i + \lambda}$$

as the ratio of joint default intensity to default intensity of firm i, we obtain

$$C^\tau(u, v) = s(s_1^{-1}(u), s_2^{-1}(v)) = \min(vu^{1-\theta_1}, uv^{1-\theta_2}).$$

The parameter vector $\theta = (\theta_1, \theta_2)$ controls the degree of dependence between the default times; we write $C^\tau = C_\theta^\tau$. If the firms default independently of each other ($\lambda = 0$ or $\lambda_1, \lambda_2 \to \infty$), then $\theta_1 = \theta_2 = 0$ and we get $C_\theta^\tau(u, v) = uv$, the product copula. If the firms are perfectly positively correlated and the firms default simultaneously ($\lambda \to \infty$ or $\lambda_1 = \lambda_2 = 0$), then $\theta_1 = \theta_2 = 1$ and $C_\theta^\tau(u, v) = u \wedge v$, the Fréchet upper bound copula. Hence,

$$uv \leq C_\theta^\tau(u, v) \leq u \wedge v, \quad \theta \in [0, 1]^2, \quad u, v \in [0, 1], \tag{2}$$

meaning that in our model defaults can only be positively related.

Besides the survival copula C^τ there exists also a unique function $K^\tau :$ $[0, 1]^2 \to [0, 1]$ such that the bivariate distribution function p of τ can be represented as

$$p(t, u) = P[\tau_1 \leq t, \tau_2 \leq u] = K^\tau(p_1(t), p_2(u))$$

where $p_i(t) = P[\tau_i \leq t] = 1 - s_i(t)$ is the distribution function of τ_i. The ('usual') copula K^τ and the survival copula C^τ describe in an equivalent way the dependence between the default times. Noting that $s(t, u) = 1 - p_1(t) - p_2(u) + p(t, u)$, we see that these copulas are related via

$$K^\tau(u, v) = C^\tau(1 - u, 1 - v) + u + v - 1$$

$$= \min([1 - v][1 - u]^{1-\theta_1}, [1 - u][1 - v]^{1-\theta_2}) + u + v - 1.$$

(2) suggests a partial ordering on the set of copulas as function-valued default correlation measures. A scalar-valued measure such as rank correlation

5

can perhaps provide more intuition about the degree of stochastic dependence between the defaults. Spearman's rank correlation ρ^S is simply the linear correlation ρ of the copula K^τ given by

$$\rho^S(\tau_1, \tau_2) = \rho(p_1(\tau_1), p_2(\tau_2)) = 12 \int_0^1 \int_0^1 K^\tau(u, v) du dv - 3$$

$$= \frac{3\theta_1\theta_2}{2\theta_1 + 2\theta_2 - \theta_1\theta_2}$$

$$= \frac{3\lambda}{3\lambda + 2\lambda_1 + 2\lambda_2} \quad (3)$$

showing that ρ^S is a function of the copula K^τ only. Notice that the same result obtains when we compute ρ^S with C^τ instead of K^τ. While Spearman's $\rho^S(\tau_1, \tau_2)$ measures the degree of monotonic default dependence, linear default time correlation given by

$$\rho(\tau_1, \tau_2) = \frac{\lambda}{\lambda + \lambda_1 + \lambda_2} \quad (4)$$

measures the degree of linear default time dependence only. Clearly $\rho \leq \rho^S$ and linear default correlation underestimates the true default dependence, cf. Figure 1, where we plot both measures as functions of the joint shock intensity λ. We fix $\lambda_1 = \lambda_2 = 0.01$, which corresponds to a one-year default probability of about 1% when firms are independent. If λ is zero, then firms default independently and $\rho = \rho^S = 0$. With increasing λ, the joint shock component of the default risk dominates the idiosyncratic component, and the default correlation increases. The relationship between ρ^S, idiosyncratic intensities, and joint shock intensities is shown in Figure 2, where ρ^S is plotted as a function of $\lambda_1 = \lambda_2$ for varying λ. Quite intuitive, (rank) default correlation is decreasing in idiosyncratic default risk, because with increasing $\lambda_1 = \lambda_2$ the idiosyncratic risk component dominates the joint shock component of default risk. If $\lambda_1 = \lambda_2 = 0$, then only the joint shock matters; if this shock occurs firms default simultaneously and the rank default correlation is one. Summarizing, we have $\rho^S, \rho \in [0, 1]$, where $\rho^S = \rho = 0$ if $\lambda = 0$ or $\lambda_1, \lambda_2 \to \infty$ (C^τ is the product copula) and $\rho^S = \rho = 1$ if $\lambda \to \infty$ or $\lambda_1 = \lambda_2 = 0$ (C^τ is the Fréchet upper bound copula).

