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## **Correlation of Default Events Some New Tools**

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## Abstract

Estimating and pricing correlation of credit deterioration is difficult, but can be handled with standard notions of correlation. The same however is not true for default events. The notion of correlation that one needs to use in dealing with credit default is fundamentally different from the notion of correlation that is useful in dealing with credit deterioration in credit portfolios or instruments. This paper provides a model of credit correlation for credit default events that describes how one can calculate (dynamic) correlations between two series of default events. “Time” is a very important factor, but default data are not measured in *equal* time intervals at all. Empirical investigation of such data sets needs a new type of model for which we obtain the distribution theory of the implied statistics. The model provided here can also be used as a way to forecast default events in one credit using the default events of other credits.

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# 1 Introduction

Credit default traders are well aware of the distinction between credit deterioration and default. There exists a retail demand for instruments of *credit deterioration*, but the interbank market prefers to trade *credit default*. In this paper we provide a model of credit correlation for credit default events.

The notion of correlation that one needs to use in dealing with credit default is fundamentally different from the notion of correlation useful in dealing with credit deterioration in credit portfolios or instruments. Credit deterioration can basically be modelled as a standard stochastic process that can be represented by a Wiener process driven (vector) diffusion process where major macroeconomic variables such as, state of the business cycle, credit tightness and (the slope of ) yield curve play an important role. Hazard models fit this framework well. Markov models of credit migration may be interpreted as dealing with credit default events but are relevant for credit deterioration also. A good survey of these models can be found in Arvanitis and Gregory[2001].

On the other hand, defaults are often not just due to lingering firm-level or macroeconomic problems that can be tracked by some hazard model or Markov Chain transition probabilities. Often default is caused by independent events that adversely affects institutions and causes major illiquidity problems. For example, it is well known in banking that insolvency can persist very long time, yet illiquidity can be fatal even for solvent institutions. Illiquidity on the other hand is often caused by “unpredictable” events. For an interesting paper dealing with the issue of credit along these lines is Madan and Unal[2001].

Estimating and pricing correlation of credit deterioration is difficult, but can be handled with standard notions of correlation. The same however is not true for *default events*. Pricing and risk managing default portfolios may require new approaches. In this paper we deal with one aspect of this problem. In particular we would like to discuss how one can calculate (dynamic) correlations between two series of default events.

Consider a credit portfolio associated with only two credits. Suppose we observe a sequence of events randomly distributed over a time axis associated with these credits. Call them *credit events*. For example, let these events be ratings changes or defaults. Then, a crucial

issue from the point of view of credit portfolio management and pricing is the extend of which the occurrence of the series of credit events in one credit may get *transmitted* to the other credit, and hence lead to correlation of credit events.

Such problems cannot be dealt with using “standard” statistical tools. First of all, the data one has consist of a series of *dates* during which the events have occurred. One cannot simply run regressions, or calculate standard correlations utilizing such data. Standard assumptions of linear time series would not be satisfied and the standard distribution theory of the statistics would not apply.

Secondly, although “time” is a very important factor, the data that one has is not measured in *equal* time intervals at all. Empirical investigation of such data sets needs (i) formulating an appropriate model and (ii) obtaining the distribution theory of the implied statistics.

This paper provides a framework where one can investigate the way a *defaults events* can be correlated. The model provided here can also be used as a may to forecast default events in one credit using the default events of other credits. The model explicitly ignores the issue of credit deterioration.

## 2 Some Heuristics

Why would correlation of defaults be of interest to market participants? The answer to this question is simple. Credit portfolios can be *decomposed* into portfolios of *credit deterioration* and portfolios of *default risk*. Retail clients may be interested in trading credit deterioration, but interbank players prefer to trade primarily default risk. This is because at the end, the more relevant risk in interbank contracts is not the one that relates to credit deterioration but it is the one that causes defaults. And here the correlation plays a crucial role as the example below will demonstrate.

Consider a regulator that is concerned with the capital adequacy due to credit risk. Given the proper measures of default risk and the correlation of defaults, the additional information that credit deterioration data may contain is of secondary relevance to this regulator. After all, for the problem at hand, banks need to put aside capital to cover *defaults*. The mark to market losses due to credit deterioration

over and above default risk is similar to market risk. The major risk of credit portfolios is default and capital need to be put aside to cover this.

