Copula from the Limit of a Multivariate Binary Model

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1 Introduction

In modelling an arrival of credit event, exponential arrival time is always assumed in the industry. However, there are many different ways in constructing joint credit event time via copula technique [1] and there is no general guideline in selecting an appropriate one. In this article, we will show the multivariate exponential copula (see [2] and [3]) is the natural choice from the limiting viewpoint of a multivariate binary model. We will also see how the KMV parameters can be used to calibrate the model. In this exposition, we assume the reader is familiar with the limiting behavior from the multi-step binary model to the exponential time model.

2 Homogenous Case with 2 Entities

Suppose we have 2 entities with a one year default indicator denoted by (D_1, D_2). We have 4 possible states $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$ by the end of the year where 0 represents survival state and 1 represents default state. Let us define the following parameters which are commonly available from the KMV database,

$$\begin{aligned} p_{1\cdot} &= \text{1 year default prob for entity 1,} \\ p_{\cdot 1} &= \text{1 year default prob for entity 2,} \\ \rho &= \text{1 year default correlation.} \end{aligned}$$

The joint distribution for the bivariate binary (D_1, D_2) is thus uniquely defined by the following probabilities:

$$\begin{aligned} p_{11} &= p_{1\cdot}p_{\cdot 1} + \rho\sqrt{p_{1\cdot}(1-p_{1\cdot})p_{\cdot 1}(1-p_{\cdot 1})}, \\ p_{10} &= p_{1\cdot} - p_{11}, \\ p_{01} &= p_{\cdot 1} - p_{11}, \\ p_{00} &= 1 - p_{1\cdot} - p_{\cdot 1} + p_{11}. \end{aligned}$$

We are going to construct continuous default arrival times which are consistent with the above probabilities. Define λ_1, λ_2 and λ_{12} by the following equations

$$\begin{aligned} e^{-\lambda_1} &= 1 - p_{1\cdot}, \\ e^{-\lambda_2} &= 1 - p_{\cdot 1}, \\ e^{-\lambda_{12}} &= p_{00}. \end{aligned}$$

We shall make the following assumption for the KMV parameters $p_{1\cdot}$, $p_{\cdot 1}$ and ρ :

$$0 \le \rho \le \rho^* = \min\left[\sqrt{\frac{p_{1\cdot}(1-p_{\cdot 1})}{p_{\cdot 1}(1-p_{1\cdot})}}, \sqrt{\frac{p_{\cdot 1}(1-p_{1\cdot})}{p_{1\cdot}(1-p_{\cdot 1})}}\right]$$

which is necessary and sufficient for the following condition

$$\max(\lambda_1, \lambda_2) \le \lambda_{12} \le \lambda_1 + \lambda_2.$$

This assumption is not too restrictive since default correlations are always positive and ρ^* is the maximum correlation allowed by the two marginal binary distributions (see [4]).

2.1 Discretization

Let $p_{1\cdot}^{(n)}$, $p_{\cdot 1}^{(n)}$, $p_{11}^{(n)}$, $p_{10}^{(n)}$, $p_{01}^{(n)}$, $p_{00}^{(n)}$ be the probabilities which are analogous to p's except these probabilities are for the smaller intervals $]\frac{i-1}{n}, \frac{i}{n}]$ for $i \in \mathbb{Z}^+$. By conserving the 1 year marginal and joint probabilities, we have the following equations for $p^{(n)}$'s:

$$(1 - p_{1\cdot}^{(n)})^n = 1 - p_{1\cdot} = e^{-\lambda_1},$$

$$(1 - p_{\cdot 1}^{(n)})^n = 1 - p_{\cdot 1} = e^{-\lambda_2},$$

$$\left(p_{00}^{(n)}\right)^n = e^{-\lambda_{12}} = p_{00}.$$

Hence we have

$$\begin{aligned}
p_{00}^{(n)} &= e^{-\frac{\lambda_{12}}{n}}, \\
p_{01}^{(n)} &= (1-p_{1\cdot}^{(n)}) - p_{00}^{(n)} = e^{-\frac{\lambda_1}{n}} - e^{-\frac{\lambda_{12}}{n}}, \\
p_{10}^{(n)} &= (1-p_{\cdot 1}^{(n)}) - p_{00}^{(n)} = e^{-\frac{\lambda_2}{n}} - e^{-\frac{\lambda_{12}}{n}}, \\
p_{11}^{(n)} &= 1 - p_{00}^{(n)} - p_{01}^{(n)} - p_{10}^{(n)} = 1 - e^{-\frac{\lambda_1}{n}} - e^{-\frac{\lambda_2}{n}} + e^{-\frac{\lambda_{12}}{n}}.
\end{aligned}$$

Thanks to the condition $\max(\lambda_1, \lambda_2) \leq \lambda_{12} \leq \lambda_1 + \lambda_2$, it is not difficult to show $p_{i,j}^{(n)} \geq 0$. Let us define the correlation parameter in the smaller interval $]\frac{i}{n}, \frac{i+1}{n}]$ as

$$\rho^{(n)} = \frac{p_{11}^{(n)} - p_{1\cdot}^{(n)} p_{\cdot 1}^{(n)}}{\sqrt{p_{1\cdot}^{(n)} p_{\cdot 1}^{(n)} (1-p_{1\cdot}^{(n)})(1-p_{\cdot 1}^{(n)})}} = \frac{e^{\frac{\lambda_1+\lambda_2-\lambda_{12}}{n}} - 1}{\sqrt{(1-e^{-\frac{\lambda_1}{n}})(1-e^{-\frac{\lambda_2}{n}})}}.$$

It can be shown that the default correlation $\rho^{(n)}$ is increasing in n and

$$\lim_{n\to\infty} \rho^{(n)} = \frac{\lambda_1 + \lambda_2 - \lambda_{12}}{\sqrt{\lambda_1 \lambda_2}} \geq \rho.$$

Remark 1 *Suppose one approximates the bivariate default process by the bivariate binary model with time steps of length $\frac{1}{n}$. It is not enough to adapt the adjusted marginal probabilities $p_{1\cdot}^{(n)}$ and $p_{\cdot 1}^{(n)}$ only. The correlation parameter should also be adjusted to $\rho^{(n)}$ for conserving the one year correlation parameter.*

Remark 2 *$\rho^{(n)}$ is increasing in n.*

Proof: Let $\phi(\lambda; t) = \frac{\lambda}{1-e^{-\lambda t}}$ where $\lambda \geq 0$ and $t > 0$. It is easy to show ϕ is increasing in λ. Define

$$f(t) = \frac{e^{(\lambda_1+\lambda_2-\lambda_{12})t} - 1}{\sqrt{(1-e^{-\lambda_1 t})(1-e^{-\lambda_2 t})}}.$$

One can show $\frac{df}{dt} \leq 0$ if and only if $\phi(\lambda_1; t) + \phi(\lambda_2; t) \geq 2\phi(\lambda_1 + \lambda_2 - \lambda_{12}; t)$. Since $\max(\lambda_1, \lambda_2) \leq \lambda_{12}$, we have $\lambda_i \geq \lambda_1 + \lambda_2 - \lambda_{12}$ and $\phi(\lambda_i; t) \geq \phi(\lambda_1 + \lambda_2 - \lambda_{12}; t)$. Therefore f is decreasing in t. The result follows since $\rho^{(n)} = f(\frac{1}{n})$.

2.2 Convergent Limit

Let τ_i be the arrival time of default for entity i and $t_1, t_2 > 0$ be given. We can choose the unique k_i such that

$$\frac{k_i}{n} < t_i \leq \frac{k_i + 1}{n}.$$

From the discrete model, one can approximate the probability

$$\mathbb{P}\left(\tau_1 > t_1,\ \tau_2 > t_2\right) \doteqdot \begin{cases} \left(p_{00}^{(n)}\right)^{k_1} (1 - p_{\cdot 1}^{(n)})^{k_2 - k_1}, & k_2 > k_1 \\ \left(p_{00}^{(n)}\right)^{k_2} (1 - p_{1\cdot}^{(n)})^{k_1 - k_2}, & k_1 > k_2 \\ \left(p_{00}^{(n)}\right)^{k_1}, & k_1 = k_2 \end{cases}.$$