Another commonly used default correlation measure is the linear correlation of the default indicator variables,

$$\rho(1_{\{\tau_1 \leq t\}}, 1_{\{\tau_2 \leq t\}}) = \frac{s(t, t) - s_1(t) s_2(t)}{\sqrt{p_1(t) s_1(t) p_2(t) s_2(t)}}. \quad (5)$$
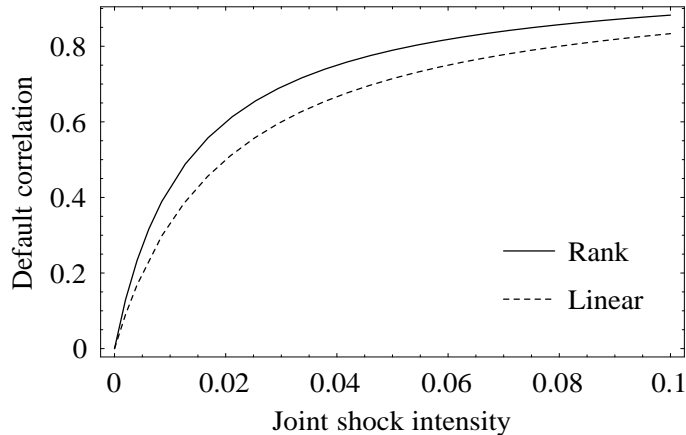
Figure 1: Rank and linear default correlation.

Figure 3 shows $\rho(1_{\{\tau_1 \leq t\}}, 1_{\{\tau_2 \leq t\}})$ over time for varying degrees of joint shock intensities λ (again we set $\lambda_1 = \lambda_2 = 0.01$). In our model, default indicator correlation is decreasing in time. This is partly in contrast to the structural model of Zhou (2001), where indicator correlation is hump-shaped.

The conclusions drawn on the basis of linear default time correlation $\rho(\tau_1, \tau_2)$ and linear default indicator correlation $\rho(1_{\{\tau_1 \leq t\}}, 1_{\{\tau_2 \leq t\}})$ should however be taken with care. Both are covariance-based and hence are only the natural dependence measures for joint elliptical random variables, cf. Embrechts, McNeil & Straumann (2001). Neither the default times nor default events are joint elliptical, and hence these measures can lead to severe misinterpretations of the true default correlation structure. ρ^S is in contrast defined on the level of the copula C^τ and therefore does not share these deficiencies.

2.2 General Multivariate Case

We now extend the default model discussed in the previous section to the general multivariate case with $n \geq 2$ firms. The default of an individual firm is driven by some idiosyncratic shock as well as other sectoral, industry, country-specific etc., or economy-wide shocks. These $m = \sum_{k=1}^{n} \binom{n}{k}$ shocks are governed by independent Poisson processes which are numbered consecutively. To indicate whether a non-firm-specific shock leads to a default of some given firm, we introduce a matrix $(a_{ij})_{n \times m}$, where $a_{ij} = 1$ if shock $j \in \{1, 2, \ldots, m\}$, modeled through the Poisson process N_j with intensity λ_j, leads to a default of firm
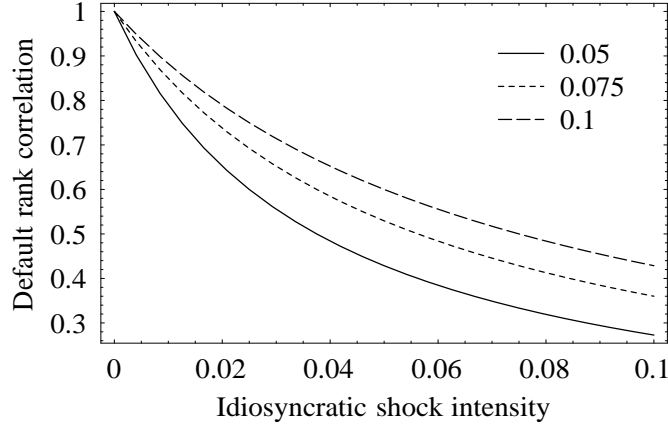
Figure 2: Rank default correlation as a function of idiosyncratic shock intensity, varying joint shock intensity.

$i \in \{1, 2, \ldots, n\}$, and $a_{ij} = 0$ otherwise. For example, for $n = 3$ firms a full specification of the model would involve

$$(a_{ij}) = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}.$$

Other specifications can be adapted to the case at hand. For example, if economy-wide shock events can be excluded, one would set $a_{i7} = 0$ for $i = 1, 2, 3$, which corresponds to bivariate dependence only.