A case in point is the following problem. Suppose a bank buys protection against an A-rated credit, from a dealer that is rated B. Would this protection *lower* the default risk of the credit portfolio and hence decrease the amount of capital that need to be put aside?

At the first sight the operation seems to be useless since, the protection is bought from a dealer that is rated *lower* than the institution insurance is sought on. How can this reduce credit risk? In fact, it does.

The reason is the following. Defaults are random events and buying default insurance from institutions with *uncorrelated* default events will dramatically reduce the credit risk even when the protection seller has a much higher default probability. For example, consider two institutions with ratings *A* and *B* respectively. Let the probability of defaults be given by:

$$P(A) = P(A \text{ defaults during interval } \Delta)$$

$$P(B) = P(B \text{ defaults during interval } \Delta)$$

And, let the *joint* probability of default equal:

$$P(B \bigcup A) = P(A \text{ and } B \text{ default during interval } \Delta).$$

If the two institutions have independent default events then we would have

$$P(B \bigcup A) = P(A)P(B).$$

We can calculate the following. According to Moody's historical default tables we have the approximations for these unconditional probabilities:

$$P(A) = 0.035$$

$$P(B) = 0.31$$

Under these conditions, if the two institutions have independent default events, then buying default protection for the *A*-rated issuer from the institution with *B*-rating, will reduce the default risk by more than 60%:

$$P(B \cup A) = (.035)(.31) = .011$$

This means that capital put aside due to credit risk can be lowered by a similar proportion.

Yet, if the default events are highly correlated, then buying protection from the  $B$ -rated bank will hardly lower the credit risk. As a matter of fact at the extreme case where the correlation is close to one, we will have:

$$P(B \cup A) \cong P(A) \cong P(B).$$

Clearly, the correlation of default events should play a crucial role in capital adequacy.

In this paper we discuss (1) A way to *test* the correlation of two default events and, (2) A way to *calculate* the correlation at a point in time and dynamically, over time.

### 3 Framework

Let a discrete (vector) time process  $\{Y_t^j : t = 1, 2, \dots\}$  represent the real economic activity that determines credit deterioration and/or default at credit  $j$  at time  $t$ . For example, one can think of  $Y_t^j$  as financing costs, productivity changes or earning shocks that affect the performance of  $j$ 'th credit.

Next assume that a series of purely default-related events occur during random time periods denoted by  $\tau_t^j$  for the  $j$ 'th credit. We are not interested in the size or type of these events. The only information we have is their time of onset,  $\tau_t^j$ .

In such settings market is, in general, interested in two possibly correlated but different risks; namely *default risk* and *credit deterioration risk*. Although the exact definition of such phenomena is difficult to precise explicitly and involves some subjective judgments, a practitioner needs to differentiate and measure these two risks. In this paper we do not deal with *how* this is done, we just assume that default events have been defined.

The properties of the two event-series  $\{\tau_{i_j}^j : i_j = 1, \dots, N_j\}$  is worth reviewing. First of all, these series form a sequence of integer valued

random variables, spread randomly over a time axis. Secondly, the  $\tau_{i_j}^j : i_j = 1, \dots, N_j$  are measurable with respect to information available at time  $t$ . This is an implication of their definition. Hence they are Markov times. Finally, and this is our main concern, they are potentially correlated over time and across credits.

In this paper our interest is in this very last point. The  $\{\tau_{i_j}^j : i_j = 1, \dots, N_j, j = 1, 2\}$  represent the times of onset of credit events. We would like to know, to what extend, if at all, the series of events of credit 1, “feed” into series of events associated with 2. Also of interest, is if one can define a sense in which one series of credit events *lead* the other and then calculate such potential “leads”. In other words we would like to measure whether or not the two credit events are “correlated” in some ways. And we would like a model where one can measure dynamic correlations between these events.