Taking the limit $n \to \infty$, we have

$$\mathbb{P}\left(\tau_1 > t_1,\ \tau_2 > t_2\right) = \begin{cases} e^{-\lambda_{12} t_1} e^{-\lambda_2 (t_2 - t_1)}, & t_1 < t_2 \\ e^{-\lambda_{12} t_2} e^{-\lambda_1 (t_1 - t_2)}, & t_1 > t_2 \\ e^{-\lambda_{12} t_1}, & t_1 = t_2 \end{cases}.$$

Thanks to the condition $\max(\lambda_1, \lambda_2) \le \lambda_{12} \le \lambda_1 + \lambda_2$ again, we can define

$$\begin{aligned} \gamma_1 &= \lambda_{12} - \lambda_2 \ge 0, \\ \gamma_2 &= \lambda_{12} - \lambda_1 \ge 0, \\ \gamma_{12} &= \lambda_1 + \lambda_2 - \lambda_{12} \ge 0, \end{aligned}$$

and the joint survival probability can be rewritten as

$$\mathbb{P}\left(\tau_1 > t_1,\ \tau_2 > t_2\right) = \exp\left(-\gamma_1 t_1 - \gamma_2 t_2 - \gamma_{12} \max(t_1, t_2)\right).$$

It is easy to check

$$\begin{aligned} \mathbb{P}\left(\tau_1 > t_1\right) &= \mathbb{P}\left(\tau_1 > t_1,\ \tau_2 > 0\right) = \exp(-\lambda_1 t_1) \\ \mathbb{P}\left(\tau_2 > t_2\right) &= \mathbb{P}\left(\tau_1 > 0,\ \tau_2 > t_2\right) = \exp(-\lambda_2 t_2) \end{aligned}$$

which are consistent with the "marginal" limiting analogy and

$$\begin{aligned} \mathbb{P}\left(\tau_1 > 1\right) &= e^{-\lambda_1} = 1 - p_{1\cdot} \\ \mathbb{P}\left(\tau_2 > 1\right) &= e^{-\lambda_2} = 1 - p_{\cdot 1} \\ \mathbb{P}\left(\tau_1 > 1,\ \tau_2 > 1\right) &= e^{-\lambda_{12}} = p_{00} \end{aligned}$$

which are consistent with the one-year KMV parameters. Suppose $\rho = 0$, we have

$$\mathbb{P}\left(\tau_1 > t_1,\ \tau_2 > t_2\right) = \exp\left(-\lambda_1 t_1 - \lambda_2 t_2\right) = \mathbb{P}\left(\tau_1 > t_1\right) \mathbb{P}(\tau_2 > t_2).$$

2.3 Another Viewpoint

Let T_i be exponentially distributed as γ_i for $i = 1, 2$, T_{12} be exponentially distributed as γ_{12} and all T's are independent. We define the default time of entity i by

$$\tau_i = \min\left(T_i, T_{12}\right)$$

where T_i is the arrival of an individual default event for entity i and T_{12} is the arrival of a joint default for both entity 1 and entity 2. Similar model has been proposed by Duffie and Singleton [3] and the joint survival function

$$\mathbb{P}(\tau_1 > t_1,\ \tau_2 > t_2) = \exp\left(-\gamma_1 t_1 - \gamma_2 t_2 - \gamma_{12}\max(t_1, t_2)\right)$$

has been found by Marshall and Olkin in [2]. This distribution is known as the bivariate exponential distribution which coincides with the limiting behavior of the bivariate model. In terms of λ_1, λ_2 and λ_{12} the density function and moment generating function can be written as