According to this default specification, we get

$$\tau_i = \inf\{t \geq 0 : \sum_{k=1}^{m} a_{ik} N_k(t) > 0\},$$

meaning that firm i defaults with intensity $\sum_{k=1}^{m} a_{ik}\lambda_k$ and

$$s_i(t) = \exp\Big(-\sum_{k=1}^{m} a_{ik}\lambda_k t \Big).$$

Arguments similar to those leading to (1) yield the joint survival function

$$s(t_1, \ldots, t_n) = P[\tau_1 > t_1, \ldots, \tau_n > t_n]$$

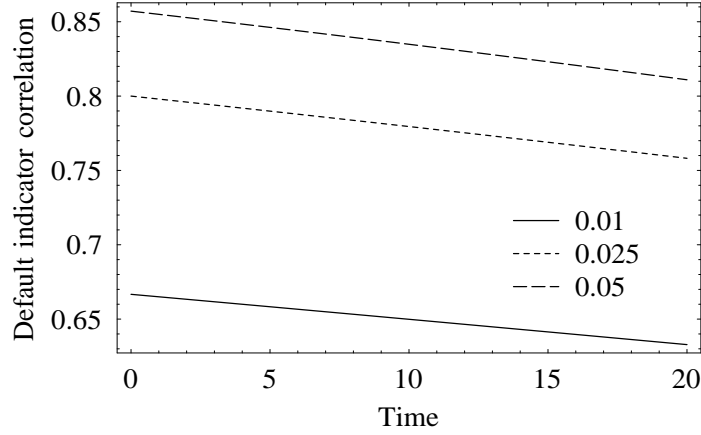$$= \exp\Big(-\sum_{k=1}^{m} \lambda_k \max(a_{1k}t_1, \ldots, a_{nk}t_n) \Big). \tag{6}$$

8

Figure 3: Default indicator correlation over time, varying joint shock intensity.

Joint default probabilities p and the associated copula K^τ can be obtained by standard arguments from s:

$$p(t_1, \ldots, t_n) = \sum_{i_1=1}^{2} \cdots \sum_{i_n=1}^{2} (-1)^{i_1+\cdots+i_n} s(v_{1i_1}, \ldots, v_{ni_n}), \qquad (7)$$

where $v_{j1} = t_j$ and $v_{j2} = 0$. The exponential survival copula associated with s can be found via $C^\tau(u_1, \ldots, u_n) = s(s_1^{-1}(u_1), \ldots, s_n^{-1}(u_n))$. Fixing some $i, j \in \{1, 2, \ldots, n\}$ with $i \neq j$, the two-dimensional marginal copula is given by

$$C^\tau(u_i, u_j) = C^\tau(1, \ldots, 1, u_i, 1, \ldots, 1, u_j, 1, \ldots, 1)$$

$$= \min(u_i^{1-\theta_i} u_j, u_i u_j^{1-\theta_j})$$

where we define, analogously to the bivariate case,

$$\theta_i = \frac{\sum_{k=1}^{m} a_{ik} a_{jk} \lambda_k}{\sum_{k=1}^{m} a_{ik} \lambda_k}, \qquad \theta_j = \frac{\sum_{k=1}^{m} a_{ik} a_{jk} \lambda_k}{\sum_{k=1}^{m} a_{jk} \lambda_k}$$

as the ratio of joint default intensity of firms i and j to default intensity of firm i or j, respectively. In analogy to (3), we can now compute Spearman's rank default time correlation matrix $(\rho_{ij}^S)_{n \times n}$ by

$$\rho_{ij}^S = \frac{3\theta_i \theta_j}{2\theta_i + 2\theta_j - \theta_i \theta_j}. \qquad (8)$$

9

2.3 Extensions

Let us briefly mention two extensions of the basic setup. Instead of assuming that shock incidents lead to immediate default events, we may suppose that shocks are not necessarily fatal. In the bivariate case, suppose an idiosyncratic shock leads to a default of firm i only with a pre-specified probability q_i. An economy-wide shock leads to a default of both firms with probability q_{11}, to a default of firm 1 only with probability q_{10}, and to a default of firm 2 only with probability q_{01}. Arguments very similar to those put forward in Section 2.1 lead us again to the exponential default time distribution

$$s(t, u) = e^{-\gamma_1 t - \gamma_2 u - \gamma(t \vee u)},$$

where $\gamma_1 = \lambda_1 q_1 + \lambda q_{10}$, $\gamma_2 = \lambda_2 q_2 + \lambda q_{01}$, and $\gamma = \lambda q_{11}$. Of course, with $q_1 = q_2 = q_{11} = 1$ we obtain model (1). The general case is analogous. Note, however, that in the non-fatal model interpretation the number of model parameters is quite high, which makes the model calibration very challenging.