## 4 Peculiarities of the correlation

Note that such questions cannot be investigated in a straightforward manner using, for example, simple correlations. First of all, the random variables  $\{\tau_{i_j}^j : i_j = 1, \dots, N_j, j = 1, 2\}$  are *increasing* for constant  $j$ . Hence one cannot calculate constant means and then calculate the correlations around this mean. Even if one can do that, it is not clear what the distribution of such a correlation would be.

Also, note that regression-like methods, or standard correlations may be very difficult to apply simply because the two  $\{\tau_{i_j}^j : i_j = 1, \dots, N_j, j = 1, 2\}$  may be available with different *number of observations*. These are periods of onset of random events and both the time interval that passes between successive  $\tau_{i_j}^j$ , fixed  $j$ , and the number of such events are random and may depend on the credit.

For example, one credit may be *AAA* and may lead to very few default events and its  $N_j$  may be very small, while the other credit may be *A* and may come with a large number of such default events. But, these two credits may *still* be positively and strongly correlated in terms of occurrence of default. It is not unimaginable to have two such credits correlated negatively.

Finally, it is quite possible that two series of default-events that belong to two *BB* credits are uncorrelated although many default events

are observed.

All this is not very surprising given that, the two series of events are in fact *point processes* that require special tools to handle. In the next section we discuss these tools.

## 5 A correlation measure

The  $\{\tau_{i_j}^j : i_j = 1, \dots, N_j, j = 1, 2\}$  can be regarded as a bivariate point process. First, we would like to see if the two  $\tau_{i_j}$  are statistically correlated for  $j = 1, 2$ . Second, in case they are, we would like to know which sequence “leads”. And finally, we would like to measure this correlation. To obtain a way one can test the extent of correlation between two point processes with possible unequal number of observations we use a framework that was pioneered by Cox(1955).

First we define the *smallest recurrence times backwards* between the first sequence with respect to the second sequence as:

$$\delta_{12}(i_2) = \inf_{i_1}[\tau_{i_1}^2 - \tau_{i_2}^1, 0] \quad \forall i_2 = 1, \dots, N_1 \quad (1)$$

Under these conditions, the process  $\{\delta_{12}(i_2)\}$  measures the distance backwards between the time of occurrence of an event in one credit with respect to the *closest* event in the other credit. Note that if there is no systematic lead-lag relations between the two occurrence times, then the  $\delta_{12}(i_2)$  will have the mean of a random variable that slices the time axis completely randomly.

On the other hand if, the default events in one credit, lead to default events in the other, then the  $\{\delta_{12}(i_2)\}$  will be “small”. In the extreme case when the two series of credit events coincide exactly in terms of their timing, the  $\{\delta_{12}(i_2)\}$  will be zero.

Thus, to obtain a test of any potential “correlation” between the two occurrence times, first the mean of  $\{\delta_{12}(i_2)\}$  should be calculated and then, the asymptotic distribution of this mean should be obtained.

Let the (sample) mean of the backward recurrence times be:

$$\hat{\delta} = \frac{1}{N_2} \sum_{i_2=1}^{N_2} \delta_{12}(i_2) \quad (2)$$



In order to evaluate the accuracy of this number, we need to obtain the asymptotic distribution of this quantity. To do this, assume that we have a sample of  $\tau_{i_j}$  during a time interval  $t \in [0, T]$ . Define the indicator functions for each  $i_j$ :

$$U(t, \tau_{i_j}^T) = \begin{cases} 1 & \text{if } 0 < t < \tau_{i_j}^T \\ 0 & \text{otherwise.} \end{cases} \quad (3)$$

Note that these functions assign a value of one to all time periods preceding a particular credit event. Accordingly, periods closer to the end of the sample will have a smaller number of  $U(.)$ 's equaling one. For such time periods, most  $U(.)$ 's will assume a value of zero.

Next, define the empirical density function

$$\hat{f}(\tau = s) = \frac{1}{T} \sum_{i=1}^{N_j} U(t, \tau_{i_j}^T) \quad (4)$$

According to the law of large numbers as  $T \rightarrow \infty$  the empirical density function will converge to the true probabilities associated with the events  $\tau_{i_j}^T = t$  for a given  $j$ .