$$\begin{aligned} dF(t_1, t_2) &= \begin{cases} (\lambda_{12} - \lambda_1)\,\lambda_1 e^{-\lambda_1 t_1} e^{-(\lambda_{12}-\lambda_1)t_2}, & t_1 > t_2 \\ (\lambda_{12} - \lambda_2)\,\lambda_2 e^{-(\lambda_{12}-\lambda_2)t_1} e^{-\lambda_2 t_2}, & t_1 < t_2 \end{cases}, \\ \Delta F(t,t) &= (\lambda_1 + \lambda_2 - \lambda_{12})\, e^{-\lambda_{12} t}. \end{aligned}$$

and

$$M(u_1, u_2) = \frac{(\lambda_{12} + u_1 + u_2)\,\lambda_1\lambda_2 + u_1 u_2\,(\lambda_1 + \lambda_2 - \lambda_{12})}{(\lambda_{12} + u_1 + u_2)\,(\lambda_1 + u_1)\,(\lambda_2 + u_2)}.$$

From this, one can show

$$\begin{aligned} \mathbb{E}[\tau_i] &= \frac{1}{\lambda_i} \\ Var[\tau_i] &= \frac{1}{\lambda_i^2} \end{aligned}$$

and

$$Cov[\tau_1, \tau_2] = \frac{\lambda_1 + \lambda_2 - \lambda_{12}}{\lambda_1\lambda_2\lambda_{12}}.$$

Hence the correlation of the default times is

$$\rho_{\tau_1,\tau_2} = \frac{\lambda_1 + \lambda_2 - \lambda_{12}}{\lambda_{12}} = \frac{\sqrt{\lambda_1\lambda_2}}{\lambda_{12}} \lim_{n\to\infty} \rho^{(n)}.$$

This bridges the conversion from the instantaneous default correlation to the default time correlation or vice versa. With the joint distribution, we can calculate tractable conditional probabilities which may be helpful in conditional sampling for simulation purpose. We can also calculate various probabilities that a risk manager may be interested in.

Example 1 *Conditional default ordering*

$$\begin{aligned} \mathbb{P}(\tau_1 < \tau_2 \mid \tau_1, \tau_2 \in]s,t]) &= \frac{1 - \frac{\lambda_2}{\lambda_{12}} - e^{-\lambda_2(t-s)} + \frac{\lambda_2}{\lambda_{12}} e^{-\lambda_{12}(t-s)}}{1 - e^{-\lambda_1(t-s)} - e^{-\lambda_2(t-s)} + e^{-\lambda_{12}(t-s)}}, \\ \mathbb{P}(\tau_1 > \tau_2 \mid \tau_1, \tau_2 \in]s,t]) &= \frac{1 - \frac{\lambda_1}{\lambda_{12}} - e^{-\lambda_1(t-s)} + \frac{\lambda_1}{\lambda_{12}} e^{-\lambda_{12}(t-s)}}{1 - e^{-\lambda_1(t-s)} - e^{-\lambda_2(t-s)} + e^{-\lambda_{12}(t-s)}}, \\ \mathbb{P}(\tau_1 = \tau_2 \mid \tau_1, \tau_2 \in]s,t]) &= \frac{\left(1 - e^{-\lambda_{12}(t-s)}\right)}{1 - e^{-\lambda_1(t-s)} - e^{-\lambda_2(t-s)} + e^{-\lambda_{12}(t-s)}}\rho_{\tau_1,\tau_2}. \end{aligned}$$

Suppose $s = 0$, $t = \infty$ and $\rho = 0$, we have the following classical results

$$\begin{aligned}
\rho_{\tau_1,\tau_2} &= 0, \\
\mathbb{P}(\tau_1 < \tau_2) &= \frac{\lambda_1}{\lambda_1 + \lambda_2}, \\
\mathbb{P}(\tau_1 > \tau_2) &= \frac{\lambda_2}{\lambda_1 + \lambda_2}.
\end{aligned}$$

3 Multivariate Extension

To fix ideas, we consider first an extension to an economy with 3 entities. Suppose $T_i, T_{ij}, T_{ijk},\ i, j, k \in \{1, 2, 3\},\ i < j < k$ are independent exponential random times with intensities γ_i, γ_{ij} and γ_{ijk}. Let us define

$$\begin{aligned}
T_i &= \text{arrival of default event for } i \text{ only}, \\
T_{ij} &= \text{arrival of joint default event for } i, j \text{ only}, \\
T_{ijk} &= \text{arrival of joint default event for } i, j, k \text{ only}.
\end{aligned}$$