Another extension concerns the variability of intensities over time. In our Poisson setup, with risk-neutrality the term structure of model-implied credit yield spreads is flat (the yield spread is in fact given by the constant intensity itself). However, in practice credit spreads vary often substantially over time. To capture these effects, in a first step the Poisson framework can be generalized to deterministically varying intensities i.e. to inhomogeneous Poisson shock arrivals. The intensity function may for example assumed to be piece-wise constant, which is a reasonably flexible approximation in certain applications. In the bivariate case, $s_i(t) = \exp(\int_0^t (\lambda_i(r) + \lambda(r)) dr)$ and

$$s(t, u) = e^{-\int_0^t \lambda_1(r) dr - \int_0^u \lambda_2(r) dr - \int_0^{t \vee u} \lambda(r) dr}.$$

In a second step, we can extend to general stochastic intensities. Such models would capture, in a realistic way, the stochastic variation in the term structure of credit spreads. One needs, however, a large and reliable data base to calibrate the parameters of such a stochastic intensity model. Now $s_i(t) = E[\exp(\int_0^t (\lambda_i(u) + \lambda(u)))]$ and

$$s(t, u) = E[e^{-\int_0^t \lambda_1(r) dr - \int_0^u \lambda_2(r) dr - \int_0^{t \vee u} \lambda(r) dr}].$$

For an extensive study of various correlation models with stochastic intensities we refer to Duffie & Singleton (1998).

3 Simulating Correlated Defaults

Based on the model of Section 2.2, we now discuss the flexible simulation of correlated default arrival times. Let us consider the following four-step algorithm, which generates default arrival times with an exponential dependence structure C^τ while allowing for arbitrary marginal default time distributions, which we take as given.

(1) Simulate an m-vector $(t_1, \ldots, t_m)$ of independent exponential shock arrival times with given parameter vector $(\lambda_1, \ldots, \lambda_m)$ where $\lambda_k > 0$. This is done by drawing, for $k \in \{1, 2, \ldots, m\}$, an independent standard uniform random variate U_k and setting

$$t_k = -\frac{1}{\lambda_k} \ln U_k.$$

Indeed, $P[t_k > T] = P[-\ln U_k / \lambda_k > T] = P[U_k \le e^{-\lambda_k T}] = e^{-\lambda_k T}$.

(2) Simulate an n-vector $(T_1, \ldots, T_n)$ of joint exponential default times by considering, for each firm $i \in \{1, 2, \ldots, n\}$, the minimum of the relevant shock arrival times:

$$T_i = \min\{t_k : 1 \le k \le m, a_{ik} = 1\}.$$

(3) Generate a sample $(v_1, \ldots, v_n)$ from the (survival) default time copula C^τ by setting, for $i \in \{1, 2, \ldots, n\}$,

$$v_i = s_i(T_i) = \exp\Big(-T_i \sum_{k=1}^m a_{ik} \lambda_k\Big).$$

(4) In order to generate an n-vector $Z = (Z_1, \ldots, Z_n)$ of correlated default arrival times with given marginal survival function q_i and exponential default dependence structure C^τ, set, for $i \in \{1, 2, \ldots, n\}$,

$$Z_i = q_i^{-1}(v_i)$$

provided that the inverse q_i^{-1} of q_i exists.

Let us emphasize that we can base our default time simulation on survival marginals q_i from a model completely different from the exponential shock

model of Section 2. For example, one may use a general stochastic intensity-based model or a structural model for q_i, or use some estimated survival function. To account for different types of idiosyncratic default risk, we can also choose different survival marginals q_i for different firms. The above algorithm then generates correlated default times Z_i with these given q_i and exponential dependence structure, i.e.

$$P[Z_1 > t_1, \ldots, Z_n > t_n] = C^\tau(q_1(t_1), \ldots, q_n(t_n)).$$

In the special case where $q_i = s_i$ for all i, the simulated default time vector Z is jointly exponentially distributed, i.e. Z has exponential marginals, an exponential copula C^τ, and its joint survival function is given by (6). In that case $Z_i = T_i$, and we have to perform steps 1 and 2 only.

From the methodology, this approach bears some interesting similarities to the approach followed by Schönbucher & Schubert (2001). In their approach the survival marginals are given by some stochastic intensity model, while one is free to choose the default copula joining marginals and joint default distribution (the copula is not directly prescribed by their single-name intensity model). In our approach, the copula of the default times is prescribed by the model structure, while one is free to choose a appropriate marginals. The exponential marginals would only be a particular choice, though a natural one.