A heuristic description of the density may be in order. The function in 4 basically assigns some weights to time periods  $0 < t < T$ . For example, if during this interval only a single  $\tau_{i_j}^T$  is observed, and if the observed value is  $T$ , then the function will assign a weight of  $\frac{1}{T}$  to all  $0 < t < T$ . If a very high number of  $\tau_{i_j}^T$  are observed, function 4 will assign a steadily declining number of weights to these periods.

Using equation 4 we can calculate the (empirical) expected value and the (empirical) variance of a typical  $\delta_{12}(i_2)$  with  $i_2$  fixed. These calculations yield the following asymptotic distribution:

$$\hat{\delta} \rightarrow \mathcal{N} \left[ \sum_{i_1=1}^{N_1} \frac{\Delta_{i_1}^2}{2T}, \quad \frac{1}{N_1} \left\{ \frac{1}{3T} \sum_{i_1=1}^{N_1} \Delta_{i_1}^3 - \frac{1}{4T^2} \left( \sum_{i_1=1}^{N_1} \Delta_{i_1}^2 \right)^2 \right\} \right] \quad (5)$$

where  $\Delta_{i_1}$  are the inter-arrival times for the exogenously given point process:

$$\Delta_{i_1} = \tau_{i_1} - \tau_{i_1-1}$$

and where the convergence is in distribution. The distribution in 5 can then be used to form accuracy tests of the proposed coreelation and lead-lag measures.

## 5.1 Results

In this paper case our purpose is to investigate whether the estimated mean recurrence time backwards between the two point processes is smaller in some statistical sense, than the recurrence times of two independent point processes. Indeed if the occurrence of credit events associated with one credit get transmitted to the other credit, we expect the  $\{\tau_{i_j, j=1,2}\}$  to occur in a “correlated” function. The test statistic proposed above will estimate this correlation

To calculate a test statistic, in our case, we need the  $\{\Delta_{i_1}\}$  and the  $\{\delta_{12}(i_2)\}$ .

### 5.1.1 The data

The empirical results provided here deal with three sovereign credits; namely Brazil, Argentina and Japan. At the outset we expect the default events observed for Brazil and Argentina to be correlated, whereas the default events for Japan be uncorrelated with the other two series. These data are shown in Tables 1-3.

From these tables we generate the  $\tau_t^j$  series, which are shown in Table 4. The estimates are shown in Table 6. We see that the series of credit-events between Argentina and Brazil is correlated at 5% level, whereas the credit-events between Japan and Argentina are statistically uncorrelated.

## 6 Estimating correlations for default events

Again, we assume that there are  $n$  credits of interest indexed by  $j$ . As discussed above, for each credit a series of default-events are observed. These events may be characterized by their *types* (marks) and by their occurrence time, We let  $\tau_i^j$  denote the time of occurrence of the  $i$ 'th event in  $j$ 'th credit. We assume that there are a finite number of event types (marks) denoted by  $\{m_k, k = 1, \dots, K\}$ . We assume that such events can occur in any order at any time.

Thus, an event is characterized by the point  $(m_k, \tau_i^j)$ . These sequence of points defines a marked point process for the  $j$ 'th economy. As usual we let the counting measures of these point processes be given by:

$N_t^j$  = Number of events observed up to time  $t$  for  $j$ th credit

Note that this counting process measures the occurrence of events of different types until time  $t$ . It is not a counting process of homogenous events only. When the time period under consideration is long and when a large number of identical events are observed one can very well disaggregate the  $N_t^j$  into counting processes for identical events.<sup>1</sup> This gives the multivariate counting process  $\{N_t^1, \dots, N_t^n\}$ . We let the  $I_t^j$  denote the information set (sigma-field) generated by the events up to time  $t$  for  $j$ 'th credit.

We have the following representation theorem for point processes:

*Theorem: Let  $N_t^j$  and  $I_t^j$  be defined as above. Then the  $\{N_t^j\}$  admits the representation:*

$$dN_t^j = \lambda_t^j dt - \sum_{k \neq j} H_k [dN_t^k - \lambda_t^k dt] + dM_t^j \quad i = 1, \dots, n$$

where the  $\lambda_t^j$  are the intensities defined as:

$$E[dN_t^j | I_t^j] = \lambda_t^j dt$$

and where  $\{M_t^j\}$  is a right-continuous Martingale measurable with respect to  $\{I_t^j\}$ . [Proof: Bremaud(1981) pages 63-70.]