Then the default time for entity i is

$$\tau_i = \min_{j \neq i,\ k \neq j} (T_i, T_{ij}, T_{ijk})$$

where $T_{ij} \triangleq T_{ji}$ if $i > j$. With these assumptions, the joint survival function can be written as

$$\mathbb{P}(\tau_1 > t_1,\ \tau_2 > t_2,\ \tau_3 > t_3) = \exp\left[-\sum_{i=1}^{3} \gamma_i t_i - \sum_{i<j} \gamma_{ij} \max(t_i, t_j) - \gamma_{123} \max(t_1, t_2, t_3)\right].$$

It is clear that similar arguments (see [2]) yield the n-dimensional exponential survival function

$$\begin{aligned}
\mathbb{P}(\tau_1 > t_1,\ \ldots,\ \tau_n > t_n) &= \exp\left[-\sum_{i=1}^{n} \gamma_i t_i - \sum_{i<j} \gamma_{ij} \max(t_i, t_j)\right. \\
&\quad - \sum_{i<j<k} \gamma_{ijk} \max(t_i, t_j, t_k) \\
&\quad \left. - \ldots - \gamma_{12\ldots n} \max(t_1, t_2, \ldots, t_n)\right]
\end{aligned}$$

We will show how to calibrate γ_i and γ_{ij} in the next section. However, other higher order parameters cannot be specified by the KMV parameters anymore. We may assume them to be 0 which has the meaning that not more than 2 defaults can happen at the same time. With this assumption we have,

$$\mathbb{P}\left(\tau_1 > t_1,\ \dots,\ \tau_n > t_n\right) = \exp\left(-\sum_{i=1}^{n}\gamma_i t_i - \sum_{i<j}\gamma_{ij}\max(t_i, t_j)\right)$$

which specifies the choice of a copula. It has also been shown that the closed form expression of the moment generating function exists (see [2]).

4 Calibration

Let ρ_{ij} be the one year default correlation for entity i and entity j and p_i be the one year marginal default probability for entity i. We can calculate

$$\lambda_i = -\ln(1-p_i),$$

$$\begin{aligned}\lambda_{ij} &= -\ln((1-p_i)(1-p_j) + \rho_{ij}\sqrt{p_i p_j (1-p_i)(1-p_j)}) \\ &= \lambda_i + \lambda_j - \ln\left(1 + \rho_{ij}\sqrt{\frac{p_i p_j}{(1-p_i)(1-p_j)}}\right).\end{aligned}$$

Similar to the bivariate case, we can define

$$\gamma_{ij} = \lambda_i + \lambda_j - \lambda_{ij} = \ln\left(1 + \rho_{ij}\sqrt{\frac{p_i p_j}{(1-p_i)(1-p_j)}}\right),$$

$$\gamma_i = \lambda_i - \sum_{j\neq i}\gamma_{ij}.$$

An additional condition for the KMV parameters to work in the multi-entity case is $\gamma_i \geq 0$ for all i. In general, p_i and $\rho_{ij}\sqrt{\frac{p_i p_j}{(1-p_i)(1-p_j)}}$ are small because p's are small and we can approximate γ_i and γ_{ij} by

$$\begin{aligned}\gamma_{ij} &\sim \rho_{ij}\sqrt{\frac{p_i p_j}{(1-p_i)(1-p_j)}}, \\ \gamma_i &\sim p_i - \sum_{j\neq i}\rho_{ij}\sqrt{\frac{p_i p_j}{(1-p_i)(1-p_j)}}.\end{aligned}$$

With these γ's, we can define the copula function or simulate according to the model in [3].