4   Estimating Shock Intensities

In this section we discuss how the model parameters can be estimated. A first step consists of the specification of the shock impact matrix (a_{ij}). In a second step the relevant (at most $m = \sum_{k=1}^n \binom{n}{k}$) shock intensities have to be estimated. Here one may be interested in risk-neutral intensities for derivatives pricing purposes, or real-world objective intensities for risk management purposes.

In order to cope with the lack of appropriate data available for the estimation of higher order shock intensities, it might be reasonable to confine to a model specification (a_{ij}) with bivariate (trivariate) dependence only, i.e. a model where at most two (three) simultaneous defaults are possible. Another assumption to account for the lack of data is to set higher order shock intensities equal for shocks leading to more than one (two, three, etc.) defaults. The simplest model which yet accounts for default correlation in a credible

way would then rely on bivariate dependence with equal intensities for joint defaults of two firms (we refer to this as bivariate symmetric dependence). We would then have to estimate $n + 1$ intensities.

4.1 Risk Neutral Intensities

Liquid bond price or single-name credit default swap quotes can be used to back out risk-neutral survival probabilities $s_i'(t)$ of each firm i for some fixed horizon t (see, for example, Hull & White (2000) for the basic procedure). We then have the relations

$$-\frac{1}{t}\ln s_i'(t) = \sum_{k=1}^{m} a_{ik}\lambda_k, \quad i = 1, 2, \ldots, n. \tag{9}$$

If quotes for a range of different bond or swap maturities t are available, we can obtain a system of m equations in the m unknown intensities λ_k. If this system does not admit a unique solution, we can choose the λ_k via some least square minimization procedure, for example. If price observations are only available for some particular date t, the relations (9) alone are not sufficient to determine all intensities uniquely; higher-order survival probabilities are needed in addition. If prices of appropriate multi-name products (such as first-to-default swaps) are quoted, one can back out risk neutral higher order survival probabilities s'. Liquid quotes are rarely available, however. Bivariate joint survival probabilities s' can be derived from single-name default swap values if the protection selling swap counterparty is itself subject to default. In connection with suitable assumptions on the model structure (a_{ij}) and the associated shock intensities, we can then exploit the relation

$$-\ln s'(t_1, \ldots, t_n) = \sum_{k=1}^{m} \lambda_k \max(a_{1k}t_1, \ldots, a_{nk}t_n), \quad t_i \geq 0$$

in addition to (9) in order to determine the shock intensities λ_k.

4.2 Objective Intensities

Real world (objective) survival probabilities $s_i(t)$ of individual firms can be estimated from historical default data. In lack of a proprietary data base, one may use long-run average values of credit rating classes, which are commonly provided by credit rating agencies. Equation (9) is then used with s_i in place

of s_i'. Data on joint defaults is however much too sparse to obtain reliable statistical estimates for joint default probabilities. Here one has to resort to other sources of suitable information.

Suppose that Moody's has supplied a Diversity Score d for the n firm portfolio under consideration. This is most common if the underlying portfolio is the collateral pool of some Collateralized Debt Obligation (CDO). That is, the original portfolio of n correlated firms is considered to be equivalent to a portfolio of d independent firms with the same survival probability $q(t)$ but a notional of (n/d) times the original notional. The diversity score d is determined such that the first two moments of the distribution of the number of defaults in these two portfolios

$$\sum_{i=1}^{n} 1_{\{\tau_i > t\}} \quad \text{and} \quad \frac{n}{d} \sum_{i=1}^{d} 1_{\{\sigma_i > t\}} \tag{10}$$

are equal for some fixed horizon t, say one year. The $1_{\{\sigma_i > t\}}$ are iid Bernoulli with success probability $q(t)$. This yields the relations

$$nq(t) = \sum_{i=1}^{n} s_i(t)$$

$$\frac{n^2}{d} q(t)(1 - q(t)) = \sum_{i=1}^{n} \left(s_i(t)(1 - s_i(t)) + \sum_{j=1, j \neq i}^{n} (s_{ij}(t,t) - s_i(t)s_j(t)) \right).$$

Given the diversity score d, one can now use these equations to obtain an estimate of the sum of the pairwise joint survival probabilities s_{ij} for firms i and j. Assuming a model (a_{ij}) with bivariate symmetric dependence, we are then able to estimate the shock intensity leading to a simultaneous default of two firms.