This theorem gives a characterization similar to the Wald representation widely used as a basis for Vector Autoregressions. Heuristically, it states that the deviations around the conditional expectations of a counting process is a Martingale. This in turn means that written in differential form, the  $dM_t^j$  have zero expectation given the information set  $I_t^j$ .

The representation given in the theorem can be used as a way of transforming a time series of a series of non-homogenous events into a

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<sup>1</sup>For example one could define point processes for ratings changes, or major earnings announcements etc.. separately.

”standard” time series. This transformation then permits estimating any dynamic correlations between sequences of events via standard statistical tools. We intend to use this Martingale Representation in estimating the (causal) relations between the point processes  $\{N_t^j\}$ . This of course necessitates some further restriction on the  $\lambda_t^j$ .

## 6.1 Estimation

In order to estimate the representation given in the Theorem above, we first write it in the following restricted form:

$$\begin{bmatrix} dN_t^1 \\ \dots \\ dN_t^n \end{bmatrix} = \begin{bmatrix} \lambda_t^1 dt \\ \dots \\ \lambda_t^n dt \end{bmatrix} + \begin{bmatrix} h_{11} & \dots & h_{1n} \\ \dots & \dots & \dots \\ h_{n1} & \dots & h_{nn} \end{bmatrix} \begin{bmatrix} dN_t^1 - \lambda_t^1 dt \\ \dots \\ dN_t^n - \lambda_t^n dt \end{bmatrix} + \begin{bmatrix} dM_t^1 \\ \dots \\ dM_t^n \end{bmatrix}$$

where the diagonal elements of  $\{h_{i,j}, i = 1 \dots n, j = 1 \dots n\}$  are assumed to be zero.

The estimation of the system of equations shown above can be done in two steps. First one estimates the intensities  $\{\lambda_t^j\}$ . Then one proceeds with the estimation of the system.

The intensities,  $\lambda_t^j$  are measurable with respect to the information set  $I_t^j$ . This means that they can be estimated separately, from the past observations of each point process history. An asymptotically *consistent* estimate will be given by:

$$\hat{\lambda}_t^j = \frac{1}{t} N_t^j$$

These estimates are then put in the system above and the  $h_{ij}$  are estimated using standard vector autoregression techniques.

## 7 Implementation

In order to estimate the correlations implied by the martingale representation shown above we need just two sets of information. First we have to obtain the data on  $\{N_t^j\}$  or equivalently on  $\tau_t^j$ .

Second, we need to estimate the  $\{\lambda_t^j\}$ . The estimation of the martingale representation will then be straightforward.

Figures I to II show the plots of  $\hat{\lambda}_t^j$  for the two chosen credits. We see the data now looks more like a “stationary” time series and hence can be treated as standard economic or financial data. In fact a Wald representation can be estimated in a straightforward fashion. In fact, using one lagged value for every point process we can write:

$$\begin{bmatrix} dN_t^1 \\ \dots \\ dN_t^n \end{bmatrix} = \begin{bmatrix} \lambda_t^1 dt \\ \dots \\ \lambda_t^n dt \end{bmatrix} + \begin{bmatrix} h_{11} & \dots & h_{1n} \\ \dots & \dots & \dots \\ h_{n1} & \dots & h_{nn} \end{bmatrix} \begin{bmatrix} dN_t^1 - \lambda_t^1 dt \\ \dots \\ dN_t^n - \lambda_t^n dt \end{bmatrix} + \begin{bmatrix} dM_t^1 \\ \dots \\ dM_t^n \end{bmatrix}$$

and then estimate the relevant parameters.

Table III shows an example. The dynamic correlations between Argentina and Brazil credit events was estimated. the Table shows the estimated coefficients and other test statistics. We see that Argentina credit events significantly explain Brazil events.

## 8 Conclusions

This note intended to illustrate aspects of investigating correlations between sequences of events that one observes among different default-events. The test statistics provided in the paper illustrated how utilization of point process methods can help determine the existence or lack of statistical correlations between default-events.

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