5 Stress Test Application

It is not difficult to find a stress scenario for the KMV default correlation in the bivariate case since we have

$$0 \le \rho \le \rho_{12}^{*} = \min \left[\sqrt{\frac{p_1(1-p_2)}{p_2(1-p_1)}}, \sqrt{\frac{p_2(1-p_1)}{p_1(1-p_2)}} \right].$$

Therefore we can always take the worse possible scenario ρ_{12}^{*} for stress test purpose. However, the problem becomes more tedious in the multivariate case. If we are using the normal copula approach, we are dealing with a set of non-linear constraints to ensure the positive semi-definiteness of the covariance matrix for the normal copula (see [4]). With the above approach, we can obtain a stress scenario in an easier fashion:

- Re-enumerate the default correlation parameters $\{\rho_{ij}\}_{i<j}$ by $\{\rho_k\}$, $k = 1, \dots, \frac{n(n-1)}{2}$.
- According to the notional amount or some risk manager's intuition, we can assign weight $w \in \mathbb{R}_{+}^{n(n-1)/2}$ for ρ.
- Solve the following mathematical programming problem

  $$\max \ w^T \rho$$

  such that

  $$\gamma_i \ge 0, \ i = 1, \dots, n$$

  $$0 \le \rho_{ij} \le \rho_{ij}^{*}, \ i < j, \ i = 1, \dots, n-1.$$

  To simplify the problem, we use the approximation $\gamma_i \sim p_i - \sum_{j \neq i} \rho_{ij} \sqrt{\frac{p_i p_j}{(1-p_i)(1-p_j)}}$ which leads to the following linear programming problem

  $$\max \ w^T \rho$$

  such that

  $$\sum_{j \neq i} \sqrt{\frac{p_j}{1-p_j}} \rho_{ij} \le \sqrt{p_i(1-p_i)}, \ i = 1, \dots, n$$

  $$0 \le \rho_{ij} \le \rho_{ij}^{*}, \ i < j, \ i = 1, \dots, n-1.$$

- The extremum point ρ can be used as a stress scenario subject to w.

Stress scenarios using $w = e_k, \ k = 1, \dots, \frac{n(n-1)}{2}$ (e's are the canonical vectors in $\mathbb{R}^{n(n-1)/2}$) can be applied successively.

6 Pricing Basket Instruments

Although the replicating property for a single entity default model is proved in [5], there is no such result available in the multi-entity default model yet. Until then, any "risk neutral pricing" in the multi-entity default model may have a large model risk because of the unknown "implied" correlations problem. We are going to propose a hybrid pricing method using the KMV parameters under the physical measure.

Let ω be a realization of the arrival of defaults according to the simulation model suggested in [3]. In particular, ω can be characterized by a vector $\overrightarrow{t}_\omega$ in $\mathbb{R}^n$ where $\overrightarrow{t}_\omega = (\tau_1(\omega), \ldots, \tau_n(\omega))$. Suppose an agent wants to price a "basket" security Y and there exists market tradable securities $\{X_{ik}\}_{k=1}^{m_i}$ which are contingent to the default of entity i only. These instruments X's can be single credit default option or corporate bond in which closed form expression for its expected value is available given the KMV edfs. By generating Monte Carlo scenarios under the physical measure, we can fit the following least square model

$$Y = \alpha + \sum_{i=1}^{n} \sum_{k=1}^{m_i} \beta_{ik}(X_{ik} - \overline{x}_{ik}) + \varepsilon$$

where α, β's are the regression parameters, $\overline{x}_{ik}$ is the known expected value of X_{ik} under the physical measure and ε is the residual risk. The fitted parameter $\widehat{\alpha}$ is the MC estimate of the expected value of Y with control variates X's (the bias can be eliminated by the splitting/batching algorithm). Suppose the agent has a preference based rule in assigning a risk premium $\pi(\varepsilon)$ on the unhedgeable risk ε through the simulation, then Y can be priced as

$$P_Y = \widehat{\alpha} + \sum_{i=1}^{n} \sum_{k=1}^{m_i} \widehat{\beta}_{ik}(P_{ik} - \overline{x}_{ik}) + \pi(\varepsilon),$$

where P_{ik} is the price of X_{ik} under the single entity "risk neutral" default model (or observable market price) and $\widehat{\beta}$ is the optimal hedge ratio.

References

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