Credit risk management software packages such as KMV's Portfolio Manager often provide pairwise default event correlations in the form of (5). Rating agencies provide default correlation matrices for industries. Nagpal & Bahar (2001) and Erturk (2000) estimate these correlations using historical default data from S&P. For estimates of rating category correlations from Moody's data, see Carty (1997). Such correlations allow to calibrate a model with (not necessarily symmetric) bivariate dependence via (5). More consistent would be the use of rank default correlations (3), for the mentioned reasons. Though admittedly hardly available, a rank default correlation matrix in connection with estimates on individual survival probabilities would allow to estimate a

model (a_{ij}) which is fully specified (i.e. a model beyond bivariate symmetric dependence).

5   Applications

5.1   Default Distribution

In credit risk management, the exponential default model can be used to measure the aggregated default risk associated with some portfolio of credits. The most comprehensive measure of that aggregated risk is the distribution of total losses due to defaults. In this section we consider such a loss distribution under the simplifying assumption that there is zero recovery in case of default. The default loss L_t at some fixed horizon t is then equal to the number of the defaulted firms $L_t = n - M_t$, where

$$M_t = \sum_{i=1}^{n} 1_{\{\tau_i > t\}}$$

is the number of firms which still operate at t. The distribution of M_t can be computed directly from the joint survival probabilities. By standard arguments we find

$$P[M_t = k] = \sum_{i=k}^{n} \binom{i}{k} (-1)^{i-k} \sum_{J \subset \{1,\ldots,n\}, |J|=i} P[\bigcap_{j \in J} \{\tau_j > t\}],$$

where the $|J|$-dimensional marginal joint survival probability $P[\bigcap_{j \in J} \{\tau_j > t\}]$ is directly available from (6) for all $J \subset \{1, \ldots, n\}$. This can be simplified if the firms in the portfolio are homogeneous and symmetric, i.e. if the default time vector is exchangeable:

$$(\tau_1, \ldots, \tau_n) \stackrel{d}{=} (\tau_{z(1)}, \ldots, \tau_{z(n)})$$

for any permutation $z(1), \ldots, z(n)$ of indices $(1, \ldots, n)$ (by $\stackrel{d}{=}$ we mean equality in distribution). Such a portfolio structure is a reasonable approximation for a well-diversified retail or SME portfolio (or at least sub-portfolios of such portfolios). We then denote by π_k the kth dimensional survival probability

$$\pi_k(t) = P[\tau_{i_1} > t, \ldots, \tau_{i_k} > t], \quad \{i_1, \ldots, i_k\} \subset \{1, \ldots, n\}, \quad 1 \leq k \leq n,$$
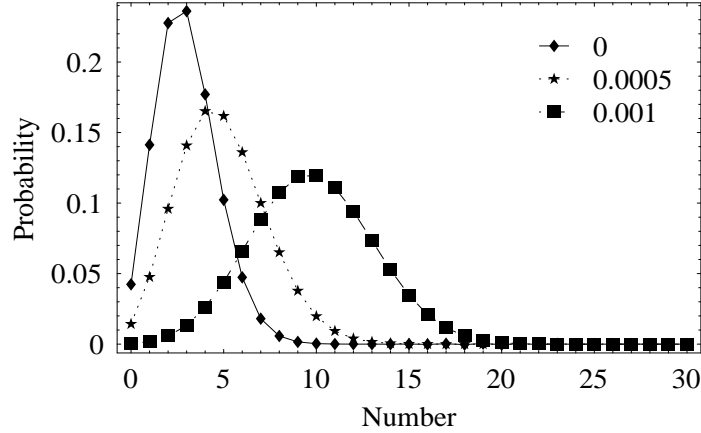
Figure 4: Distribution of the number of defaults, varying joint shock intensity.

which is the probability of survival of any arbitrarily selected subgroup of k firms by horizon t. We then have simply

$$P[M_t = k] = \sum_{i=0}^{n-k}(-1)^i \frac{n!}{i!k!(n-k-i)!}\pi_{k+i}(t).$$

To illustrate the effect of default correlation on the default distribution, let us consider an exponential model (a_{ij}) with symmetric bivariate dependence, where firms default individually with intensity λ, and shocks leading to a default of any two firms simultaneously arrive with intensity $\bar{\lambda}$. Then

$$\pi_k(t) = \exp(-(\lambda k + \bar{\lambda}[\binom{k}{2} + k(n-k)])t).$$

Figure 4 graphs the distribution of the number of defaults $P[L_t = k] = P[M_t = n - k]$ for a ten year time horizon with $n = 30$ firms, where the joint shock intensity $\bar{\lambda}$ is varied. We hold the one-year default probability fixed at one percent. It is clear that the expected number of defaults $E[L_t]$ is increasing in the systemic shock probability. This effect is shown in Figure 5. It is interesting that the variance of the number of defaults is hump-shaped: the variance is increasing in $\bar{\lambda}$ up to a certain point only; after that point it is decreasing in $\bar{\lambda}$. The joint shock intensity $\bar{\lambda}$ starts to dominate the idiosyncratic risk component λ after it has reached some critical level. Then a default is mainly due to a systemic shock and the probability of a systemic default of all firms increases,
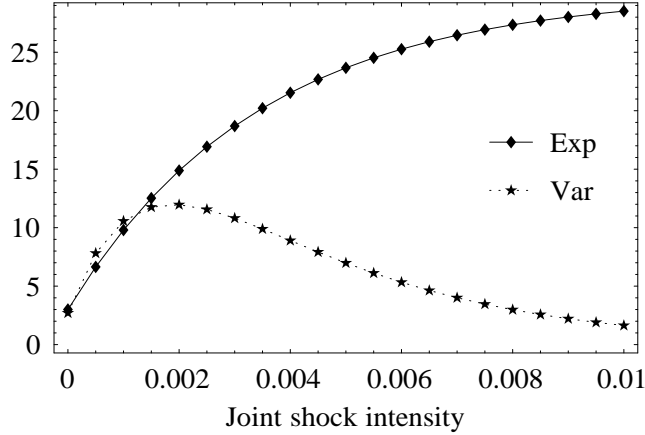
16

Figure 5: Expected defaults and variance of total defaults as a function of joint shock intensity.

while the probability of few defaults vanishes. That means that for increasing $\bar{\lambda}$ all the default probability mass will be shifted towards the right end point of the default distribution (see Figure 6 in that respect). For sufficiently high $\bar{\lambda}$ it is almost certain that all firms default before the horizon, and the variance vanishes.

5.2   First-to-Default Baskets

The exponential model can also be applied to the valuation of multi-name credit derivatives. Commonly traded are kth-to-default basket swaps, which pay some specified amount if at least k defaults in the reference bond basket occur before the maturity of the contract. As an example, let us consider a binary first-to-default swap, which involves the payment of one unit of account upon the *first* default in the reference portfolio in exchange for a periodic payment (the swap spread). The swap spread is paid up to the maturity T of the swap or the first default, whichever is first. The index set of the reference portfolio is $\{1, 2, \ldots, n\}$.

Let us denote by $\tau = \min_i(\tau_i)$ the first-to-default time. We have

$$P[\tau > t] = \exp\left(-t\sum_{k=1}^{m} \lambda_k \max(a_{1k}, \ldots, a_{nk})\right).$$

Assuming that investors are risk-neutral (i.e. P is some risk-neutral probabil-
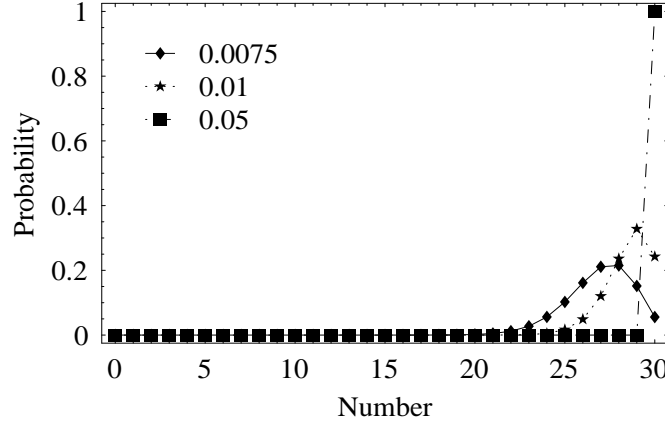
17

Figure 6: Distribution of the number of defaults, varying joint shock intensity.

ity), the value c of the contingent leg of the swap is at time zero given by

$$c = E[e^{-\int_0^\tau r_s ds} 1_{\{\tau \le T\}}]$$

where $(r_t)_{t \ge 0}$ is the riskless short rate. Supposing for simplicity that $r_t = r > 0$ for all t, we get

$$c = \int_0^\infty e^{-ru} 1_{\{u \le T\}} P[\tau \in du] = \Lambda \int_0^T e^{-(r+\Lambda)u} du = \frac{\Lambda}{r + \Lambda}(1 - e^{-(r+\Lambda)T})$$

where $\Lambda = \sum_{k=1}^m \lambda_k \max(a_{1k}, \dots, a_{nk})$. If the (constant) swap spread R is paid at dates $t_1 < t_2 < \dots < t_j = T$, then the fee leg has a value of

$$f = \sum_{i:t_i \le T} E[e^{-\int_0^{t_i} r_s ds} R 1_{\{\tau > t_i\}}]$$

$$= R \sum_{i:t_i \le T} e^{-(r+\Lambda)t_i}$$

where for the second line we invoke the assumption of constant short rates. We neglect any accrued swap spread here. The value of the fee leg paid by the protection buyer compensates the protection seller for paying one unit of account upon the first default in the reference portfolio. The swap spread R is therefore such that $c = f$ at inception of the contract $(t = 0)$.

For illustration, we assume that the reference portfolio has a bivariate symmetric dependence structure with idiosyncratic shock intensity λ and joint
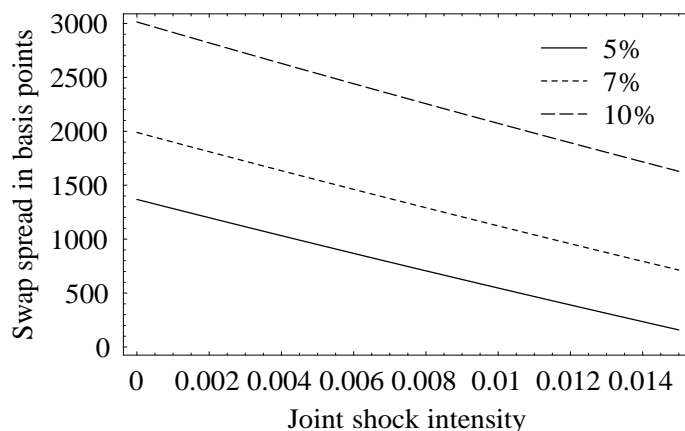
18

Figure 7: Swap spread as a function of the joint shock intensity, varying individual default probability.

shock intensity $\bar{\lambda}$. Symmetry and homogeneity are quite realistic for first-to-default baskets. Then $\Lambda = \lambda n + \bar{\lambda}\binom{n}{2}$. Supposing that $n = 5$, $r = 0$, $T = 1$, $t_1 = 0.5$, and $t_2 = 1$ (i.e. semi-annual coupon payments), in Figure 7 we plot the swap spread R as a function of the joint shock intensity $\bar{\lambda}$ for varying one-year default probabilities (while increasing $\bar{\lambda}$, we decrease λ such that the default probability remains constant). Since the likelihood of a payment by the protection seller is increasing in individual firms' default probabilities, the spread is increasing in these default probabilities. The spread is decreasing in $\bar{\lambda}$, which is also intuitively clear: for increasing (positive) default correlation the probability of multiple defaults increases and the degree of default protection provided by a first-to-default swap is diminished. With zero correlation the premium is at its maximum, because the likelihood of multiple defaults is at its minimum (given that negative correlation is excluded).

Finally note that we can use the distribution of the number of defaults derived in the last subsection to analyze general k-th-to-default baskets.

References

Carty, Lea (1997), Moody's rating migration and credit quality correlation 1920-1996. Special Comment, Moody's Investors Service.

Duffie, Darrell & Kenneth J. Singleton (1998), Simulating correlated defaults. Working Paper, GSB, Stanford University.

Duffie, Darrell & Nicolae Garleanu (2001), 'Risk and valuation of collateralized debt obligations', *Financial Analyst's Journal* **57**(1), 41–59.

Embrechts, Paul, Alexander J. McNeil & Daniel Straumann (2001), Correlation and dependence in risk management, *in* M.Dempster, ed., 'Risk management: value at risk and beyond', Cambridge University Press, Cambridge.

Erturk, Erkan (2000), 'Default correlation among investment grade borrowers', *Journal of Fixed Income* **9**(4), 55–60.

Giesecke, Kay (2002), Compensator based simulation of correlated defaults. Working Paper, Department of Economics, Humboldt-Universität zu Berlin.

Hull, John & Alan White (2000), 'Valuing credit default swaps i: No counterparty default risk', *Journal of Derivatives* **8**, 29–40.

Lindskog, Filip & Alexander McNeil (2001), Common poisson shock models: Applications to insurance and credit risk modeling. Working Paper, RiskLab, ETH Zürich.

Marshall, Albert W. & Ingram Olkin (1967), 'A multivariate exponential distribution', *Journal of the American Statistical Association* **62**, 30–44.

Nagpal, Krishan & Reza Bahar (2001), 'Measuring default correlation', *Risk* **14**(3), 129–132.

Nelsen, Roger (1999), *An Introduction to Copulas*, Springer-Verlag, New York.

Schönbucher, Philipp & Dirk Schubert (2001), Copula-dependent default risk in intensity models. Working Paper, Universität Bonn.

Zhou, Chunsheng (2001), 'An analysis of default correlation and multiple defaults', *Review of Financial Studies* **14**(2), 555–576.